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ON THE GEOMETRY OF SUBSPACES IN EUCLIDEAN n-SPACES*

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Introduction. This paper consists of four sections. In §1 the intersections of manifolds in E^n are studied. Theorem 1 characterizes the case when an intersection of two manifolds is nonempty, and gives explicit expressions for that intersection. Theorems 2 and 3 give the intersection of two subspaces and the projector thereon. Applications are then given for systems of equations in partitioned form (Corollary 2) and for the inverses of partitioned matrices (Corollaries 3, 4). Section 2 applies the results of §1 to ill-conditioned systems of linear equations. Partitioning the equations is used to obtain a sequence of approximate solutions (63), for which an error bound is given by (64). In §3 an application is given to the generalized inverse of a product of projectors. Section 4 is independent of the rest of the paper: it extends the concept of inclination between subspaces in E^n, as introduced by Zassenhaus [24]. Thus we show that if the subspace $S_j = R(A_j^*)$, $j = 1, 2$, then the eigenvalues of $A_1A_2^\dagger A_2A_1^\dagger$ are the squares of the cosines of the angles of inclination between the subspaces S_1, S_2 (Theorem 4).

Notations. For an $m \times n$ complex matrix A we denote by:

A^* the *conjugate transpose* of A,
$A^\dagger$ the generalized inverse of A (see [20]),
$R(A)$ the *range space* of A,
$N(A)$ the *null space* of A,
$\| A \| = \max \{\sqrt{\lambda}: \lambda \text{ an eigenvalue of } A^*A\}$, the *spectral norm* of A.

This norm is consistent with the Euclidean vector norm, e.g., [16, p. 44].

Let E^n denote the n-dimensional complex vector space. For a subspace L of E^n we denote by:

P_L the *projector* on L, i.e., $P_L = P_L^2 = P_L^*$ and $L = R(P_L)$,
$L^\perp$ the *orthogonal complement* of L,
$\{x + L\} = \{x + y : y \in L\}$ the manifold parallel to L through x,
$AL = \{Ax : x \in L\}$.

The following facts are often used in this paper:

$$AA^\dagger = P_{R(A)}, \qquad \text{(e.g., see [5])}, \tag{1}$$

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$$A^{\dagger}A = P_{R(A^*)}, \tag{2}$$

$$I - AA^{\dagger} = P_{N(A^*)}, \tag{3}$$

$$I - A^{\dagger}A = P_{N(A)}. \tag{4}$$

1. On intersections of manifolds.

THEOREM 1. *Let L, M be any two subspaces of E^n, and let x, y be any two vectors in E^n. Then the intersection of manifolds*

$$\{x + L\} \cap \{y + M\} \tag{5}$$

is nonempty if and only if

$$(P_L + P_M)(P_L + P_M)^{\dagger}(y - x) = y - x, \tag{6}$$

in which case

$$\begin{aligned} \{x + L\} \cap \{y + M\} &= x + P_L(P_L + P_M)^{\dagger}(y - x) + L \cap M \\ &= y - P_M(P_L + P_M)^{\dagger}(y - x) + L \cap M. \end{aligned} \tag{7}$$

Proof. The manifold (5) is nonempty if and only if there are vectors u, v in E^n such that

$$x + P_L u = y + P_M v. \tag{8}$$

Rewriting (8) as

$$(P_L, -P_M)\begin{pmatrix} u \\ v \end{pmatrix} = y - x, \tag{9}$$

we recall [20, Theorem 2] that (9) is solvable if and only if

$$(P_L, -P_M)(P_L, -P_M)^{\dagger}(y - x) = y - x. \tag{10}$$

Using the result $A^{\dagger} = A^*(AA^*)^{\dagger}$ with $A = (P_L, -P_M)$, we verify that

$$\begin{aligned} (P_L, -P_M)^{\dagger} &= \begin{pmatrix} P_L \\ -P_M \end{pmatrix} \left[(P_L, -P_M) \begin{pmatrix} P_L \\ -P_M \end{pmatrix} \right]^{\dagger} \\ &= \begin{pmatrix} P_L \\ -P_M \end{pmatrix} (P_L + P_M)^{\dagger}. \end{aligned} \tag{11}$$

Therefore,

$$(P_L, -P_M)(P_L, -P_M)^{\dagger} = (P_L + P_M)(P_L + P_M)^{\dagger}, \tag{12}$$

and (6) follows from (10) and (12). If (9) is solvable, its solutions are

$$\begin{pmatrix} u \\ v \end{pmatrix} = (P_L, -P_M)^{\dagger}(y - x) + N(P_L, -P_M), \tag{13}$$

which may be substituted in either side of (8) to give (5):

$$\begin{aligned}\{x + L\} \cap \{y + M\} \\ (14) \qquad &= x + (P_L, 0)[(P_L, -P_M)^\dagger(y - x) + N(P_L, -P_M)] \\ &= y + (0, P_M)[(P_L, -P_M)^\dagger(y - x) + N(P_L, -P_M)].\end{aligned}$$

From (11) we have

$$(15) \qquad \begin{aligned}(P_L, 0)(P_L, -P_M)^\dagger &= P_L(P_L + P_M)^\dagger, \\ (0, P_M)(P_L, -P_M)^\dagger &= -P_M(P_L + P_M)^\dagger.\end{aligned}$$

It is also clear that

$$(16) \qquad (P_L, 0)N(P_L, -P_M) = L \cap M = (0, P_M)N(P_L, -P_M).$$

Finally, (7) follows from substituting (15) and (16) in (14).

Remark 1. An alternative proof that

$$\{x + L\} \cap \{y + M\} \neq \varnothing \quad \text{if and only if} \quad (P_L + P_M)(P_L + P_M)^\dagger(y - x) = y - x$$

is as follows: the set (5) is nonempty if and only if

$$(17) \qquad y - x \in L + M.$$

From (1) it follows that

$$(18) \qquad P_{L+M} = (P_L + P_M)(P_L + P_M)^\dagger,$$

and (6) follows from (17) and (18).

Remark 2. Let A, B be two m-rowed complex matrices. The intersection $R(A) \cap R(B)$ is nonempty, being an intersection of two subspaces of E^m. Similarly to (16) we have

$$(19) \qquad \begin{aligned}R(A) \cap R(B) &= (A, 0)N(A, -B) \\ &= (0, B)N(A, -B).\end{aligned}$$

Remark 3. The following theorem expresses the intersection of two subspaces in terms of their projectors.

THEOREM 2. *Let L, M be any two subspaces of E^n. Then*

$$(20) \qquad \begin{aligned}L \cap M &= P_L x - P_L(P_L + P_M)^\dagger(P_L x - P_M y) \\ &= P_M y + P_M(P_L + P_M)^\dagger(P_L x - P_M y),\end{aligned}$$

where x, y range over E^n.

Proof. From (16) we have

$$
\begin{aligned}
L \cap M &= (P_L, 0)N(P_L, -P_M) \\
&= (P_L, 0)\left[I - (P_L, -P_M)^\dagger(P_L, -P_M)\right]\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{by (4)} \\
&= (P_L, 0)\left[I - \begin{pmatrix} P_L \\ -P_M \end{pmatrix}(P_L + P_M)^\dagger(P_L, -P_M)\right]\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{by (11)} \\
&= P_L x - P_L(P_L + P_M)^\dagger(P_L x - P_M y).
\end{aligned} \tag{21}
$$

The alternative statement in (20) is similarly proved.

Another expression of $L \cap M$ in terms of P_L, P_M is given in the following theorem due to Nakano [18]; see also Pyle [22] and Altman [1], [2] for extensions and applications.

THEOREM 3 (Nakano). *Let L, M be any two subspaces of E^n. Then*

$$P_{L\cap M} = \lim_{k\to\infty} (P_L P_M)^k. \tag{22}$$

Returning to manifolds we have as an easy consequence of Theorem 1 the following corollary.

COROLLARY 1. *Let L, M be two subspaces of E^n which satisfy*

$$L \cap M = \{0\}, \tag{23}$$

$$\dim L + \dim M = n. \tag{24}$$

Then, for any two vectors x, y in E^n, the intersection (5) *consists of the single vector*

$$
\begin{aligned}
\{x + L\} \cap \{y + M\} &= \{x + P_L(P_L + P_M)^\dagger(y - x)\} \\
&= \{y - P_M(P_L + P_M)^\dagger(y - x)\}.
\end{aligned} \tag{25}
$$

Proof. The intersection (5) is nonempty for any pair x, y because (24) implies that $(P_L, -P_M)$ is of rank n and therefore (9) is solvable for all $(y - x)$. Using (23) and (7) we get (25).

An application of Theorem 1 to the solution of linear equations is made in the following corollary.

COROLLARY 2. *Let A, B be two complex matrices with n columns, and let the two systems of equations*

$$Ax = a, \tag{26}$$

$$Bx = b \tag{27}$$

be solvable and possess a common solution. Then

$$\begin{aligned}\binom{A}{B}^{\dagger}\binom{a}{b} &= A^{\dagger}a + (P_{N(A)} - P_{N(A)\cap N(B)}) \\ &\qquad\cdot(P_{N(A)} + P_{N(B)})^{\dagger}(B^{\dagger}b - A^{\dagger}a) \\ &= B^{\dagger}b - (P_{N(B)} - P_{N(A)\cap N(B)}) \\ &\qquad\cdot(P_{N(A)} + P_{N(B)})^{\dagger}(B^{\dagger}b - A^{\dagger}a).\end{aligned} \tag{28}$$

Proof. The solution manifolds of (26), (27) are, respectively,

$$R = A^{\dagger}a + N(A), \tag{29}$$

$$S = B^{\dagger}b + N(B). \tag{30}$$

Consider the joint system

$$\binom{A}{B}x = \binom{a}{b}, \tag{31}$$

whose solution manifold is nonempty (because (26) and (27) have a solution in common) and is given by:

$$T = \binom{A}{B}^{\dagger}\binom{a}{b} + N\binom{A}{B} = \binom{A}{B}^{\dagger}\binom{a}{b} + N(A) \cap N(B). \tag{32}$$

Clearly,

$$T = R \cap S, \tag{33}$$

and using (7) with $x = A^{\dagger}a$, $y = B^{\dagger}b$, $L = N(A)$ and $M = N(B)$, we get

$$\begin{aligned} T &= A^{\dagger}a + P_{N(A)}(P_{N(A)} + P_{N(B)})^{\dagger}(B^{\dagger}b - A^{\dagger}a) + N(A) \cap N(B) \\ &= B^{\dagger}b - P_{N(B)}(P_{N(A)} + P_{N(B)})^{\dagger}(B^{\dagger}b - A^{\dagger}a) + N(A) \cap N(B).\end{aligned} \tag{34}$$

In (32) the manifold T is represented as the subspace $N(A) \cap N(B)$ translated to the point

$$\binom{A}{B}^{\dagger}\binom{b}{a},$$

which is in

$$R\left(\binom{A}{B}^{\dagger}\right) = R\left(\binom{A}{B}^{*}\right) = R(A^*) + R(B^*) = (N(A) \cap N(B))^{\perp}. \tag{35}$$

To bring (34) to this form, we must project the vector therein on $R(A^*) + R(B^*)$. Thus,

$$\begin{aligned}\binom{A}{B}^{\dagger}\binom{a}{b} &= P_{R(A^*)+R(B^*)}(A^{\dagger}a + P_{N(A)}(P_{N(A)} + P_{N(B)})^{\dagger}(B^{\dagger}b - A^{\dagger}a)) \\ &= P_{R(A^*)+R(B^*)}(B^{\dagger}b - P_{N(B)}(P_{N(A)} + P_{N(B)})^{\dagger}(B^{\dagger}b - A^{\dagger}a)).\end{aligned} \tag{36}$$

But

$$P_{R(A^*)+R(B^*)}A^\dagger = A^\dagger, \tag{37}$$

$$P_{R(A^*)+R(B^*)}B^\dagger = B^\dagger, \tag{38}$$

$$\begin{aligned} P_{R(A^*)+R(B^*)}P_{N(A)} &= (I - P_{N(A)\cap N(B)})P_{N(A)} \\ &= P_{N(A)} - P_{N(A)\cap N(B)}, \end{aligned} \tag{39}$$

$$\begin{aligned} P_{R(A^*)+R(B^*)}P_{N(B)} &= (I - P_{N(A)\cap N(B)})P_{N(B)} \\ &= P_{N(B)} - P_{N(A)\cap N(B)}. \end{aligned} \tag{40}$$

Substituting (37)–(40) in (36) we get (28).

Corollary 2 can be used to obtain a representation for the generalized inverse of the partitioned matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ as follows.

COROLLARY 3. *Let A, B, be two complex matrices with n columns, and let the intersection of the subspaces $R(A^*)$, $R(B^*)$ consist of the zero vector alone. Then*

$$\begin{aligned} \begin{pmatrix} A \\ B \end{pmatrix}^\dagger &= (A^\dagger, 0) + (P_{N(A)} - P_{N(A)\cap N(B)})(P_{N(A)} + P_{N(B)})^\dagger(-A^\dagger, B^\dagger) \\ &= (0, B^\dagger) - (P_{N(B)} - P_{N(A)\cap N(B)})(P_{N(A)} + P_{N(B)})^\dagger(-A^\dagger, B^\dagger). \end{aligned} \tag{41}$$

Proof. From $R(A^*) \cap R(B^*) = \{0\}$ it follows that $N(A) + N(B) = E^n$, so that the manifolds (29) and (30) have a nonempty intersection for every pair of vectors a, b. The conditions of Corollary 2 are thus satisfied, and (41) follows from (28) and the arbitrariness of a, b.

Other representations for the generalized inverse of a partitioned matrix were given by Penrose [21], Greville [13], [14], Cline [8], [9] and Rohde [23].

An important special case of (41) occurs when the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is nonsingular.

COROLLARY 4. *If the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is nonsingular, then*

$$\begin{aligned} \begin{pmatrix} A \\ B \end{pmatrix}^{-1} &= (A^\dagger, 0) + P_{N(A)}(P_{N(A)} + P_{N(B)})^{-1}(-A^\dagger, B^\dagger) \\ &= (0, B^\dagger) - P_{N(B)}(P_{N(A)} + P_{N(B)})^{-1}(-A^\dagger, B^\dagger). \end{aligned} \tag{42}$$

Proof. From the nonsingularity of $\begin{pmatrix} A \\ B \end{pmatrix}$ the following two facts follow:

$$P_{N(A)\cap N(B)} = 0, \tag{43}$$

$$P_{N(A)} + P_{N(B)} \text{ is nonsingular.} \tag{44}$$

Equation (42) follows now from (41).

Remark 4. If the subspaces $N(A)$, $N(B)$ are orthogonal, i.e., if

$$P_{N(A)} + P_{N(B)} = P_L \tag{45}$$

for some subspace L, then (28) yields

$$\begin{aligned}\begin{pmatrix} A \\ B \end{pmatrix}^{\dagger} \begin{pmatrix} a \\ b \end{pmatrix} &= A^{\dagger}a + P_{N(A)}B^{\dagger}b \\ &= B^{\dagger}b + P_{N(B)}A^{\dagger}a.\end{aligned} \tag{46}$$

Indeed, (46) follows from (28) by noting that

$$P_L^{\dagger} = P_L, \tag{47}$$

$$P_{N(A)}P_L = P_{N(A)}, \quad P_{N(B)}P_L = P_{N(B)} \qquad \text{from (45)}, \tag{48}$$

$$P_{N(A)}A^{\dagger} = 0, \qquad P_{N(B)}B^{\dagger} = 0. \tag{49}$$

In particular, if the subspaces $N(A)$, $N(B)$ are orthogonal complements, i.e., if

$$P_{N(A)} + P_{N(B)} = I, \tag{50}$$

then $\begin{pmatrix} A \\ B \end{pmatrix}$ is nonsingular and

$$\begin{pmatrix} A \\ B \end{pmatrix}^{-1} = (A^{\dagger}, B^{\dagger}). \tag{51}$$

Example. Let

$$A = (1, 1),$$

$$B = (1, 2),$$

$$\begin{pmatrix} A \\ B \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

$$A^{\dagger} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad P_{N(A)} = I - A^{\dagger}A = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$B^{\dagger} = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad P_{N(B)} = \frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix},$$

$$P_{N(A)} + P_{N(B)} = \frac{1}{10}\left[\begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} + \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}\right] = \frac{1}{10}\begin{pmatrix} 13 & -9 \\ -9 & 7 \end{pmatrix},$$

$$(P_{N(A)} + P_{N(B)})^{-1} = \begin{pmatrix} 7 & 9 \\ 9 & 13 \end{pmatrix}.$$

Using (42) we obtain

$$\begin{pmatrix} A \\ B \end{pmatrix}^{-1} = (A^\dagger, 0) + P_{N(A)}(P_{N(A)} + P_{N(B)})^{-1}(-A^\dagger, B^\dagger)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 7 & 9 \\ 9 & 13 \end{pmatrix}\frac{1}{10}\begin{pmatrix} -5 & 2 \\ -5 & 4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

or, alternatively,

$$\begin{pmatrix} A \\ B \end{pmatrix}^{-1} = (0, B^\dagger) - P_{N(B)}(P_{N(A)} + P_{N(B)})^{-1}(-A^\dagger, B^\dagger)$$

$$= \frac{1}{5}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 7 & 9 \\ 9 & 13 \end{pmatrix}\frac{1}{10}\begin{pmatrix} -5 & 2 \\ -5 & 4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} -10 & 6 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

2. Application to ill-conditioned equations. Using the representation (42), it follows that

$$(52) \qquad \begin{pmatrix} A \\ B \end{pmatrix}^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = A^\dagger a + P_{N(A)}(P_{N(A)} + P_{N(B)})^{-1}(B^\dagger b - A^\dagger a),$$

which may be applied in the solution of the system

$$(31) \qquad \begin{pmatrix} A \\ B \end{pmatrix} x = \begin{pmatrix} a \\ b \end{pmatrix},$$

where the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is ill-conditioned, but A is a well-behaving submatrix, say AA^* is well-conditioned. Such a submatrix may be found by diagonalizing and stopping at a stage when all the rows below the pivot row contain elements whose moduli are less than or equal to some prescribed positive number (see, e.g., Gavurin [12]). Thus the ill-conditioning of (31) affects only the part

$$(53) \qquad P_{N(A)}(P_{N(A)} + P_{N(B)})^{-1}(B^\dagger b - A^\dagger a)$$

of the solution (52), being inherent in the matrix $(P_{N(A)} + P_{N(B)})$. But since the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is nonsingular, we have

$$(54) \qquad N(A) \cap N(B) = \{0\}$$

so that

$$(55) \qquad \| P_{N(A)} + P_{N(B)} \| < 2;$$

and therefore,

$$P_{N(A)} + P_{N(B)} = I + K, \tag{56}$$

where K is Hermitian and

$$\| K \| < 1. \tag{57}$$

Therefore (see, e.g., [16, p. 54]),

$$(P_{N(A)} + P_{N(B)})^{-1} = \sum_{j=0}^{\infty} (-1)^j K^j, \tag{58}$$

where the ill-conditioning of $(P_{N(A)} + P_{N(B)})$ results in a slow convergence of (58). Using (58), (2) and (4), the solution (52) is

$$\begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = A^{\dagger}a + (I - A^{\dagger}A) \sum_{j=0}^{\infty} (-1)^j K^j (B^{\dagger}b - A^{\dagger}a); \tag{59}$$

and by using $A^{\dagger} = A^{\dagger}AA^{\dagger}$,

$$\begin{aligned} \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} &= A^{\dagger}a + (I - A^{\dagger}A)B^{\dagger}b \\ &\quad + (I - A^{\dagger}A) \sum_{j=1}^{\infty} (-1)^j K^j (B^{\dagger}b - A^{\dagger}a). \end{aligned} \tag{60}$$

An upper bound on the error associated with the approximate solution

$$x_0 = A^{\dagger}a + (I + A^{\dagger}A)B^{\dagger}b \tag{61}$$

is given by

$$\left\| \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} - x_0 \right\| \leqq \frac{\| K \|}{1 - \| K \|} \| B^{\dagger}b - A^{\dagger}a \|; \tag{62}$$

this follows from (60) by using

$$\| (I - A^{\dagger}A) \| \leqq 1$$

(see, e.g., [4]).

More generally, an upper bound on the error associated with replacing (59) by the partial sum

$$x_k = A^{\dagger}a + (I - A^{\dagger}A) \sum_{j=0}^{k} (-1)^j K^j (B^{\dagger}b - A^{\dagger}a) \tag{63}$$

is given by

$$\left\| \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} - x_k \right\| \leqq \frac{\| K \|^{k+1}}{1 - \| K \|} \| B^{\dagger}b - A^{\dagger}a \|. \tag{64}$$

Remark 5. The expansion (60) in powers of K should not be taken seriously as a computational method for solving an ill-conditioned system. However, (60) sheds light on some special cases, e.g., $\| K(B^\dagger b - A^\dagger a) \| \approx 0$, where the solution of (31) may be well-behaved although the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is ill-conditioned.

Remark 6. For a discussion of ill-conditioned systems in partitioned form, see [6].

3. Application to the generalized inverse of a product of projectors. The generalized inverses of matrix products received much attention in the literature, mainly because the relation

$$(AB)^{-1} = B^{-1}A^{-1} \tag{65}$$

cannot be extended to singular matrices in a straightforward manner. Conditions for

$$(AB)^\dagger = B^\dagger A^\dagger \tag{66}$$

to hold were given by Greville [15]; see also Erdelyi [10], Katz [17], Pearl [19] and Arghiriade [3]. An actual representation for $(AB)^\dagger$ was given by Cline [7] as follows:

$$(AB)^\dagger = B_1^\dagger A_1^\dagger, \quad \text{where} \quad B_1 = A^\dagger AB, \quad A_1 = AB_1B_1^\dagger. \tag{67}$$

Another representation follows as a special case from a more general result of Foulis [11]:

$$(AB)^\dagger = P_{R((AB)^*)}B^\dagger(P_{R(A^*)}P_{R(B)})^\dagger A^\dagger P_{R(AB)} . \tag{68}$$

The computational significance of (68) is impaired by the fact, as shown by Householder [16, p. 9], that the direct computation of $X^\dagger$ requires little more than the computation of $P_{R(X)}$ and of $P_{R(X^*)}$. Nevertheless, both (67) and (68) give insight into the structure of the operator $(AB)^\dagger$.

If both A, B are projectors, then both representations (67) and (68) reduce to $(PQ)^\dagger = (PQ)^\dagger$. The results of §1 will now be applied to obtain a representation for $(PQ)^\dagger$, where P, Q are projectors.

LEMMA 1. *Let P, Q be two arbitrary projectors in E^n. Then for any vector x either*

$$PQx = 0, \tag{69}$$

or

$$(PQ)^n x \neq 0 \quad \textit{for} \quad n = 1, 2, \cdots . \tag{70}$$

Proof. Clearly it is enough to prove the lemma for $n = 2$. We show first that

$$PQx \neq 0 \quad \text{implies} \quad QPQx \neq 0. \tag{71}$$

This follows from

$$QPQx = QPPQx \tag{72}$$

and the fact that the operator QP is nonsingular when restricted to $R((QP)^*) = R(PQ)$. Similarly,

$$QPQx \neq 0 \quad \text{implies} \quad PQPQx \neq 0, \tag{73}$$

and the lemma follows from (71) and (73).

COROLLARY 5. *Let P, Q be two arbitrary projectors in E^n. Then for any*

$$y \in R(PQ) \tag{74}$$

we have

$$\{y + N(QP)\} \cap R(QP) = \{(PQ)^\dagger y\}. \tag{75}$$

Proof. From Lemma 1 it follows that

$$R(QP) \cap N(QP) = \{0\}; \tag{76}$$

for if

$$0 \neq x = QPu, \tag{77}$$

$$QPx = QPQPu = 0, \tag{78}$$

then Lemma 1 is contradicted. Using (76) and

$$\dim R(QP) + \dim N(QP) = n, \tag{79}$$

it follows from Corollary 1 that the intersection

$$\{y + N(QP)\} \cap R(QP) \tag{80}$$

consists of a single vector. We will show this vector to be $(PQ)^\dagger y$. Indeed,

$$(PQ)^\dagger y \in R((PQ)^*) = R(QP), \tag{81}$$

say,

$$(PQ)^\dagger y = QPz \quad \text{for some } z. \tag{82}$$

Using (74) and (82) we obtain

$$Py = y = (PQ)(PQ)^\dagger y = PQQPz = PQPz = P(PQ)^\dagger y \tag{83}$$

so that

$$y - (PQ)^\dagger y \in N(P) \subset N(QP). \tag{84}$$

From (81) and (84) we conclude that

$$(PQ)^\dagger y \in \{y + N(QP)\} \cap R(QP). \tag{85}$$

This completes the proof.

Corollary 5 gives $(PQ)^{\dagger}y$ for all $y \in R(PQ)$ as the intersection (80). A representation for $(PQ)^{\dagger}$ follows now by the explicit construction of (80) using (25).

COROLLARY 6. *Let P, Q be two arbitrary projectors in E^n. Then*

$$(86)\qquad \begin{aligned}(PQ)^{\dagger} &= P_{R(QP)}(P_{R(QP)} + P_{N(QP)})^{\dagger}P_{R(PQ)}\\ &= (PQ)^{\dagger}PQ[(PQ)^{\dagger}PQ + (I - PQ(PQ)^{\dagger})]^{\dagger}PQ(PQ)^{\dagger}.\end{aligned}$$

Proof. The proof follows from (75) by setting in the first equality of (25),

$$(87)\qquad x = 0,\quad y = PQ(PQ)^{\dagger}y \in R(PQ),\quad L = R(QP),\quad M = N(QP).$$

The second equality in (86) follows from the first by using

$$(88)\qquad P_{R(QP)} = P_{R((QP)^*)} = (PQ)^{\dagger}PQ,$$

$$(89)\qquad P_{N(QP)} = I - P_{R((QP)^*)} = I - P_{R(PQ)} = I - PQ(PQ)^{\dagger}.$$

The above discussion shows that every representation for $(PQ)^{\dagger}$ must involve the projectors $P_{R(QP)}$, $P_{N(QP)}$, i.e., the matrix $(PQ)^{\dagger}$ itself. Since in (68) one needs $(P_{R(A^*)}P_{R(B)})^{\dagger}$, it follows generally that the knowledge of $A^{\dagger}$, $B^{\dagger}$ is not of much help when $(AB)^{\dagger}$ is sought.

4. On the inclination between subspaces. H. Zassenhaus introduced in [24] a set of cosines of the angles of inclination between two linear subspaces S_1, S_2 of E^n as follows: Let

$$(90)\qquad q_j = \dim S_j \quad \text{for} \quad j = 1, 2,$$

and let

$$(91)\qquad q_1 \leqq q_2 .$$

Let the row vectors of the $q_j \times n$ matrix A_j form a basis of S_j, $j = 1, 2$. The eigenvalues of the matrix

$$(92)\qquad f(A_1, A_2) = A_1A_2^*(A_2A_2^*)^{-1}A_2A_1^*(A_1A_1^*)^{-1}$$

are shown by Zassenhaus [24] to be real nonnegative and $\leqq 1$, and are then interpreted as the squares of the cosines of the angles of inclination between S_1 and S_2.

In what follows we will give Zassenhaus' definition a symmetric form, avoiding (91) and expressing the fact that the inclination between S_1 and S_2 is the same as that between S_2 and S_1. Also the matrices A_j, whose rows span S_j, will not be assumed to be of full row rank.

LEMMA 2. *If T is an $r \times r$ matrix and if S is an $n \times r$ matrix of rank r, then the matrices T, $STS^{\dagger}$ have the same nonzero eigenvalues.*

Proof. From rank $S = r$ we conclude that

$$S^\dagger S = I. \tag{93}$$

Thus from

$$Tx = \lambda x \tag{94}$$

it follows that

$$(STS^\dagger)Sx = \lambda Sx \tag{95}$$

Conversely (94) follows from (95), when premultiplied by $S^\dagger$. The proof is completed by noting that

$$\text{rank } T = \text{rank } (STS^\dagger). \tag{96}$$

LEMMA 3. *Let A_1, A_2 be matrices of respective dimensions $m_1 \times n$, $m_2 \times n$ and respective ranks r_1, r_2. Let $\tilde{A}_j$ be a submatrix of A_j, consisting of r_j linearly independent rows, $j = 1, 2$. Then the nonzero eigenvalues of $\tilde{A}_1\tilde{A}_2^\dagger\tilde{A}_2\tilde{A}_1^\dagger$ and $A_1A_2^\dagger A_2A_1^\dagger$ are the same.*

Proof. One can write

$$A_j = B_j\tilde{A}_j \quad \text{for} \quad j = 1, 2, \tag{97}$$

where the matrices B_j are of full column rank. From (97) it follows that

$$A_j^\dagger = \tilde{A}_j^\dagger B_j^\dagger \quad \text{for} \quad j = 1, 2, \tag{98}$$

so that

$$\begin{aligned} A_1A_2^\dagger A_2A_1^\dagger &= B_1\tilde{A}_1\tilde{A}_2^\dagger B_2^\dagger B_2\tilde{A}_2\tilde{A}_1^\dagger B_1^\dagger \\ &= B_1\tilde{A}_1\tilde{A}_2^\dagger\tilde{A}_2\tilde{A}_1^\dagger B_1^\dagger \end{aligned} \tag{99}$$

since $B_2^\dagger B_2 = I$.

Taking $T = \tilde{A}_1\tilde{A}_2^\dagger\tilde{A}_2\tilde{A}_1^\dagger$ and $S = B_1$ in Lemma 2, we conclude from (99) that $\tilde{A}_1\tilde{A}_2^\dagger\tilde{A}_2\tilde{A}_1^\dagger$ and $A_1A_2^\dagger A_2A_1^\dagger$ have the same nonzero eigenvalues.

THEOREM 4. *Let S_1, S_2 be two arbitrary subspaces in E^n, and let A_1, A_2 be two matrices satisfying*

$$S_j = R(A_j^*) \quad \textit{for} \quad j = 1, 2. \tag{100}$$

Then the nonzero eigenvalues of the matrix

$$A_1A_2^\dagger A_2A_1^\dagger \tag{101}$$

are the squares of the cosines of the angles of inclination between S_1 and S_2 in the sense of Zassenhaus.

Proof. The proof follows from Lemma 3, by noting that Zassenhaus' definition involves the eigenvalues of

$$\tilde{A}_1\tilde{A}_2^*(\tilde{A}_2\tilde{A}_2^*)^{-1}\tilde{A}_2\tilde{A}_1^*(\tilde{A}_1\tilde{A}_1^*)^{-1} = \tilde{A}_1\tilde{A}_2^\dagger\tilde{A}_2\tilde{A}_1^\dagger, \tag{102}$$

which coincide with the nonzero eigenvalues of $A_1A_2^\dagger A_2A_1^\dagger$. Finally, the nonzero eigenvalues of $(A_1A_2^\dagger)(A_2A_1^\dagger)$ and of $(A_2A_1^\dagger)(A_1A_2^\dagger)$ are the same so that the result is independent of the order in which the subspaces are taken; thus (91) can be avoided.

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