The Rocky Mountain Journal of Mathematics **30**(2000), 411-434

STABILITY OF BROCARD POINTS OF POLYGONS

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Abstract. A continuous nested sequence of similar triangles converging to the Brocard point of a given triangle is investigated. All these triangles have the same Brocard point. For polygons, the Brocard point need not exist, but there is always a limit object for an analogously defined nested sequence of inner polygons. This limit object is a Brocard point if and only if the inner polygons are all similar to the original polygon. The similarity of two distinct inner polygons already suffices. In that case, all the inner polygons have the same Brocard point.

1. Introduction

The positive Brocard point of a triangle $A_1A_2A_3$ is the unique point Ω within the triangle such that the angle between $A_i\Omega$ and A_iA_{i+1} is the same for all i modulo 3. This is illustrated in Figure 2 (where the vertices are denoted A, B, C). The earliest easily accessible reference to the Brocard point that we are aware of is [1]. According to Honsberger [7], the Brocard point was already known to Crelle, Jacobi and others at the beginning of the 19th century. Indeed, the historically more accurate name of **Crelle-Brocard point** is used by Mitrinovic, Pecaric and Volenec [16] (where other references to both older and contemporary work are also given).

Traditionally, the Brocard point was constructed by ruler and compass: see Honsberger [7], Johnson [8], Shively [17]. An entirely different approach to generate the Brocard point, by an infinite limit process, was taken by Yff in [18]. Another infinite limit process to generate the Brocard point was described by the present authors in [2]. In this latter paper the limit process was defined for arbitrary convex polygons, and it yields the Brocard point whenever it exists, as in Figure 3. (For n–gons the Brocard point is defined analogously with the triangle case as above, with arbitrary n instead of 3.) In the present paper we analyze the Brocard point limit process described in [2], both for triangles and general convex polygons.

For yet another approach to the generation of triangle centers, in fact placing [1] in a general framework, see Kimberling [9]–[14]. Note also that the infinite process we consider in [2], and in the present paper, is based on non-concurrent cevians converging to concurrency. The Brocard point theory has already been linked to Ceva's Theorem by a proof of Abi-Khuzam's inequality due to Veldkamp, Stroeker and Hoogland (see [16]). Very recent work on polygonal generalizations of Ceva's Theorem includes Grünbaum and Shephard [3]–[6], where further references are given.

Notation and terminology. All points and sets are in the real Euclidean plane $\mathbb{R}^2$.

For any two points X, Y, the directed segment (vector) from X to Y is denoted by $\overrightarrow{XY}$, and its **length** by $|XY|$.

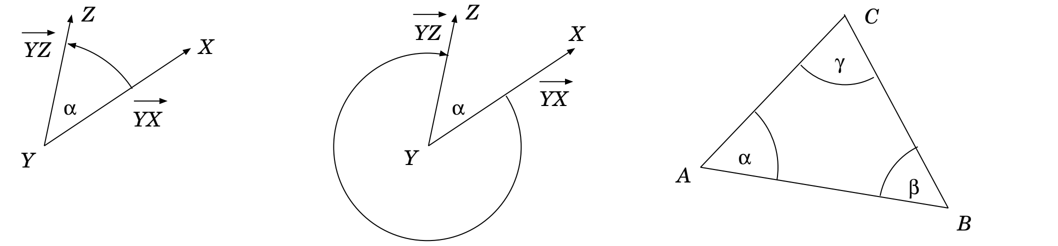
Given three distinct points X, Y, Z, the vector $\overrightarrow{YX}$ can be rotated around Y to the direction of $\overrightarrow{YZ}$ in two ways, see Figure 1(a),(b). The **signed angle** $\angle XYZ$ is the smaller of these two rotations, and is positive if the rotation is counterclockwise, negative otherwise.

If the points X, Y, Z are collinear, with Y between X and Z, we define $\angle XYZ = \pi$.

The **absolute angle**, or just **angle**, is the absolute value of $\angle XYZ$. Absolute angles are denoted by lower case Greek letters.

For the triangle ABC of Figure 1(c) it follows that

$$\angle BAC + \angle CBA + \angle ACB = \alpha + \beta + \gamma \ , \quad \angle ABC + \angle BCA + \angle CAB = -(\alpha + \beta + \gamma) \ .$$



(a) $\angle XYZ \ = \ \alpha$ (b) $\angle ZYX \ = \ -\alpha$ (c) The (absolute) angles of a triangle

FIGURE 1. Signed and absolute angles

Definition 1. A **direct similarity** is a map $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that:
(a) there is a positive real number t, called the **stretch ratio** of h, such that

$$|h(x)h(y)| \ = \ t\,|xy| \ , \quad \text{for any two points } x, y \tag{1}$$

(b) for any three distinct points x, y, z:

$$\angle h(x)h(y)h(z) \ = \ \angle xyz \tag{2}$$

Remark 1. Note that Definition 1(a) implies that h is injective. In fact, the elementary theory of similarities tells us the following (see e.g. [15]):
(a) Direct similarities are bijective maps: $\mathbb{R}^2 \to \mathbb{R}^2$, and form a group under composition, with the identity map as neutral element.
(b) The image, under a direct similarity, of any convex set is convex.
(c) Each direct similarity is determined by its action on any two distinct points.

A convex polygon in the Euclidean plane $\mathbb{R}^2$ can be represented in two ways:

- An **intersection** of finitely many halfplanes.
- A **convex hull** of finitely many points, the **vertices** of the polygon.

In the latter case we assume that the set of vertices $\{V_1, \cdots, V_n\}$ is minimal, and ordered, enumerated clockwise or counterclockwise. The indices $1, 2, \cdots, n$ are understood modulo n, so that $V_n = V_0, V_{n+1} = V_1$, etc.

Definition 2. Let Π and Π' be two polygons with the same number of vertices, enumerated counterclockwise as $V_1, \cdots, V_n$ and $V_1', \cdots, V_n'$, respectively. Then Π and Π' are called **similar** if there is a direct similarity h such that $h(V_i) = V_i'$, $i = 1, \cdots, n$. We write $\Pi \sim \Pi'$, the corresponding vertex sequences being understood.

Remark 2.
(a) Definition 2 can be restated as follows: Two polygons Π and Π' with vertices enumerated counterclockwise $V_1, V_2, \cdots, V_n$ and $V_1', V_2', \cdots, V_n'$ respectively, are similar if:

- Corresponding signed angles are equal,

$$\angle\, V_{i-1}V_iV_{i+1} \ = \ \angle\, V_{i-1}'V_i'V_{i+1}' \ , \quad i = 1, \cdots, n$$

- Corresponding sides have equal ratios,

$$\frac{|V_i'V_{i+1}'|}{|V_iV_{i+1}|} = \text{constant} \ , \quad i = 1, \cdots, n$$

Date: December 7, 1998. Revised May 5, 1999.
1991 *Mathematics Subject Classification.* Primary 51M15, 51M20; Secondary 51M04, 52C05.
Key words and phrases. Triangle Geometry, Brocard Angle, Brocard Points, Convex Polygons, Similarity.
The work of the second named author was partially supported by ONR grant N00014-92-J-1375 and by RUTCOR.

(b) Polygon similarity is an order specific property. For example, a triangle ABC is in general not similar to the triangle BCA or to CBA. However, if $ABC \sim A'B'C'$ then $BCA \sim B'C'A'$ and also $CBA \sim C'B'A'$.

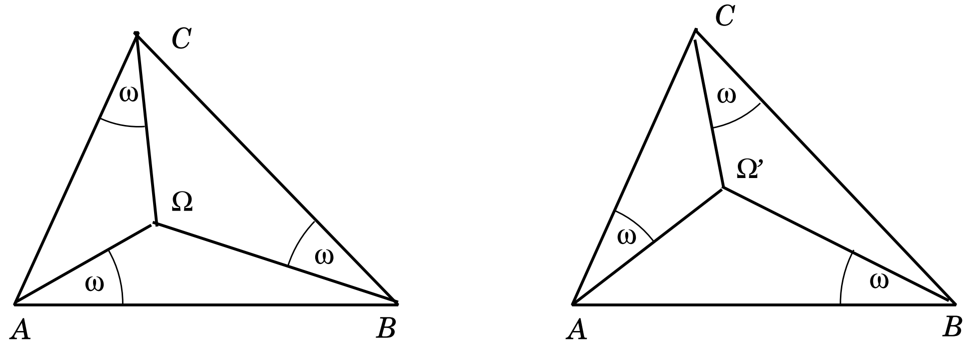
Remark 3. Let A, B, C, D be 4 distinct points, and consider the 4 triangles ABC, ABD, ACD, BCD. If any three of the points A, B, C, D are collinear, they define a degenerate triangle (segment).

Let A', B', C', D' be a set of corresponding points. If any two of the triangle pairs are similar, say

$$ABC \sim A'B'C' \text{ and } ABD \sim A'B'D'$$

then the other two pairs are similar,

$$ACD \sim A'C'D' \text{ and } BCD \sim B'C'D'$$



(a) The point Ω and angle ω (b) The point Ω' and angle ω

FIGURE 2. The Brocard angle ω, and two Brocard points Ω, Ω' of a triangle ABC.

2. The Brocard transformation

Given a triangle ABC, there is a unique angle ω and a unique point Ω such that

$$\omega = \angle AC\Omega = \angle BA\Omega = \angle CB\Omega ,$$

see Figure 2(a). The angle ω is called the **Brocard angle** and the point Ω is the (**positive**) **Brocard point** of the triangle. The negative Brocard point, Ω', is the isogonal conjugate of Ω, and

$$\omega = \angle \Omega' AC = \angle \Omega' BA = \angle \Omega' CB ,$$

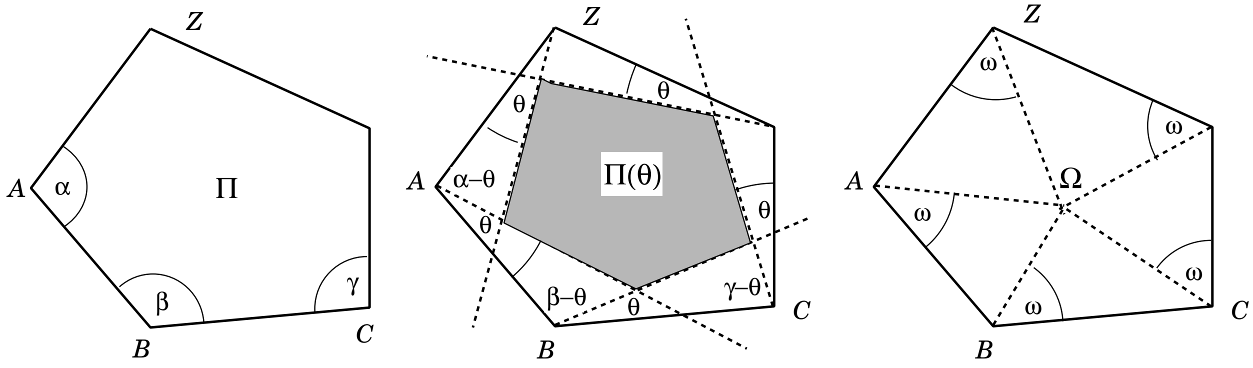
see Figure 2(b). The two Brocard points coincide if the triangle is equilateral, in which case $\omega = \frac{\pi}{6}$.

We study Brocard points and angles for polygons, using the following transformation (introduced in [2]).

Definition 3. Let the convex polygon Π have n vertices $V_1, V_2, \cdots, V_n$ (numbered counter-clockwise) and let θ be an angle not exceeding the smallest angle of the polygon. For $i = 1, \cdots, n$ let

- $L_i(\theta)$ be the line through V_i with (counter-clockwise) angle θ from the direction $\overrightarrow{V_i V_{i+1}}$,
- $L_i^+(\theta)$ be the closed half-plane defined by $L_i(\theta)$, which:
  - if $\theta > 0$, excludes the next vertex V_{i+1}, and
  - if $\theta = 0$, includes the vertex V_{i+2}.

(a) The (**positive**) **Brocard transform** $\Pi(\theta)$ is the intersection (possibly empty) of the n half-planes $L_i^+(\theta)$, $i = 1, \cdots, n$.
(b) The (**positive**) **Brocard angle** ω is the largest angle θ with nonempty $\Pi(\theta)$ (see Remark 4(c)).
(c) If $\Pi(\omega)$ is a singleton $\{\Omega\}$, and if all lines $\{L_i(\omega) : i = 1, \cdots, n\}$ intersect at Ω, then Ω is called the (**positive**) **Brocard point** of Π.
The **negative** Brocard transforms, angle and point are defined analogously, and have analogous properties. It therefore suffices to study the positive Brocard objects, and the adjective **positive** can be omitted, as we do below.
The Brocard transformation is illustrated in Figure 3(b), for the polygon Π with vertices $A, B, \cdots, Z$. The Brocard point and angle are shown in Figure 3(c).



(a) A polygon Π with vertices $A, B, \cdots, Z$ (b) A Brocard transform $\Pi(\theta)$ (c) The Brocard point Ω and angle ω

FIGURE 3. Illustration of the Brocard transformation of polygons.

Remark 4. (a) The Brocard transforms $\Pi(\theta)$ are closed convex polygons, by their definition as intersections of finitely many closed half-planes.
(b) $\Pi(0)$ coincides with Π. If Π is nonempty, it follows that $\Pi(\theta)$ is nonempty for all sufficiently small θ.
(c) The Brocard transformation is monotone in the sense that

$$0 \leq \theta_1 \leq \theta_2 \quad \Longrightarrow \quad \Pi(\theta_2) \subseteq \Pi(\theta_1) \subseteq \Pi$$

so that

$$\Pi(\theta) = \bigcap_{0 \leq \alpha \leq \theta} \Pi(\alpha) \tag{3}$$

showing that the Brocard angle ω is well-defined. Its existence follows by a standard compactness argument.
(d) If Π is an n–polygon, the polygon $\Pi(\omega)$ is either

- a singleton, the intersection of n lines $L_i(\omega)$, or
- a singleton, the intersection of fewer than n lines, or
- a line segment.

The Brocard point exists only in the first case.

Our main results are:
Existence: Given a polygon Π, the following statements are equivalent:

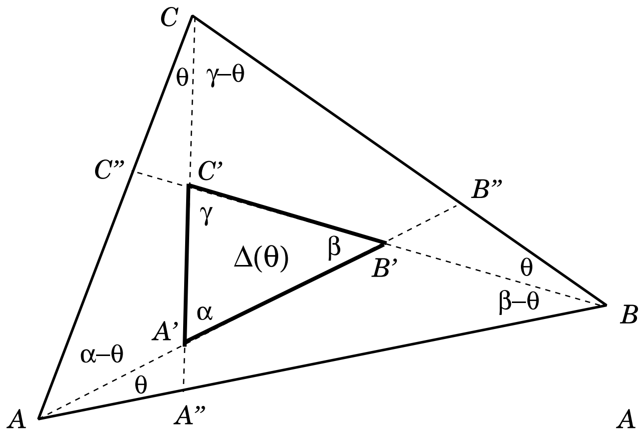
- Π has a Brocard point
- $\Pi \sim \Pi(\theta)$ for some $0 < \theta < \omega$
- $\Pi \sim \Pi(\theta)$ for all $0 \leq \theta < \omega$
- $\Pi(\theta_1) \sim \Pi(\theta_2)$ for some $0 \leq \theta_1 < \theta_2 < \omega$

By definition, similar Brocard transforms $\Pi(\theta)$ have the same number of vertices $V_1(\theta), \cdots, V_n(\theta)$. We number corresponding vertices consistently as $V_i(\theta) := L_{i-1}(\theta) \cap L_i(\theta)$.
Stability: If the polygon Π has a Brocard point, then all polygons $\{\Pi(\theta) : 0 \leq \theta < \omega\}$ have the same Brocard point.
Triangles, considered in § 3, present a special case: both (positive and negative) Brocard points exist, and the (positive and negative) Brocard angles are equal. The proof of the stability result for triangles is particularly simple, see Theorem 1.

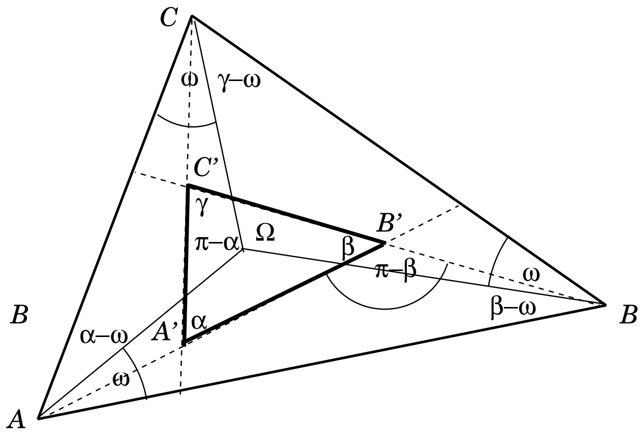
For general polygons, considered in § 4, neither existence of Brocard points, nor equality of Brocard angles, is guaranteed.

3. TRIANGLES

Let Δ be a triangle with vertices A, B, C and let θ be any angle smaller than the Brocard angle ω of Δ. The **Brocard transform** $\Delta(\theta)$ is illustrated in Figure 4.



(a) $\Delta(\theta)$ positioned in the original triangle.



(b) $\Delta(\theta)$ and the Brocard point Ω

FIGURE 4. Illustration of the triangle $\Delta(\theta)$.

The original triangle corresponds to $\theta = 0$ and is denoted $\Delta(0)$. The triangles $\Delta(0)$ and $\Delta(\theta)$ are similar, and therefore all triangles $\{\Delta(\theta) : 0 \le \theta < \omega\}$ are similar.

We denote the area of the triangle $\Delta(\theta)$ by $A(\theta)$. The ratio $A(\theta)/A(0)$ is therefore the square of the ratio of lengths of corresponding sides of $\Delta(\theta)$ and $\Delta(0)$,

$$\frac{A(\theta)}{A(0)} = k^2(\theta) \tag{4a}$$

$$\text{where} \quad k(\theta) = \frac{|A'B'|}{|AB|}\ , \quad \text{see Figure 4}\ . \tag{4b}$$

The factor $k(\theta)$ is calculated here twice, first in terms of the three angles α, β and γ (Lemma 1), then in terms of the Brocard angle ω (Lemma 2).

Lemma 1.

$$k(\theta) = \cos\theta - \ell\sin\theta\ , \quad \text{where } \ell = \cot\beta + \frac{\sin\beta}{\sin\alpha\sin\gamma} \tag{5}$$

Proof. Using the notation of Figure 4(a),

$$\begin{aligned}
|A'B'| &= |AB'| - |AA'| \\
|AB'| &= |AB|\,\frac{\sin(\beta-\theta)}{\sin\beta} \\
|AA'| &= |AC|\,\frac{\sin\theta}{\sin\alpha} = |AB|\,\frac{\sin\beta}{\sin\gamma}\,\frac{\sin\theta}{\sin\alpha} \\
\therefore\ |A'B'| &= |AB|\left(\frac{\sin(\beta-\theta)}{\sin\beta} - \frac{\sin\beta\,\sin\theta}{\sin\alpha\,\sin\gamma}\right) \\
\therefore\ k(\theta) &= \frac{\sin(\beta-\theta)}{\sin\beta} - \frac{\sin\beta\,\sin\theta}{\sin\alpha\,\sin\gamma}\ , \quad \text{by (4b)}\ , \\
&= \cos\theta - \sin\theta\left(\cot\beta + \frac{\sin\beta}{\sin\alpha\sin\gamma}\right)
\end{aligned} \tag{6}$$

□

Lemma 2.

$$k(\theta) = \frac{\sin(\omega-\theta)}{\sin\omega} \tag{7}$$

Proof. Using the notation of Figure 4(b),

$$\begin{aligned}
|AA'| &= |AC|\,\frac{\sin\theta}{\sin\alpha} = |A\Omega|\,\frac{\sin(\pi-\alpha)}{\sin\omega}\,\frac{\sin\theta}{\sin\alpha} \\
&= |AB|\,\frac{\sin(\beta-\omega)}{\sin\beta}\,\frac{\sin\alpha}{\sin\omega}\,\frac{\sin\theta}{\sin\alpha} \\
\therefore\ |A'B'| &= |AB'|-|AA'| = |AB|\left(\frac{\sin(\beta-\theta)}{\sin\beta}-\frac{\sin\theta}{\sin\omega}\,\frac{\sin(\beta-\omega)}{\sin\beta}\right)\,, \quad \text{by (6)}\,. \\
\therefore\ k(\theta) &= \frac{|A'B'|}{|AB|} = \frac{\sin(\beta-\theta)}{\sin\beta}-\frac{\sin\theta}{\sin\omega}\,\frac{\sin(\beta-\omega)}{\sin\beta} \\
&= \cos\theta-\cot\omega\ \sin\theta \qquad (8) \\
&= \frac{\sin(\omega-\theta)}{\sin\omega}\,.
\end{aligned}$$

□

Remark 5. By comparing (5) and (8) we obtain the following well known identity, giving the Brocard angle in terms of the angles of the triangle,

$$\begin{aligned}
\cot\omega &= \cot\beta+\frac{\sin\beta}{\sin\alpha\,\sin\gamma} = \cot\beta+\frac{\sin(\alpha+\gamma)}{\sin\alpha\,\sin\gamma} \\
&= \cot\alpha+\cot\beta+\cot\gamma \qquad (9)
\end{aligned}$$

The following result is needed in the sequel.

Lemma 3. Let ω be the Brocard angle of a triangle with angles α, β and γ. Then

$$\frac{\sin(\beta-\omega)}{\sin\omega}=\frac{\sin^2\beta}{\sin\alpha\ \sin\gamma} \qquad (10)$$

Proof.

$$\begin{aligned}
\frac{\sin(\beta-\omega)}{\sin\omega} &= \frac{\sin\beta\ \cos\omega-\sin\omega\ \cos\beta}{\sin\omega} = \sin\beta\ \cot\omega-\cos\beta \\
&= \sin\beta\ (\cot\alpha+\cot\beta+\cot\gamma)-\cos\beta\,, \quad \text{by (9)}\,, \\
&= \sin\beta\ \left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}+\frac{\cos\gamma}{\sin\gamma}\right)-\cos\beta \\
&= \sin\beta\ \left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\gamma}{\sin\gamma}\right) = \sin\beta\ \frac{\cos\alpha\ \sin\gamma+\cos\gamma\ \sin\alpha}{\sin\alpha\ \sin\gamma} \\
&= \sin\beta\ \frac{\sin(\alpha+\gamma)}{\sin\alpha\ \sin\gamma} \\
&= \frac{\sin^2\beta}{\sin\alpha\ \sin\gamma}\,.
\end{aligned}$$

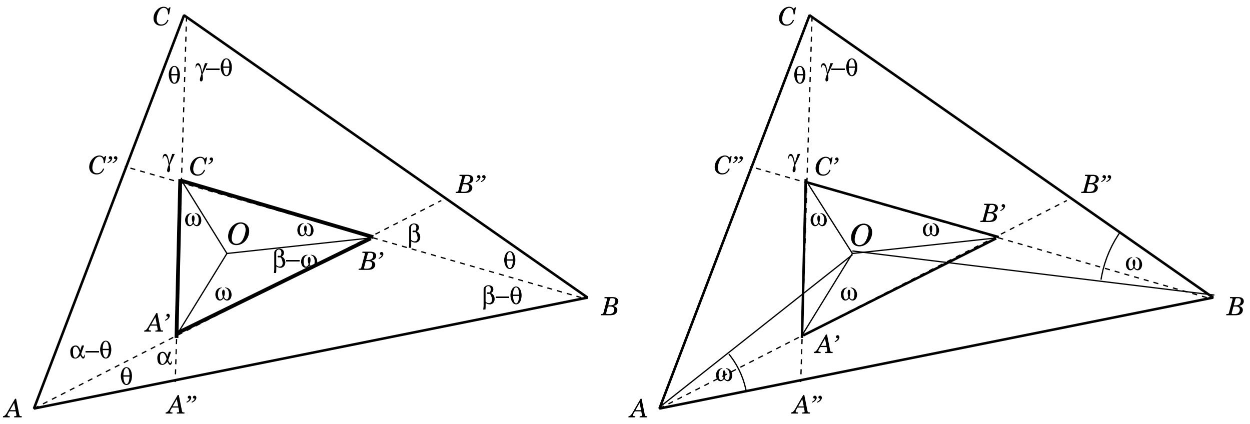
□

Given a triangle Δ with Brocard angle ω, consider all Brocard transforms $\{\Delta(\theta) : 0 \le \theta < \omega\}$ of Δ. These triangles are similar, and therefore have the same Brocard angle ω. We prove that they also share the (positive) Brocard point Ω, i.e. Ω is stable under the Brocard transformations.

As θ increases from 0 to ω, the triangles $\Delta(\theta)$ shrink, and their areas $A(\theta)$ satisfy,

$$\frac{A(\theta)}{A(0)}=\frac{\sin^2(\omega-\theta)}{\sin^2\omega} \qquad (11)$$

see (4a) and (7). In particular, $A(\omega)=0$ i.e. $\Delta(\omega)$ is a point, which by definition is the positive Brocard point Ω of Δ.



(a) O is a Brocard point of $\Delta(\theta)$. (b) The triangles $AA'O$ and $BB'O$ are similar.

FIGURE 5. Illustration of Theorem 1.

Theorem 1. Given a triangle Δ with Brocard angle ω and a (positive) Brocard point Ω, all the triangles $\{\Delta(\theta) : 0 \leq \theta < \omega\}$ have the same Brocard point.

Proof. We prove that Δ and $\Delta(\theta)$ have the same (positive) Brocard point, for any $0 < \theta < \omega$.

Let O be the positive Brocard point of $\Delta(\theta)$, see Figure 5(a). It suffices to show the equality of the signed angles

$$\angle BAO = \angle CBO \,. \tag{12}$$

To prove (12) we show that the triangles $AA'O$ and $BB'O$ are similar, see Figure 5(b).

A repeated application of the sine-rule gives

$$\begin{aligned} \frac{|AA'|}{|AC|} &= \frac{\sin\theta}{\sin\alpha} \\ \frac{|BB'|}{|AB|} &= \frac{\sin\theta}{\sin\beta} \\ \therefore \quad \frac{|AA'|}{|BB'|} &= \frac{|AC|}{|AB|}\,\frac{\sin\beta}{\sin\alpha} = \frac{\sin^2\beta}{\sin\alpha\ \sin\gamma} \\ \text{also} \quad \frac{|A'O|}{|B'O|} &= \frac{\sin(\beta-\omega)}{\sin\omega} \end{aligned}$$

From the last two equations and (10) we conclude

$$\frac{|AA'|}{|A'O|} = \frac{|BB'|}{|B'O|}$$

showing that the triangles $AA'O$ and $BB'O$ are similar. □

4. POLYGONS

Theorem 2. Let Π be a nonempty convex n–polygon. Then the following are equivalent:
(a) Π has a Brocard point.
(b) $\Pi \sim \Pi(\theta)$ for all $0 \leq \theta < \omega$ where ω is the Brocard angle of Π.
(c) $\Pi \sim \Pi(\theta)$ for some $0 < \theta < \omega$.
(d) There exist two angles $0 \leq \theta_1 < \theta_2 < \omega$ such that $\Pi(\theta_1)$ and $\Pi(\theta_2)$ are similar n-polygons.
If these conditions hold, all Brocard transforms $\{\Pi(\theta) : 0 \leq \theta < \omega\}$ have the same Brocard point.

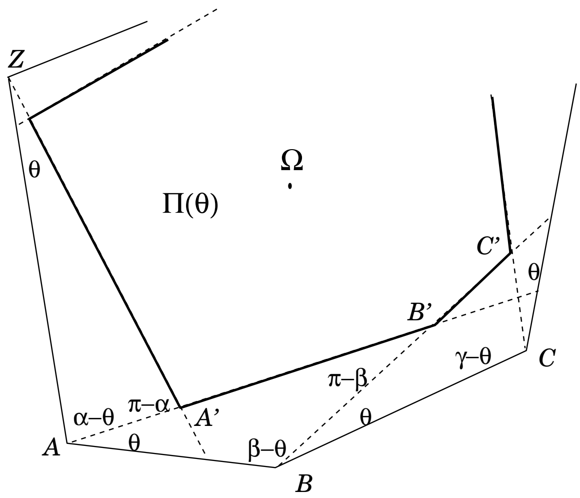


FIGURE 6. $\Pi(\theta)$ positioned in the original polygon.

Proof.
(a) $\Longrightarrow$ (b): Let Ω be the Brocard point of Π. Then

$$\begin{aligned}
|A'B'| &= |AB'| - |AA'| \,, \quad \text{see Figure 6}\,, \\
|AB'| &= |AB| \, \frac{\sin(\beta-\theta)}{\sin\beta} \\
|AA'| &= |AZ| \, \frac{\sin\theta}{\sin\alpha} \\
&= |A\Omega| \, \frac{\sin\alpha}{\sin\omega} \, \frac{\sin\theta}{\sin\alpha} \\
&= |AB| \, \frac{\sin(\beta-\omega)}{\sin\beta} \, \frac{\sin\alpha}{\sin\omega} \, \frac{\sin\theta}{\sin\alpha} \\
\therefore \quad \frac{|A'B'|}{|AB|} &= \frac{\sin(\beta-\theta)}{\sin\beta} - \frac{\sin\theta \, \sin(\beta-\omega)}{\sin\omega \, \sin\beta} \\
&= \cos\theta - \cot\omega \, \sin\theta
\end{aligned}$$

proving that $\Pi \sim \Pi(\theta)$.
(b) $\Longrightarrow$ (c): Clear.
(c) $\Longrightarrow$ (d): Take $\theta_1 = 0$, $\theta_2 = \theta$.
(d) $\Longrightarrow$ (a): Let $0 \le \theta_1 < \theta_2 < \omega$ be such that $\Pi(\theta_1) \sim \Pi(\theta_2)$. The difference of these two angles is denoted by

$$\delta := \theta_2 - \theta_1 \tag{13}$$

Let h be the direct similarity

$$\Pi(\theta_2) \,=\, h(\Pi(\theta_1)) \tag{14}$$

The vertices of Π, $\Pi(\theta_1)$, $\Pi(\theta_2)$ are denoted by $A, B, C, \cdots, Z$; $A_1, B_1, C_1, \cdots, Z_1$; $A_2, B_2, C_2, \cdots, Z_2$ respectively, where

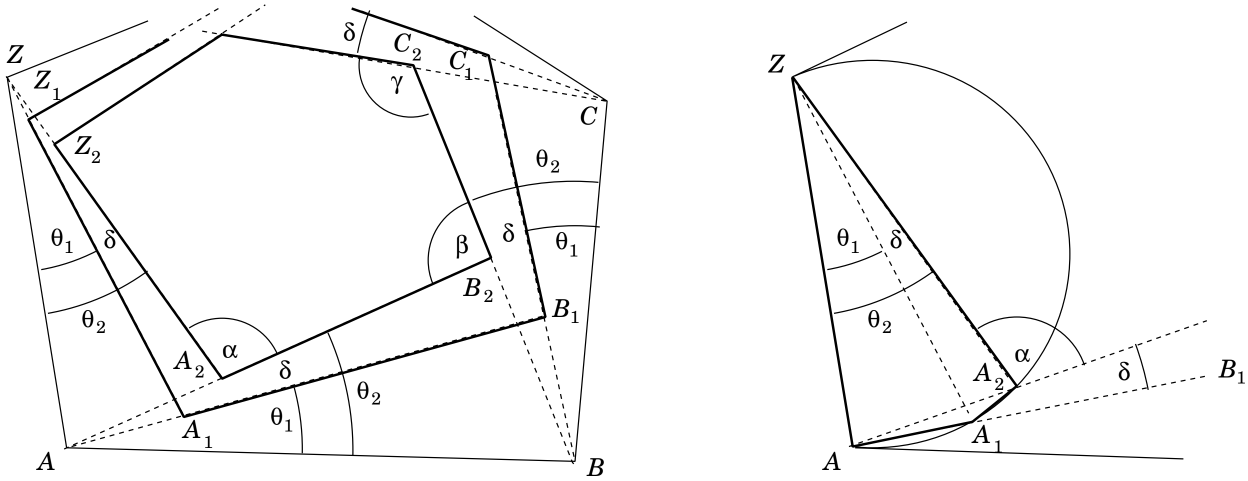
$$A_2 \,=\, h(A_1)\,,\ B_2 \,=\, h(B_1)\,, \cdots\ Z_2 \,=\, h(Z_1)$$

see Figure 7(a). The angles of the similar polygons $\Pi(\theta_1)$, $\Pi(\theta_2)$ are denoted α, β, $\gamma \cdots$.

Corresponding sides in $\Pi(\theta_1)$, $\Pi(\theta_2)$ are related by

$$\frac{|A_2B_2|}{|A_1B_1|} \,=\, t \tag{15}$$

where $0 < t < 1$ is the stretch ratio of the similarity h.



(a) The polygons Π and $\Pi(\theta_1) \sim \Pi(\theta_2)$. (b) The points A , A_1 , A_2 , Z are concyclic.

FIGURE 7. Illustration of Theorem 2.

Define the sequence of similar polygons $\{\Pi_n\}$ by

$$\Pi_n := h(\Pi_{n-1}) = h^{n-1}(\Pi_1) , \quad \text{with } \Pi_1 := \Pi(\theta_1) . \tag{16}$$

Clearly the polygons Π_n are nested, and become smaller as n increases. We prove that there is a point Ω such that:

$$\lim_{n\to\infty} \Pi_n = \{\Omega\} \tag{17}$$

$$\angle BA\Omega = \angle CB\Omega = \cdots = \angle AZ\Omega \tag{18}$$

i.e. Ω is the Brocard point of the original polygon Π.

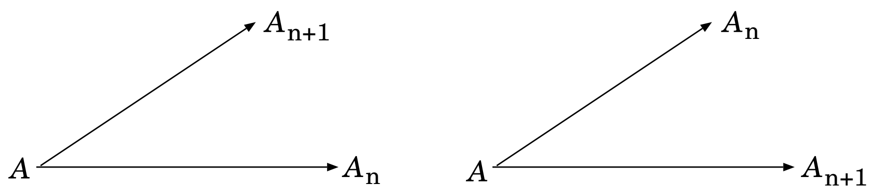
Proof of (17): Denote the vertices of Π_n by $\{A_n, B_n, \cdots, Z_n\}$. Then

$$|A_nA_{n+1}| = |h^{n-1}(A_1)h^{n-1}(A_2)| = t^{n-1}|A_1A_2|$$

shows that $|A_nA_{n+1}| \to 0$, i.e. the sequence $\{A_n\}$ converges to some point, say Ω. But

$$|A_nB_n| = |h^{n-1}(A_1)h^{n-1}(B_1)| = t^{n-1}|A_1B_1|$$

shows that $|A_nB_n| \to 0$. Therefore the sequence $\{B_n\}$ converges to the same point. Similarly all vertices of Π_n converge to Ω, proving (17).



(a) Positive angle. (b) Negative angle.

FIGURE 8. The signed angle $\angle A_nAA_{n+1}$.

Proof of (18): The angle $\angle BA\Omega$ is the sum

$$\angle BA\Omega = \theta_1 + \sum_{n=1}^{\infty} \angle A_nAA_{n+1} \tag{19}$$

where some of the signed angles in the right side may be negative, see e.g. Figure 8(b). We prove (18) by showing the equality of the signed angles

$$\angle A_nAA_{n+1} = \angle B_nBB_{n+1} = \cdots = \angle Z_nZZ_{n+1}$$

which we prove by establishing the similarity of triangles

$$AA_nA_{n+1} \sim BB_nB_{n+1} \sim \cdots \sim ZZ_nZ_{n+1} \tag{20}$$

proved by induction on n.
Verification of (20) for $n = 1$: The equality of angles $\angle ZA_1A = \angle ZA_2A$ implies that the 4 points A, A_1, A_2, Z are concyclic, see Figure 7(b). We can thus compute the angles

$$\angle A_2A_1A = \pi - \theta_2 \tag{21}$$

$$\therefore \quad \angle B_1A_1A_2 = \theta_2 \tag{22}$$

$$\angle AA_2A_1 = \pi - (\pi - \theta_2) - \delta = \theta_1 \ , \text{ by (13)}. \tag{23}$$

It follows from (21) and (23) that the angles of the triangle AA_1A_2 depend only on θ_1 and θ_2, showing the similarity of triangles

$$AA_1A_2 \sim BB_1B_2 \sim \cdots \sim ZZ_1Z_2 \ .$$

verifying (20) for $n = 1$.
The inductive step: Assume (20) for n, and we'll prove it for $n+1$. We first prove the similarity of the triangles

$$A_nA_{n+1}A_{n+2} \sim B_nB_{n+1}B_{n+2} \sim \cdots \sim Z_nZ_{n+1}Z_{n+2} \tag{24}$$

for all n. Since $A_nA_{n+1}A_{n+2} = h^{n-1}(A_1A_2A_3)$, $B_nB_{n+1}B_{n+2} = h^{n-1}(B_1B_2B_3) \cdots$ it is enough to prove (24) for $n = 1$, i.e.,

$$A_1A_2A_3 \sim B_1B_2B_3 \sim \cdots \sim Z_1Z_2Z_3 \tag{25}$$

Since $B_2A_2A_3 = h(B_1A_1A_2)$ we have, see Figure 9,

$$\angle B_2A_2A_3 = \angle B_1A_1A_2 = \theta_2 \ , \quad \text{by (22)}$$

$$\therefore \quad \angle A_3A_2A_1 = \theta_1 + \pi - \theta_2 = \pi - \delta \tag{26}$$

Moreover, since $A_2 = h(A_1)$, $A_3 = h(A_2)$,

$$\frac{|A_2A_3|}{|A_1A_2|} = t \tag{27}$$

Since (26) and (27) depend only on θ_1, θ_2 and the similarity h, it follows that all triangles in (25) are similar.

Combining (20) and (24),

$$AA_nA_{n+1} \sim BB_nB_{n+1} \sim \cdots$$
$$A_nA_{n+1}A_{n+2} \sim B_nB_{n+1}B_{n+2} \sim \cdots$$

it follows from Remark 3 that

$$AA_{n+1}A_{n+2} \sim BB_{n+1}B_{n+2} \sim \cdots$$

which is (20) for $n+1$.

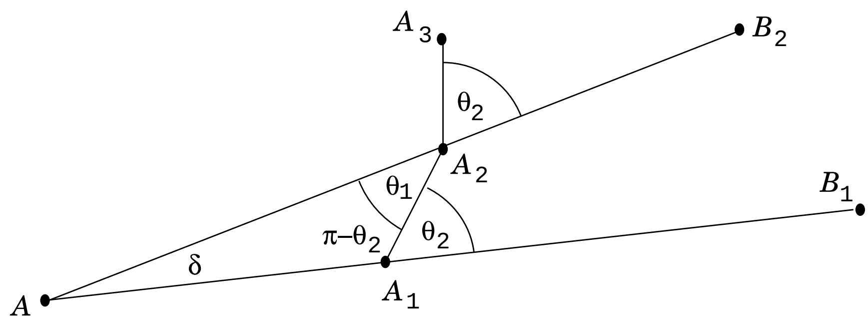


Figure 9. The triangle $A_1A_2A_3$.

Finally, we prove that if Π has a Brocard point, then all Brocard transforms $\{\Pi(\theta) : 0 \leq \theta < \omega\}$ have the same Brocard point. Let $\Pi(\theta)$ be one such transform. Because it is similar to Π, it has a Brocard point $\bar{\Omega}$, see Figure 10(a). We prove that $\bar{\Omega}$ is a Brocard point of Π by showing that

$$\angle BA\bar{\Omega} = \angle CB\bar{\Omega} = \cdots$$

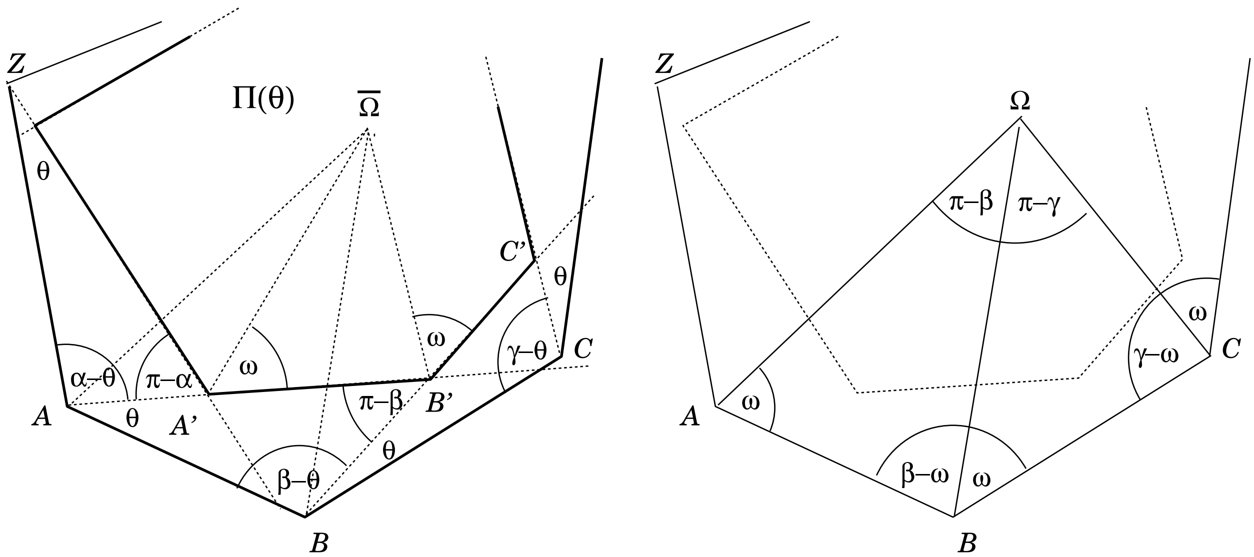
which will follow from the similarity of triangles

$$AA'\bar{\Omega} \sim BB'\bar{\Omega} \sim \cdots \tag{28}$$

Since the angles $\angle\bar{\Omega}A'A = \angle\bar{\Omega}B'B = \cdots = \pi - \omega$, it is enough to prove the equality of ratios of corresponding sides, say

$$\frac{|BB'|}{|CC'|} = \frac{|B'\bar{\Omega}|}{|C'\bar{\Omega}|} \tag{29}$$

Applying the sine rule to the triangle ABB' (see Figure 10(a)),



(a) $\bar{\Omega}$ of $\Pi(\theta)$. (b) Ω of Π.

FIGURE 10. The Brocard points of $\Pi(\theta)$ and Π.

$$\frac{|BB'|}{|AB|} = \frac{\sin\theta}{\sin\beta}$$

Analogously,

$$\frac{|CC'|}{|BC|} = \frac{\sin\theta}{\sin\gamma}$$

Therefore

$$\frac{|BB'|}{|CC'|} = \frac{|AB|}{|BC|}\frac{\sin\gamma}{\sin\beta} \tag{30}$$

Let now Ω be the Brocard point of Π, see Figure 10(b). Then

$$\begin{aligned}
\frac{|\Omega B|}{|BC|} &= \frac{\sin(\gamma-\omega)}{\sin(\pi-\gamma)} = \frac{\sin(\gamma-\omega)}{\sin\gamma} \\
\frac{|\Omega B|}{|BC|} &= \frac{|\Omega B|}{|AB|}\,\frac{|AB|}{|BC|} = \frac{\sin\omega}{\sin(\pi-\beta)}\,\frac{|AB|}{|BC|} = \frac{\sin\omega}{\sin\beta}\,\frac{|AB|}{|BC|} \\
\therefore\quad \frac{\sin(\gamma-\omega)}{\sin\gamma} &= \frac{\sin\omega}{\sin\beta}\,\frac{|AB|}{|BC|} \\
\text{or}\quad \frac{\sin(\gamma-\omega)}{\sin\omega} &= \frac{\sin\gamma}{\sin\beta}\,\frac{|AB|}{|BC|} \qquad (31)\\
&= \frac{|BB'|}{|CC'|}\,, \quad \text{by (30)}\,. \qquad (32)
\end{aligned}$$

Applying the sine rule to the triangle $\bar{\Omega}B'C'$, we get

$$\frac{\sin(\gamma-\omega)}{\sin\omega} = \frac{|B'\bar{\Omega}|}{|C'\bar{\Omega}|}$$

which, combined with (32), proves (29). □

If a convex polygon Π with internal angles $\alpha_1, \cdots, \alpha_n$ has a positive Brocard point with positive Brocard angle ω, then it is not difficult to show (see e.g. [2]) that

$$\sin^n\omega = \prod_i \sin(\alpha_i - \omega)$$

Thus, for a polygon Π that has a positive Brocard point, the positive Brocard angle ω is fully determined by the internal angles α_i. Further, if Π also has a negative Brocard point, then the negative and positive Brocard angles are the same.

However, the existence of a positive Brocard point does not imply that a negative Brocard point also exists. For example, consider the non-regular hexagon of Figure 11 with vertices $V_1 = (0,-1)$, $V_2 = (1,-1)$, $V_3 = (3/2,-1/2)$, $V_4 = (3/2,1/2)$, $V_5 = (1,1/2)$ and where the vertex V_6 is the intersection, with negative ordinate, of the line through V_5 with slope 1, and the unit circle with center $(0,0)$. This hexagon has a positive Brocard point and a positive Brocard angle of $\pi/4$, and it can easily be verified that the negative Brocard angle is less than $\pi/4$. A different example appears in [2].

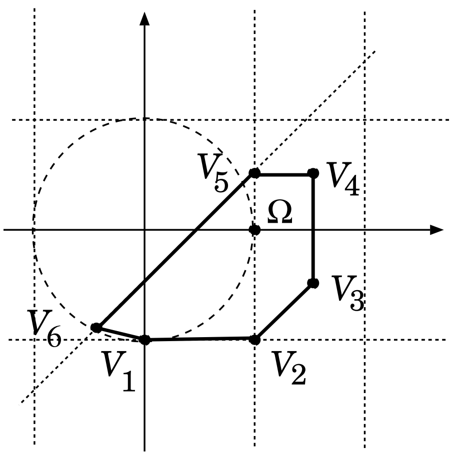


FIGURE 11. A non-regular hexagon with different Brocard angles.

Note. We thank the referee for pointing out to us the infinite limit process described by Yff [18] to construct the Brocard point. The referee has also noted that the ratios $k(\theta)$ of (4a), (4b) and (7) appear in [18] as the ratios of similarity of the inscribed Miquel triangles that constitute the possible starting points of Yff's infinite limit process.

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