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ON MATRICES OF INDEX ZERO OR ONE*

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Notations and preliminaries. We shall let C^n denote n-dimensional complex vector space; $C^{m\times n}$, the $m \times n$ complex matrices; $C_r^{m\times n}$, the same with rank r.

For any $A \in C^{m\times n}$, let

A^* denote the conjugate transpose of A,

$R(A)$, the range of A,

$N(A)$, the null space of A,

I_n, the $n \times n$ identity matrix.

For any subspace $L \subset C^n$ let P_L be the perpendicular projection on L. For any $A \in C^{n\times n}$ define ind A, the index of A, as the smallest nonnegative integer k such that rank A^k = rank A^{k+1}.

For $A \in C^{m\times n}$ consider the matrix equations

(1) $$AXA = A,$$

(2) $$XAX = X,$$

(3) $$(AX)^* = AX,$$

(4) $$(XA)^* = XA;$$

and if $m = n$, also

(1^k) $$A^kXA = A^k \text{ for some integer } k > 1,$$

(5) $$AX = XA.$$

Any $X \in C^{n\times m}$ which solves equations $(i), (j), \cdots, (l)$ from among $(1), (1^k), (2), \cdots, (5)$ is called an $\{i, j, \cdots, l\}$-*inverse* of A.

The set of $\{i, j, \cdots, l\}$-inverses of A is denoted by $A\{i, j, \cdots, l\}$.

For any $A \in C^{m\times n}$, $A\{1, 2, 3, 4\}$ is nonempty and unique: $A\{1, 2, 3, 4\} \equiv A^\dagger$, the *Moore-Penrose generalized inverse* of A (see [10], [1]).

For any $A \in C^{n\times n}$, $A\{1^k, 2, 5\}$ is nonempty if and only if $k \geqq$ ind A, in which case it is unique: $A\{1^k, 2, 5\} \equiv A^D$, the *Drazin pseudoinverse* of A (see [4], [3], [6], [7]).

For any $A \in C^{n\times n}$, $A\{1, 2, 5\}$ is nonempty if and only if ind $A = 0$ or 1, in which case it is unique: $A\{1, 2, 5\} \equiv A^\#$, the *group inverse* of A (see [5], [11], [13]). $A^\#$, whenever it exists, thus coincides with A^D which exists for all $A \in C^{n\times n}$.

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$A \in C_r^{n \times n}$ is an *EPr matrix* if and only if $N(A) = N(A^*)$ or, equivalently, if and only if $A\{1, 2, 3, 4, 5\}$ is nonempty, i.e., $A^\dagger = A^\#$. EPr matrices were introduced in [12] and further studied in [8], [9].

This note deals with the class of matrices $A \in C^{n \times n}$ with $\operatorname{ind} A = 0$ or 1, i.e., $\operatorname{rank} A = \operatorname{rank} A^2$. A theorem characterizing matrices in this class is followed by a characterization of their ranges (Corollary 1) and of the subclass of EPr matrices (Corollary 2). These characterizations use limits in much the same way that nonzero scalars α may be characterized by the existence of $\lim_{\lambda \to 0} (\lambda + \alpha)^{-1}$.

Results. A characterization of the class of $A \in C^{n \times n}$ with $\operatorname{ind} A = 0$ or 1 is given in the following theorem.

THEOREM. *Let* $A \in C^{n \times n}$. *Then*

$$\operatorname{rank} A = \operatorname{rank} A^2 \tag{6}$$

if and only if the limit

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} A \tag{7}$$

exists, in which case

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} A = AA^\#. \tag{8}$$

Remark. Here $\lambda \to 0$ means $\lambda \to 0$ through any neighborhood of 0 in C which excludes the nonzero eigenvalues of $-A$.

Proof. Let $\operatorname{rank} A = r$ and let A be written as

$$A = BG, \quad \text{where } B \in C_r^{n \times r}, \quad G \in C_r^{r \times n}.$$

Cline [3] has shown that (6) is equivalent to the nonsingularity of GB, in which case

$$A^\# = A(GB)^{-2}G$$

and so

$$AA^\# = B(GB)^{-1}G. \tag{9}$$

It is easily verified that

$$(\lambda I_n + A)^{-1} A = B(\lambda I_r + GB)^{-1} G \tag{10}$$

whenever the inverses in question exist. The existence of the limit (7) is thus equivalent to the existence of

$$\lim_{\lambda \to 0} (\lambda I_r + GB)^{-1},$$

which, in turn, is equivalent to the nonsingularity of GB and thus to (6). Equation (8) follows then from (10) and (9).

COROLLARY 1. *Let* $b \in C^n$ *and let* $A \in C^{n \times n}$ *satisfy* $\operatorname{rank} A = \operatorname{rank} A^2$. *Then* $b \in R(A)$ *if and only if the limit*

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} b$$

exists, in which case

$$\lim_{\lambda\to 0}(\lambda I_n + A)^{-1}b = A^{\#}b.$$

Proof. Writing $b \in C^n$ as

$$b = AA^{\#}b + (I_n - AA^{\#})b,$$

we verify by using the identity

$$(\lambda I_n + A)^{-1}(I_n - AA^{\#}) = \lambda^{-1}(I_n - AA^{\#})$$

that

$$\lim_{\lambda\to 0}(\lambda I_n + A)^{-1}b = A^{\#}b + \lim_{\lambda\to 0}\lambda^{-1}(I - AA^{\#})b,$$

which exists if and only if $(I - AA^{\#})b = 0$; this is equivalent to $b \in R(A)$.

If A is an EPr matrix, then clearly ind $A = 0$ or 1. A characterization of EPr matrices is given in the following corollary.

COROLLARY 2. *Let* $A \in C_r^{n\times n}$. *Then* A *is EPr if and only if*

$$\lim_{\lambda\to 0}(\lambda I_n + A)^{-1}P_{R(A)} = A^{\dagger}. \tag{11}$$

The proof follows from the theorem since $A^{\dagger} = A^{\#}$ if and only if A is EPr.
As an application of these results consider the next corollary.

COROLLARY 3 (see [2]). *Let* $A \in C^{m\times n}$. *Then*

$$\lim_{\lambda\to 0}(\lambda I_n + A^*A)^{-1}A^* = A^{\dagger}. \tag{12}$$

Proof.

$$\begin{aligned}\lim_{\lambda\to 0}(\lambda I_n + A^*A)^{-1}A^* &= \lim_{\lambda\to 0}(\lambda I_n + A^*A)^{-1}P_{R(A^*A)}A^* && (\text{since } R(A^*) = R(A^*A))\\ &= (A^*A)^{\dagger}A^* && (\text{by (11) since } A^*A \text{ is EPr})\\ &= A^{\dagger} && (\text{e.g., [10]}).\end{aligned}$$

Remark. The limit in (12) is not restricted to real nonnegative λ as in previous versions of this result, e.g., [1], [2].

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