Matrix Volume and its Applications

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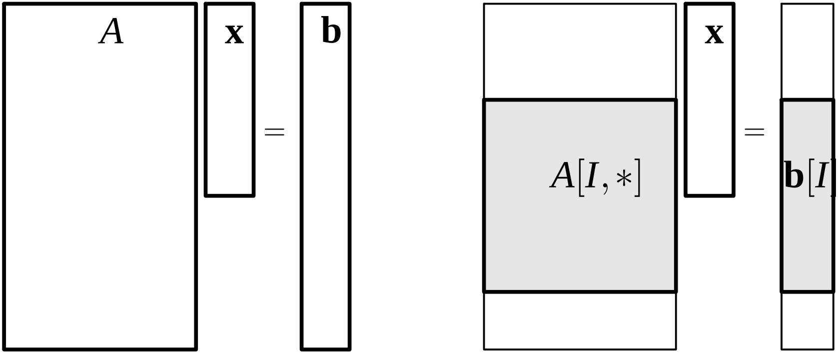
Outline



Least squares solution $\mathbf{x}^* := \arg\min \|A\mathbf{x} - \mathbf{b}\|$

I–basic solution $\mathbf{x}_I := A[I,*]^{-1}\mathbf{b}[I]$.

A. Ben-Tal & M. Teboulle, [10]

Consider a system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

$A \in \mathbb{R}_r^{m\times r}$, $\mathbf{b} \in \mathbb{R}^m$, and its least squares solution (LSS)

$$\mathbf{x}^* := \arg\min \|A\mathbf{x} - \mathbf{b}\|.$$

For any $I = \{i_1, i_2, \cdots, i_r\} \subset 1{:}m$, with $A[I,*]$ nonsingular, the I–basic solution is

$$\mathbf{x}_I := A[I,*]^{-1}\mathbf{b}[I].$$

Then LSS is a convex combination of the basic solutions

$$\mathbf{x}^* = \sum_I \lambda_I \mathbf{x}_I, \quad \lambda_I \propto \det{}^2 A[I,*]$$

$$\lambda_I = \frac{\det^2 A[I,*]}{\sum_J \det^2 A[J,*]}$$

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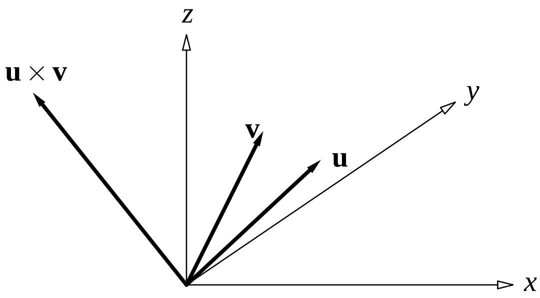
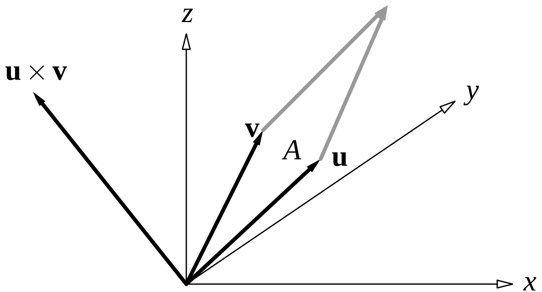
Why determinants? recall the cross product $\mathbf{u} \times \mathbf{v}$



Let $\mathbf{u} = (u_i)$, $\mathbf{v} = (v_i) \in \mathbb{R}^3$, and the cross product

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\,\mathbf{i} + (u_1 v_3 - u_3 v_1)\,\mathbf{j} + (u_1 v_2 - u_2 v_1)\,\mathbf{k}.$$

$\mathbf{u} \times \mathbf{v}$ is the (signed) area of $\lozenge\{\mathbf{u}, \mathbf{v}\}$



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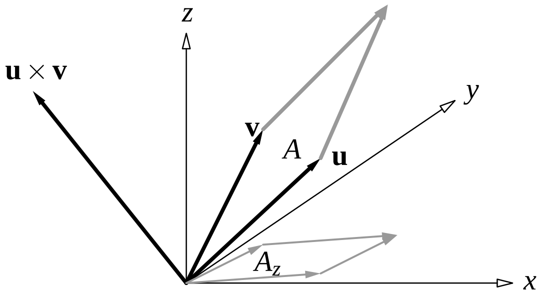
The area A of $\lozenge\{\mathbf{u}, \mathbf{v}\}$ is

$$A = \|\mathbf{u} \times \mathbf{v}\|$$

Why $\sum_J \det^2 A[J,*]$? A Pythagorean theorem for areas



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$$A = \|\mathbf{u}\times\mathbf{v}\| = \sqrt{A_x^2 + A_y^2 + A_z^2},$$ where $A_z = |\, u_1 v_2 - u_2 v_1 \,|$

is the area of the projection of $\Diamond\{\mathbf{u},\mathbf{v}\}$ on the xy–plane, etc.

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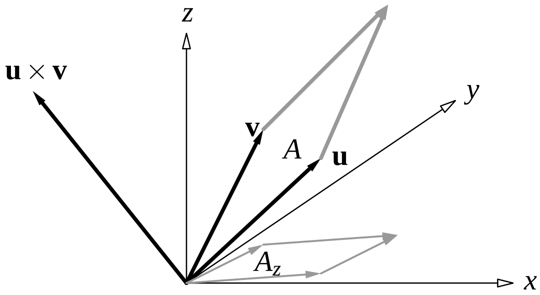

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Notation:

$$1{:}n := \{1, 2, \cdots, n-1, n\}$$
$$\mathbf{Q}(r,n) = \{I = \{i_1, \cdots, i_r\} \in 1{:}n \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\}$$
$$\mathbf{I}(A) = \{I \in \mathbf{Q}(r,m) \mid \text{rank}\, A[I,*] = r\}$$
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Let $A \in \mathbb{R}_r^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and consider the minimum norm least squares solution (MNLSS) $\mathbf{x}^*$ of

$$A\mathbf{x} = \mathbf{b}$$

For any $(I,J) \in \mathbf{M}(A)$, the solution $\mathbf{x}_{IJ}$ of

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L. Berg, [11]

The MNLSS, $\mathbf{x}^* = A^\dagger \mathbf{b}$, is a convex combination of the $\mathbf{x}_{IJ}$'s

$$\mathbf{x}^* = \sum_{(I,J)\in\mathbf{M}(A)} \lambda_{IJ} \widehat{A[I,J]}^{-1} \mathbf{b}[I]$$

with convex weights proportional to $\det{}^2 A[I,J]$,

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Theorem

If $A \in \mathbb{R}_r^{m\times n}$ then

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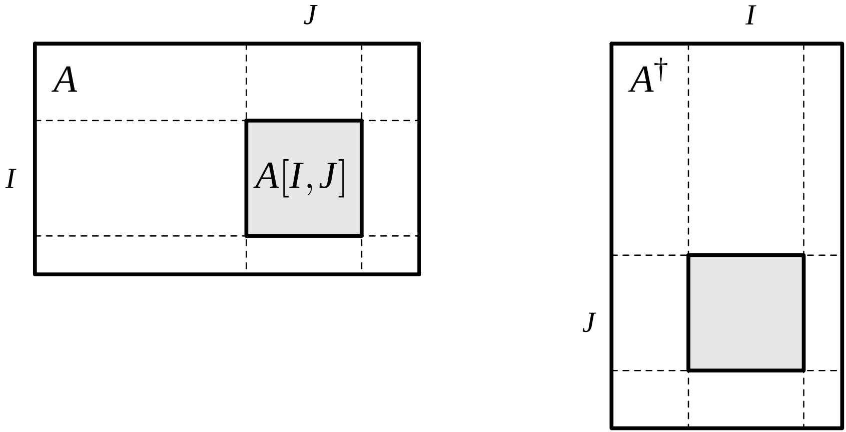
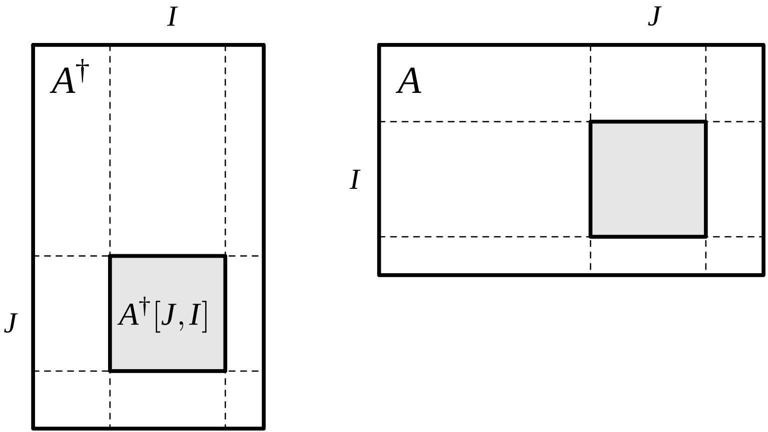


$$A = A^{\dagger\dagger} = \sum_{(J,I)\in \mathbf{M}(A^\dagger)} \lambda_{JI} \widehat{A^\dagger[J,I]^{-1}}, \quad \lambda_{JI} \propto \det{}^2 A^\dagger[J,I]$$



Outline

Definitions of $\text{vol}(A)$, $A \in \mathbb{R}_r^{m\times n}$

Let $A \in \mathbb{R}_r^{m\times n}$. The volume of A, $\text{vol}(A)$, is defined as

$$\text{vol}(A) := \begin{cases} 0, & r = 0; \\ \sqrt{\sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 A[I,J]}, & r > 0. \end{cases}$$

Equivalently, if $r > 0$,

$$\text{vol}(A) := \prod_{i=1}^{r} \sigma_i$$

the product of the singular values of A.

Given $A \in \mathbb{R}_r^{m\times n}$, every unit cube in $\mathbf{R}(A^T)$ is mapped by A into a parallelepiped in $\mathbf{R}(A)$ of volume $\text{vol}(A)$

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If $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\} \subset \mathbb{R}^n$ are l.i., the (signed) volume of the parallelepiped $\Diamond\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ is given by the determinant

$$\operatorname{vol}(\Diamond\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}) = \det W, \quad W = (\mathbf{w}_1, \cdots, \mathbf{w}_n),$$
$$\text{and} \quad \operatorname{vol}(W) = |\det W| .$$

If $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\} \subset \mathbb{R}^n$ are l.i., $k \leq n$, the volume of the parallelepiped $\Diamond\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ is given by the Gram determinant,

$$\operatorname{vol}^2(\Diamond\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}) = \det W^T W, \quad W = (\mathbf{w}_1, \cdots, \mathbf{w}_k),$$
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Full rank factorizations

A full rank factorization (FRF) of $A \in \mathbb{R}_r^{m \times n}$ is

$$A = CR, \; C \in \mathbb{R}_r^{m \times r}, \; R \in \mathbb{R}_r^{r \times n}$$

If $A = CR$ is a FRF,

$$\mathbf{I}(A) = \mathbf{I}(C)$$
$$\mathbf{J}(A) = \mathbf{J}(R)$$
$$\mathbf{M}(A) = \mathbf{I}(A) \times \mathbf{J}(A)$$

$(I \in \mathbf{I}(A), J \in \mathbf{J}(A) \implies A[I,J] = C[I,*]R[*,J]$ nonsingular)

Examples: SVD, QR, CUR
The results below hold for any FRF.

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Volume of FRF

The volume of $A \in \mathbb{R}_r^{m \times n}$ is $\quad \mathrm{vol}\,(A) := \sqrt{\sum_{(I,J)\in \mathbf{M}(A)} \det{}^2 A[I,J]}$

Theorem

If $A \in \mathbb{R}_r^{m\times n}$, $r > 0$, and $A = CR$ is any FRF, then

$$\mathrm{vol}\,(A) = \mathrm{vol}\,(C)\,\mathrm{vol}\,(R)$$

Proof.

$$\begin{aligned}
\mathrm{vol}^2(A) &= \sum_{(I,J)\in \mathbf{M}(A)} \det{}^2 A[I,J] = \sum_{(I,J)\in \mathbf{M}(A)} \det{}^2 C[I,*]R[*,J] \\
&= \sum_{(I,J)\in \mathbf{M}(A)} \det{}^2 C[I,*]\,\det{}^2 R[*,J] \\
&= \left(\sum_{I\in \mathbf{I}(A)} \det{}^2 C[I,*]\right)\left(\sum_{J\in \mathbf{J}(A)} \det{}^2 R[*,J]\right) \\
&= \mathrm{vol}^2(C)\,\mathrm{vol}^2(R).
\end{aligned}$$

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Proof.

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\operatorname{vol}^2(A) &= \sum_{(I,J)\in\mathbf{M}(A)} \det^2 A[I,J] = \sum_{(I,J)\in\mathbf{M}(A)} \det^2 C[I,*]R[*,J] \\
&= \sum_{(I,J)\in\mathbf{M}(A)} \det^2 C[I,*] \det^2 R[*,J] \\
&= \left(\sum_{I\in\mathbf{I}(A)} \det^2 C[I,*]\right)\left(\sum_{J\in\mathbf{J}(A)} \det^2 R[*,J]\right) \\
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Singular values

Theorem

If $A \in \mathbb{R}_r^{m \times n}$ *with singular values* $\{\sigma_i \mid i \in 1{:}r\}$, *then*

$$\mathrm{vol}\,(A) = \prod_{i=1}^{r} \sigma_i$$

Proof.

Given a singular value decomposition (SVD) of A,

$$A = U\Sigma V^T$$

with $\Sigma = \mathrm{diag}(\sigma_i) \in \mathbb{R}_r^{r \times r}$, $U^T U = V^T V = I_r$

$$\begin{aligned} \therefore \ \mathrm{vol}\,(A) &= \mathrm{vol}\,(U)\,\mathrm{vol}\,(\Sigma V^T) \\ &= \mathrm{vol}\,(U)\,\mathrm{vol}\,(\Sigma)\,\mathrm{vol}\,(V^T) \\ &= \prod_{i=1}^{r} \sigma_i \end{aligned}$$

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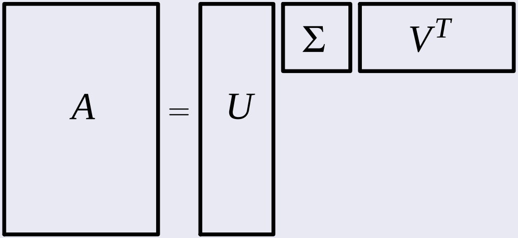
SVD of $A \in \mathbb{R}_r^{m\times n}$

$$A = U\Sigma V^T, \quad \Sigma = \text{diag}\{\sigma_i \mid i \in 1{:}r\},\ U^TU = V^TV = I_r$$



$$U = (\mathbf{u}_1, \cdots, \mathbf{u}_r),\ V = (\mathbf{v}_1, \cdots, \mathbf{v}_r)$$

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i,\ i \in 1{:}r$$

$$A : \Box\{\mathbf{v}_1, \cdots, \mathbf{v}_r\} \to \Box\{\sigma_1 \mathbf{u}_1, \cdots, \sigma_r \mathbf{u}_r\}, \text{ of volume } \prod_{i=1}^{r} \sigma_i = \text{vol}(A)$$

A maps any unit $\Box$ in $\mathbf{R}(A^T)$ into a $\Diamond$ of volume $\text{vol}(A)$ in $\mathbf{R}(A)$

SVD of $A \in \mathbb{R}_r^{m\times n}$

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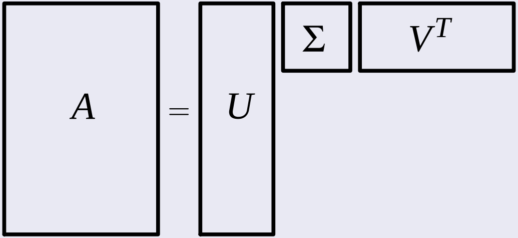

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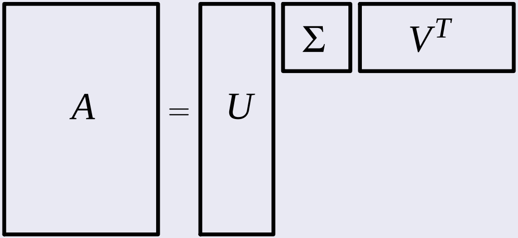

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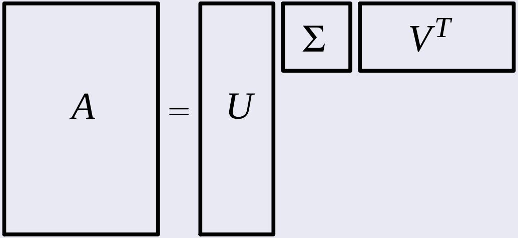

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Outline

Principal angles & vectors

Let L, M be subspaces in $\mathbb{R}^n$, $\dim L = \ell \leq \dim M = m$.
The principal angles between L and M,

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_\ell \leq \frac{\pi}{2}$$

are computed recursively as follows

$$\cos \theta_i = \frac{\langle \mathbf{x}_i, \ \mathbf{y}_i \rangle}{\|\mathbf{x}_i\| \, \|\mathbf{y}_i\|} = \max \left\{ \frac{\langle \mathbf{x}, \ \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \ \middle| \ \begin{array}{ll} \mathbf{x} \in L, & \mathbf{x} \perp \mathbf{x}_k, \\ \mathbf{y} \in M, & \mathbf{y} \perp \mathbf{y}_k, \end{array} \quad k \in 1{:}i-1 \right\}$$

where

$$(\mathbf{x}_i, \ \mathbf{y}_i) \ \in \ L \times M, \ i \in 1{:}\ell$$

are the corresponding pairs of principal vectors. We also define

$$\mathbf{sin}\{L, M\} := \prod_{i=1}^{\ell} \sin \theta_i, \qquad \mathbf{cos}\{L, M\} := \prod_{i=1}^{\ell} \cos \theta_i.$$

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The determinant of $A = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$ satisfies

$$|\det A| \le \prod_{i=1}^{n} \|\mathbf{v}_i\|$$

equality $\iff$ the vectors are orthogonal, or include zero.

Theorem

Let $A = (A_1, A_2)$, $A_1 \in \mathbb{R}_\ell^{n\times n_1}$, $A_2 \in \mathbb{R}_m^{n\times n_2}$ *, rank* $A = \ell + m$ *. Then*

$$\text{vol}_{\ell+m}(A) = \text{vol}_\ell(A_1)\,\text{vol}_m(A_2)\,\mathbf{sin}\{\mathbf{R}(A_1), \mathbf{R}(A_2)\}\ .$$

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Let $A = (A_1, A_2) \in \mathbb{R}^{n\times n}$*, with* $A_1 \in \mathbb{R}^{n\times\ell}$*,* $A_2 \in \mathbb{R}^{n\times m}$*. Then*

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Orthogonal projections

$V := \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ set of vectors in $\mathbb{R}^n$
$S := \text{span}\{V\}$, the subspace spanned by V
$\dim S = r$
Any $\mathbf{w} \in \mathbb{R}^n$ can be written as $\mathbf{w} = \mathbf{w}_S + \mathbf{w}_{S^\perp}$.

Theorem

Let V, S be as above. Then, for any $\mathbf{w} \in \mathbb{R}^n$,

$$\|\mathbf{w}_{S^\perp}\| = \frac{\text{vol}_{r+1}(\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{w})}{\text{vol}_r(\mathbf{v}_1, \cdots, \mathbf{v}_k)} \ ,$$

where $(\mathbf{v}_1, \cdots, \mathbf{v}_k)$ is the matrix with $\mathbf{v}_j$ as columns.

Proof.

If $\mathbf{w} \in S$, $0 = 0$. If $\mathbf{w} \notin S$ then

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For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|,$$

with equality if and only if $\mathbf{u}$, $\mathbf{v}$ are collinear.

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Outline

Exterior products

$V =$ finite–dimensional linear space over field F

An exterior product is an operation $\wedge : V \times V \to V$ that is
(a) anti–commutative, $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$
(b) $(\lambda \cdot \mathbf{u}) \wedge \mathbf{v} = \lambda \cdot (\mathbf{u} \wedge \mathbf{v})$
(c) distributive in both variables:

$$(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}$$
$$\mathbf{w} \wedge (\mathbf{u} + \mathbf{v}) = \mathbf{w} \wedge \mathbf{u} + \mathbf{w} \wedge \mathbf{v}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \lambda \in F$.

$\overset{k}{\wedge} V$ = the k_{th}–exterior space over V, spanned by all exterior products $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$ of k elements in V

$$\dim \overset{k}{\wedge} \mathbb{R}^n = \binom{n}{k}$$

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Compound matrices

V, U = finite–dimensional linear spaces
$\mathbf{L}(V,U)$ = the linear transformations: $V \to U$
Linear transformations $\longleftrightarrow$ their matrix representations.

For $V = \mathbb{R}^n$, $U = \mathbb{R}^m$,
$A \in \mathbb{R}_r^{m\times n}$, $r > 0$, $k \in 1{:}r$,
the k–compound matrix of A is the matrix representing the linear transformation $C_k(A) \in \mathbf{L}(\overset{k}{\wedge} V, \overset{k}{\wedge} U)$, defined by

$$C_k(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k) := A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_k, \quad \forall\,\{\mathbf{x}_i\} \subset V,$$

The compound matrix $C_k(A)$ is $\binom{m}{k} \times \binom{n}{k}$ of rank $\binom{r}{k}$.

In particular, $C_r(A)$ is $\binom{m}{r} \times \binom{n}{r}$ of rank 1.

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Compound matrices

If $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $X \in \mathbb{C}^{n\times n}$ then:

$$C_1(A) = A$$

$$\det C_k(X) = (\det X)^{\binom{n-1}{k-1}}, \quad \text{in particular} \;\; C_n(X) = \det X$$

$$C_k(AB) = C_k(A)C_k(B)$$

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Maple program for $C_k(A)$

```
with(LinearAlgebra):with(combinat):

Compound:=proc(A,k)
local m,n,i,j,rr,cc,P,Q:   global CO:
m:=RowDimension(A):n:=ColumnDimension(A)
P:=choose(m,k):Q:=choose(n,k):
rr:=numbcomb(m,k):cc:=numbcomb(n,k):
CO:=Matrix(rr,cc):
for i from 1 to rr do
convert(P[i],list):
for j from 1 to cc do
convert(Q[j],list):
CO(i,j):=evalf(Determinant(SubMatrix(A,P[i],Q[j])))
od:od:
print(CO):
end:
```

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$$A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_k = C_k(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k),$$

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If A is $n \times n$ then $C_n(A) = \det(A)$, i.e.,

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$A \in \mathbb{R}_r^{m\times n} \implies C_k(A)$ is $\binom{m}{k} \times \binom{n}{k}$ of rank $\binom{r}{k}$

Let $A \in \mathbb{R}^{m\times n}$. The k–compound $C_k(A)$ is defined by

$$A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_k = C_k(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k),$$

for all $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\} \subset \mathbb{R}^n$.

If A is $n \times n$ then $C_n(A) = \det(A)$, i.e.,

$$A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_n = \det(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n),$$

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Let $A \in \mathbb{R}_r^{m\times n}$. Then

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for all $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\} \subset \mathbf{R}(A^T)$.

Plücker coordinates, [60]

To any subspace $W \subset \mathbb{R}^n$, $\dim W = r$, there corresponds a 1–dimensional subspace $\overset{r}{\wedge} W \subset \overset{r}{\wedge} \mathbb{R}^n$, spanned by

$$\mathbf{w}^{\wedge} = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r$$

where $\{\mathbf{w}_1, \cdots, \mathbf{w}_r\}$ is any basis of W.

The $\binom{n}{r}$ components of $\mathbf{w}^{\wedge}$ (determined up to a multiplicative constant) are the Plücker coordinates of W.

Let $A \in \mathbb{R}_r^{m \times n}$ and $A = U\Sigma V^T$ its SVD,

$$U = (\mathbf{u}_1, \cdots, \mathbf{u}_r),\ V = (\mathbf{v}_1, \cdots, \mathbf{v}_r)$$

Plücker coordinates of $\mathbf{R}(A)$ and $\mathbf{R}(A^T)$ are

$$\mathbf{u}^{\wedge} = C_r(U) = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r, \quad \mathbf{v}^{\wedge} = C_r(V) = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$$

If A is square then

$$\cos\{\mathbf{R}(A), \mathbf{R}(A^T)\} = \cos\angle\{\mathbf{u}^{\wedge}, \mathbf{v}^{\wedge}\}$$

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Plücker coordinates preserve angles

Theorem

Let $\mathbf{U}, \mathbf{V}$ *be subspaces of* $\mathbb{R}^n$*, with the columns of*

$$U = (\mathbf{u}_1, \cdots, \mathbf{u}_r),\ V = (\mathbf{v}_1, \cdots, \mathbf{v}_r)$$

as bases, and Plücker coordinates

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Then

$$\mathbf{cos}\{\mathbf{U}, \mathbf{V}\} = \cos \angle\{\mathbf{u}^\wedge, \mathbf{v}^\wedge\}$$

WLOG assume columns o.n., angles $\leq \frac{\pi}{2}$. Then

$$\det(U^T V) = \mathbf{cos}\{\mathbf{U}, \mathbf{V}\},$$ also,

$$\det(U^T V) = C_r(U^T V) = C_r(U)^T C_r(V) = \langle \mathbf{u}^\wedge, \mathbf{v}^\wedge \rangle$$

□

Plücker coordinates preserve angles

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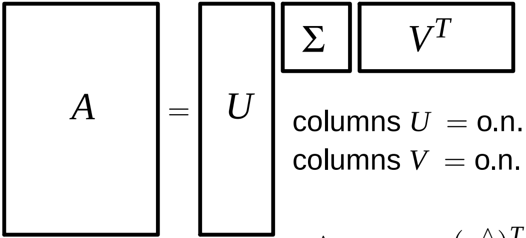
SVD of $A \in \mathbb{R}_r^{m\times n}$

$$A = U \Sigma V^T$$

columns U $=$ o.n. basis of $\mathbf{R}(A)$
columns V $=$ o.n. basis of $\mathbf{R}(A^T)$

SVD of $A \in \mathbb{R}_r^{m\times n}$ and $C_r(A)$



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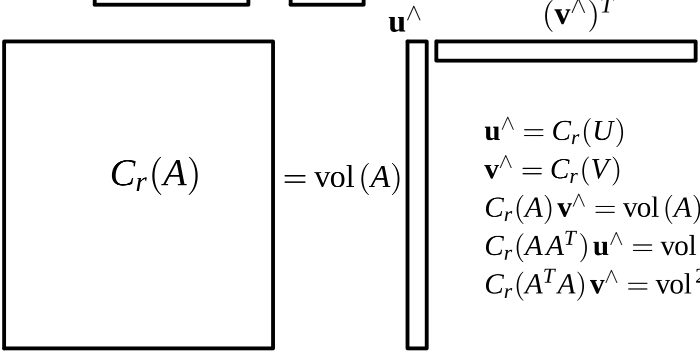
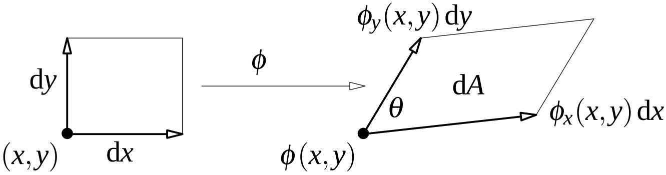
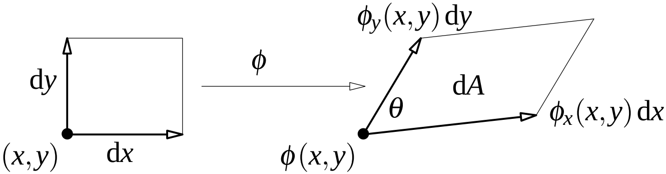


$$\mathbf{u}^\wedge = C_r(U)$$
$$\mathbf{v}^\wedge = C_r(V)$$
$$C_r(A)\,\mathbf{v}^\wedge = \mathrm{vol}\,(A)\,\mathbf{u}^\wedge$$
$$C_r(AA^T)\,\mathbf{u}^\wedge = \mathrm{vol}^2(A)\,\mathbf{u}^\wedge$$
$$C_r(A^TA)\,\mathbf{v}^\wedge = \mathrm{vol}^2(A)\,\mathbf{v}^\wedge$$

Outline



$$E := \phi_x \cdot \phi_x = \|\phi_x\|^2$$
$$F := \phi_x \cdot \phi_y = \|\phi_x\|\|\phi_y\| \cos\theta$$
$$G := \phi_y \cdot \phi_y = \|\phi_y\|^2$$
$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2 \text{ (1st fundamental form)}$$
$$dA = \|\phi_x \times \phi_y\|\,dx\,dy = \|\phi_x\|\|\phi_y\| \sin\theta\,dx\,dy$$
$$= \sqrt{EG - F^2}\,dx\,dy$$
$$\therefore \text{ Area}\,\phi(U) = \iint_U \sqrt{EG - F^2}\,dx\,dy$$



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$$

$$\mathrm{d}s^2 = E\,\mathrm{d}x^2 + 2F\,\mathrm{d}x\,\mathrm{d}y + G\,\mathrm{d}y^2 \text{ (1st fundamental form)}$$

$$
\begin{aligned}
\mathrm{d}A &= \|\phi_x \times \phi_y\|\,\mathrm{d}x\,\mathrm{d}y = \|\phi_x\|\|\phi_y\| \sin\theta\,\mathrm{d}x\,\mathrm{d}y \\
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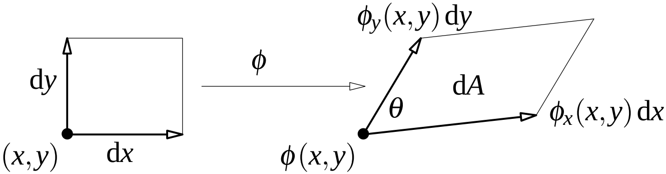

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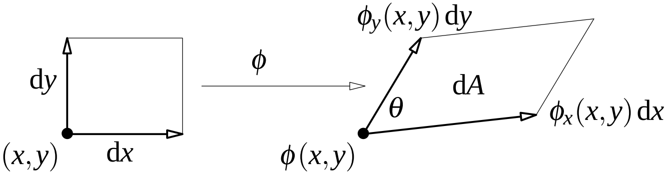

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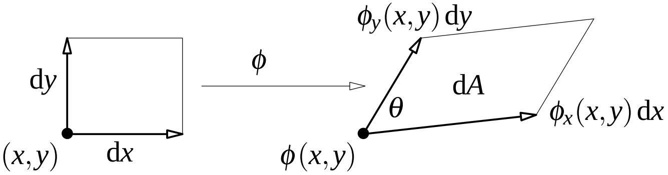

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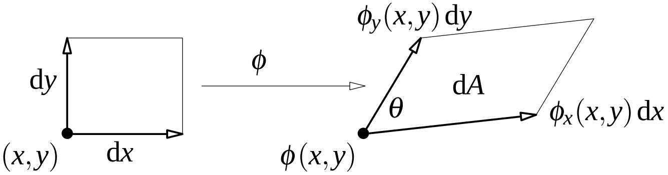
Monge patch: Surface in $\mathbb{R}^3$ given by $z = f(x,y)$



$$\phi(x,y) = (x,y,f(x,y))$$

$$J_\phi(x,y) = \frac{\partial(x,y,z)}{\partial(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix} = (\phi_x \ \phi_y)$$

$$EG - F^2 = \|\phi_x\|^2\|\phi_y\|^2 - (\phi_x \cdot \phi_y)^2 = 1 + f_x^2 + f_y^2 = \text{vol}^2(J_\phi(x,y))$$

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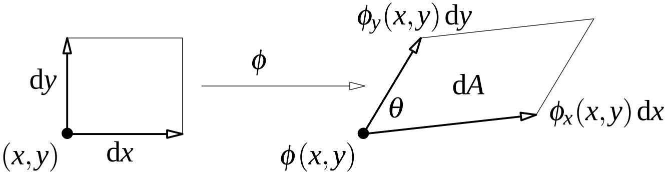

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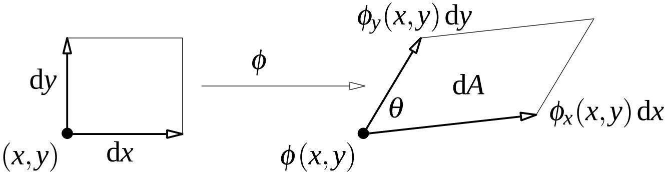

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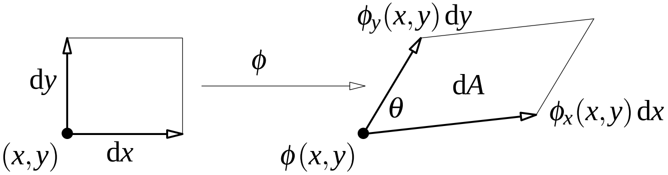

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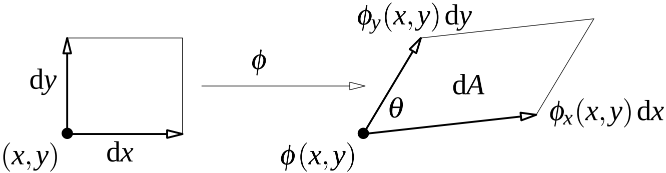

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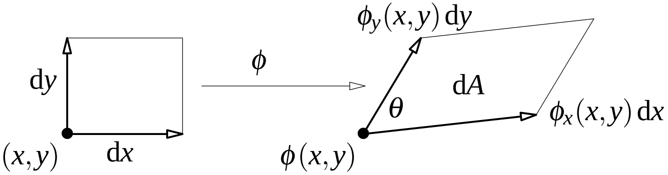

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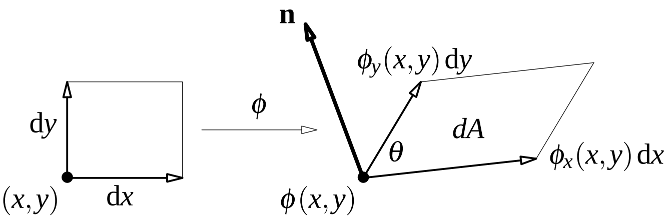
The unit normal $\mathbf{n} = \mathbf{n}(x,y)$ at $\phi(x,y)$



$$\mathbf{n} = \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|}$$

$$J_\phi(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix} \implies \mathbf{n} \propto \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$$

$$\therefore\ \mathbf{n} = \frac{1}{\operatorname{vol}(J_\phi(x,y))} \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}.$$

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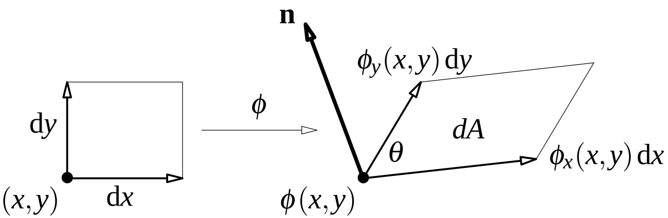

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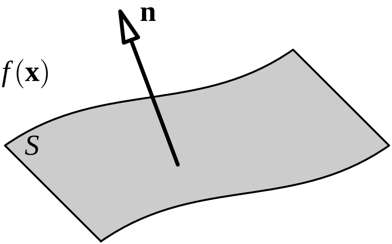
$$x_{n+1} = f(x_1, x_2, \cdots, x_n) = f(\mathbf{x})$$

$$S = \phi(U),\ U \subset \mathbb{R}^n$$

$$\phi(\mathbf{x}) = \begin{pmatrix} \mathbf{x} \\ f(\mathbf{x}) \end{pmatrix}$$



$$J_\phi(\mathbf{x}) = \begin{pmatrix} I \\ \nabla f(\mathbf{x})^T \end{pmatrix}, \quad \text{vol}\,(J_\phi(\mathbf{x})) = \sqrt{1 + \|\nabla f(\mathbf{x})\|^2}, \quad [13]$$

$$J_\phi(\mathbf{x}) = \begin{pmatrix} I \\ \nabla f(\mathbf{x})^T \end{pmatrix} \implies \mathbf{n} \propto \begin{pmatrix} -\nabla f(\mathbf{x}) \\ 1 \end{pmatrix} = \nabla(x_{n+1} - f(\mathbf{x}))$$

$$\therefore\ \mathbf{n} = \frac{1}{\sqrt{1 + \|\nabla f(\mathbf{x})\|^2}} \begin{pmatrix} -\nabla f(\mathbf{x}) \\ 1 \end{pmatrix}$$

n–dimensional surface, given by $\phi : \mathbb{R}^n \to \mathbb{R}^{n+1}$

$$x_{n+1} = f(x_1, x_2, \cdots, x_n) = f(\mathbf{x})$$

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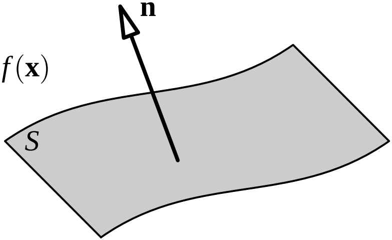

$$J_\phi(\mathbf{x}) = \begin{pmatrix} I \\ \nabla f(\mathbf{x})^T \end{pmatrix}, \quad \mathrm{vol}\,(J_\phi(\mathbf{x})) = \sqrt{1 + \|\nabla f(\mathbf{x})\|^2}, \quad [13]$$

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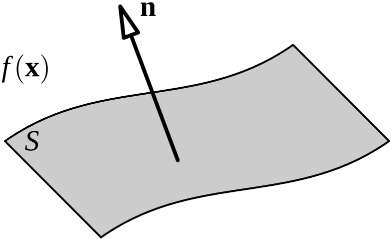

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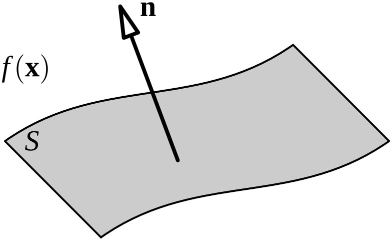

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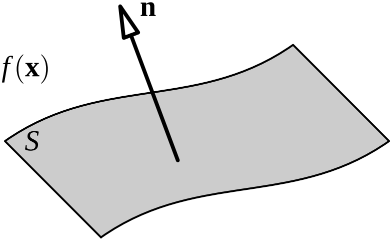

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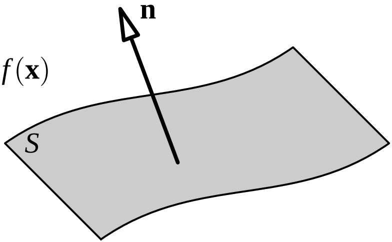

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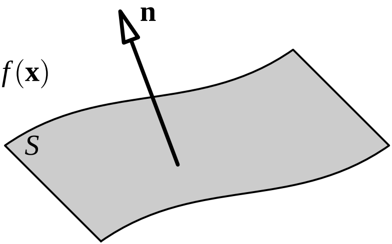

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Outline

- The change of variables formula is

$$\int_{\phi(U)} F(\mathbf{v})\,\mathrm{d}\mathbf{v} = \int_U (F\circ\phi)(\mathbf{u})\,\left|\det J_\phi(\mathbf{u})\right|\,\mathrm{d}\mathbf{u} \qquad \text{(A)}$$

  $U, V \subset \mathbb{R}^n$,
  $\phi : U \to V$ is a sufficiently well-behaved function,
  F is integrable on $\phi(U)$,
  $\mathrm{d}\mathbf{x}$ is the volume element $|\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n|$, and
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$$J_\phi := \left(\frac{\partial \phi_i}{\partial u_j}\right), \quad \text{also denoted} \quad \frac{\partial(v_1, v_2, \cdots, v_n)}{\partial(u_1, u_2, \cdots, u_n)},$$

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C.G.J. Jacobi, [41]

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- The volume of an $m \times n$ matrix of rank r is

$$\mathrm{vol}\,A := \sqrt{\sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 A[I,J]}$$

$A[I,J]$ is the submatrix of A with rows I and columns J, $\mathbf{M}(A)$ is the index set of $r \times r$ nonsingular submatrices of A.

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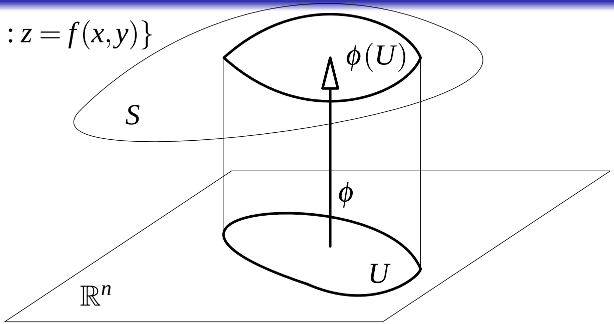
$S = \{(x,y,z) : z = f(x,y)\}$



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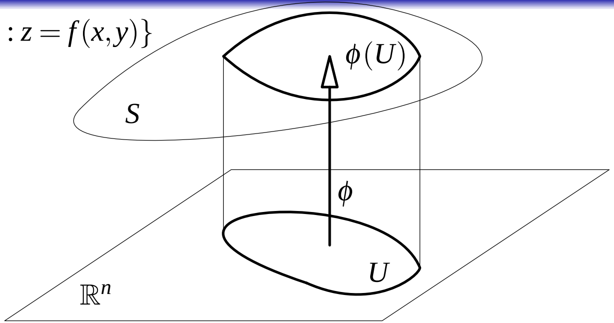

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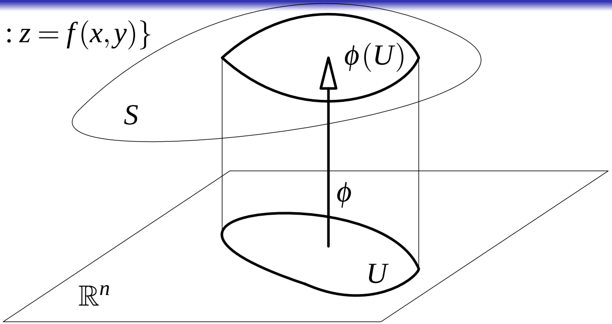

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Let $U \subset \mathbb{R}^2$, $\phi : U \to \mathbb{R}^3$

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Example: Cylindrical coordinates

Let S be a surface in $\mathbb{R}^3$ represented by $z = z(r, \theta)$ where $\{r, \theta\}$ are polar coordinates, or by the mapping

$$\phi(r, \theta) \longrightarrow (x, y, z) = (r \cos\theta, r \sin\theta, z(r, \theta))$$

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$$\therefore \; J_\phi(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

$$\therefore \; \mathrm{vol}\,(J_\phi) = \sqrt{r^2 + r^2\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r\sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}\,,$$

An integral over a domain $V \subset S$ is therefore

$$\iint\limits_{\phi(U)} F(x,y,z)\,\mathrm{d}S =$$

$$\iint\limits_{U} F(r\cos\theta, r\sin\theta, z(r,\theta))\, r\sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}\,\mathrm{d}r\,\mathrm{d}\theta\,.$$

Example: Regular simplex in $\mathbb{R}^n$

Consider the simplex

$$\Delta_n = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \le 1,\ \mathbf{x} \ge \mathbf{0}\},\ \text{with volume } V_n,$$

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Then $V_n = \dfrac{1}{n!}$, by induction $\therefore\ A_j = V_{n-1} = \dfrac{1}{(n-1)!}$.

To calculate A_0, let $\phi : F_j \to A_0$ be given by $x_n = 1 - \sum_{j \ne i = 1}^{n-1} x_i$.

$$\therefore\ \operatorname{vol}(J_\phi) = \sqrt{n}. \qquad \therefore\ A_0 = \int_{F_j} \operatorname{vol}(J_\phi)\, d\mathbf{x} = \sqrt{n}\, A_j.$$

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Then by similarity, the volume of $\Delta_n(\theta)$,

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Example: Radon transform

Let $\mathbf{H}(\mathbf{v},p)$ be a ("non–vertical") hyperplane in $\mathbb{R}^n$

$$\mathbf{H}(\mathbf{v},p) := \{\mathbf{x} : <\mathbf{v},\mathbf{x}> = p\} = \left\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} v_i x_i \;=\; p\right\}, \quad (v_n \neq 0)$$

$$\mathbf{H}(\mathbf{v},p) = \phi(\mathbb{R}^{n-1}), \quad x_n := \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i$$

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The Radon transform $(\mathbf{R}F)(\mathbf{v},p)$ of a function $F : \mathbb{R}^n \to \mathbb{R}$ is,

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Example: Integrals over $\mathbb{R}^n$

$\mathbb{R}^n$ is a union of (parallel) hyperplanes,

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Outline

The unit ball & sphere in $\mathbb{R}^n$

$\|\cdot\|$ the Euclidean norm,
$\mathbf{B}_n(r) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le r\}$; $\mathbf{B}_n(1) = \mathbf{B}_n$, the unit ball in $\mathbb{R}^n$,
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For $n = 2, 3, \cdots$,

$$\mathrm{d}V_n(r) = V_n'(r)\,dr = A_n(r)\,\mathrm{d}r\ ,$$

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$$\mathbf{S}_n^+ = \phi(\mathbf{B}_{n-1}), \quad \phi = (\phi_1, \phi_2, \cdots, \phi_n)$$

$$\phi_i(x_1, x_2, \cdots, x_{n-1}) = x_i \ , \ i \in 1{:}n-1 \ ,$$

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The Jacobi matrix and its volume,

$$J_\phi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{x_1}{x_n} & -\frac{x_2}{x_n} & \cdots & -\frac{x_{n-1}}{x_n} \end{pmatrix}$$

$$\operatorname{vol} J_\phi = \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{x_i}{x_n}\right)^2} = \frac{1}{|x_n|} = \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}} \ .$$

$\mathbf{S}_n^+$ the upper hemisphere

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The unit sphere in $\mathbb{R}^n$ (cont'd)

The area A_n is twice the area of the "upper hemisphere":

$$A_n = 2 \int\limits_{\mathbf{B}_{n-1}} \frac{\mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1}}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}} = 2 \int\limits_{r=0}^{1} \frac{\mathrm{d}V_{n-1}(r)}{\sqrt{1-r^2}}$$

$$= 2 \int\limits_{r=0}^{1} \frac{A_{n-1}(r)\,\mathrm{d}r}{\sqrt{1-r^2}} = 2 \int\limits_{r=0}^{1} \frac{A_{n-1}\, r^{n-2}\,\mathrm{d}r}{\sqrt{1-r^2}}$$

$$\therefore\ \frac{A_n}{A_{n-1}} = 2 \int\limits_{r=0}^{1} \frac{r^{n-2}\,\mathrm{d}r}{\sqrt{1-r^2}}$$

$$\therefore\ A_n = \frac{2\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

using well–known properties of the beta function,

$$B(p,q) := \int_0^1 (1-x)^{p-1} x^{q-1}\,\mathrm{d}x$$

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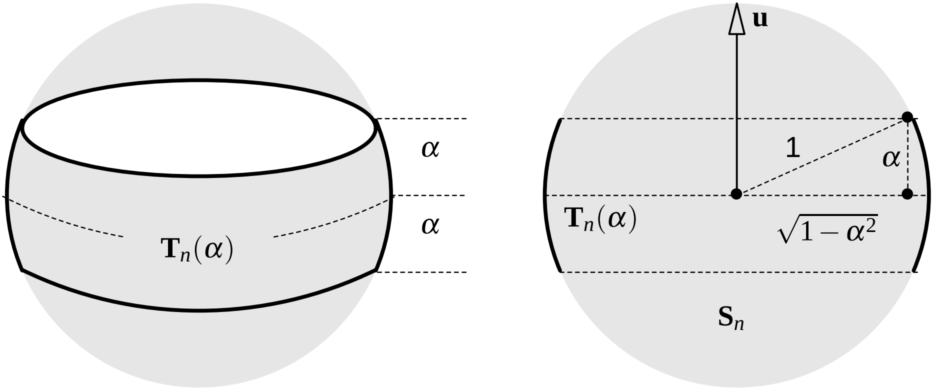
The unit sphere $\mathbf{S}_n$ and an equatorial belt $\mathbf{T}_n(\alpha)$, $\alpha > 0$

$$\mathbf{T}_n(\alpha) = \{\mathbf{x} \in \mathbf{S}_n : -\alpha \le x_n \le \alpha\}$$



Prob $\{\mathbf{X} \in \mathbf{T}_n(\alpha)\}$, $\mathbf{X} \sim \mathbf{U}(S_n)$

$$A(\mathbf{T}_n(\alpha)) := \text{area of } \mathbf{T}_n(\alpha)$$

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$$\begin{aligned} \therefore\ \text{Prob}\{\mathbf{X} \in \mathbf{T}_n(\alpha)\} &= \frac{A_{n-1}}{A_n}\int_{1-\alpha^2}^{1} x^{(n-3)/2}(1-x)^{-1/2}\,\mathrm{d}x\,, \\ &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}\int_{1-\alpha^2}^{1} x^{(n-3)/2}(1-x)^{-1/2}\,\mathrm{d}x\,. \end{aligned}$$

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Prob $\{\mathbf{X} \in \mathbf{T}_n(\frac{k}{\sqrt{n}})\}$ for $\mathbf{X} \sim \mathbf{U}(S_n)$

n	$k=1$	$k=2$	$k=3$
2	.5		
5	.6260990336	.9838699099	
10	.6565636037	.9632125020	.9999914613
100	.6802515257	.9550652747	.9976960345
1000	.6824473395	.9545539777	.9973400661

Values of Prob $\{\mathbf{X} \in \mathbf{T}_n(\frac{k}{\sqrt{n}})\}$ for some k, n

Theorem

Let $\mathbf{u} \in \mathrm{S}_n$ be fixed, $\mathbf{x} \sim \mathbf{U}(S_n)$. Then, as $n \to \infty$,

$$\text{Prob}\left\{|\langle \mathbf{u}, \mathbf{x}\rangle| \le \frac{k}{\sqrt{n}}\right\} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-x^2/2}\, \mathrm{dx}\,.$$

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Outline

Probability density of a function of RV's

$(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ RV's with a given joint density $f_{\mathbf{X}}(x_1, \cdots, x_n)$,

$h : \mathbb{R}^n \to \mathbb{R}$ well behaved, in articular $\dfrac{\partial h}{\partial x_n} \neq 0$,

It is required to find the density of the RV

$$\mathbf{Y} = h(\mathbf{X}_1, \cdots, \mathbf{X}_n).$$

Solve for x_n,

$$x_n = h^{-1}(y|x_1, \cdots, x_{n-1})$$

Change variables from $\{x_1, \cdots, x_n\}$ to $\{x_1, \cdots, x_{n-1}, y\}$, and use

$$\det \left(\frac{\partial(x_1, \cdots, x_n)}{\partial(x_1, \cdots, x_{n-1}, y)} \right) = \frac{\partial h^{-1}}{\partial y}$$

to get the density of $\mathbf{Y} = h(\mathbf{X}_1, \cdots, \mathbf{X}_n)$

$$f_{\mathbf{Y}}(y) = \int_{\mathbb{R}^{n-1}} f_{\mathbf{X}}(x_1, \cdots, x_{n-1}, h^{-1}(y|x_1, \cdots, x_{n-1})) \left| \frac{\partial h^{-1}}{\partial y} \right| \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}$$

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A surface integral on $\mathbf{V}(y)$

Let $\mathbf{V}(y)$ be the surface in $\mathbb{R}^n$ given by

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Theorem

If the ratio

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then

$$f_{\mathbf{Y}}(y) = A \int_{\mathbf{V}(y)} f_{\mathbf{X}}$$

Proof. Compare

$$f_{\mathbf{Y}}(y) = \int_{\mathbb{R}^{n-1}} f_{\mathbf{X}}(x_1,\cdots,x_{n-1},h^{-1}(y|x_1,\cdots,x_{n-1}))\left|\frac{\partial h^{-1}}{\partial y}\right| \mathrm{d}x_1\cdots\mathrm{d}x_{n-1}$$

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Condition (C)

the ratio

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Functions $h : \mathbb{R}^n \to \mathbb{R}$ that satisfy (C):

$$h(x_1, \cdots, x_n) = \sum_{i=1}^{n} v_i x_i, \quad v_n \neq 0$$

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$$h(x_1, \cdots, x_n) = \exp\left\{\sum_{i=1}^{n} x_i\right\}$$

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Check condition (C) for $h(x_1,\cdots,x_n) := \sum_{i=1}^{n} v_i x_i,\ v_n \neq 0$

$$x_n = h^{-1}(y|x_1,\cdots,x_{n-1}) := \frac{y}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i$$

with $\dfrac{\partial h^{-1}}{\partial y} = \dfrac{1}{v_n}$, $\sqrt{1+\sum_{i=1}^{n-1}\left(\dfrac{\partial h^{-1}}{\partial x_i}\right)^2} = \sqrt{1+\sum_{i=1}^{n-1}\left(\dfrac{v_i}{v_n}\right)^2} = \dfrac{\|\mathbf{v}\|}{|v_n|}$

The density of $\mathbf{Y} = \sum v_i \mathbf{X}_i$ is the integral of $f_{\mathbf{X}}$ on the hyperplane

$$\mathbf{H}(\mathbf{v},y) := \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} v_i x_i = y\right\}$$

$$\therefore\ f_{\mathbf{Y}}(y) = \frac{1}{\|\mathbf{v}\|}(\mathbf{R}f_{\mathbf{X}})(\mathbf{v},y)$$

$$= \frac{1}{|v_n|}\int_{\mathbb{R}^{n-1}} f_{\mathbf{X}}\left(x_1,\cdots,x_{n-1},\frac{y}{v_n} - \sum_{i=1}^{n-1}\frac{v_i}{v_n}x_i\right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1}$$

Check condition (C) for $h(x_1, \cdots, x_n) := \sum_{i=1}^{n} v_i x_i, \; v_n \neq 0$

$$x_n = h^{-1}(y|x_1, \cdots, x_{n-1}) := \frac{y}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i$$

with $\quad \dfrac{\partial h^{-1}}{\partial y} = \dfrac{1}{v_n}, \quad \sqrt{1 + \sum_{i=1}^{n-1} \left(\dfrac{\partial h^{-1}}{\partial x_i}\right)^2} = \sqrt{1 + \sum_{i=1}^{n-1} \left(\dfrac{v_i}{v_n}\right)^2} = \dfrac{\|\mathbf{v}\|}{|v_n|}$

The density of $\mathbf{Y} = \sum v_i \mathbf{X}_i$ is the integral of $f_{\mathbf{X}}$ on the hyperplane

$$\mathbf{H}(\mathbf{v}, y) := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} v_i x_i = y \right\}$$

$$\therefore \; f_{\mathbf{Y}}(y) = \frac{1}{\|\mathbf{v}\|} (\mathbf{R} f_{\mathbf{X}})(\mathbf{v}, y)$$

$$= \frac{1}{|v_n|} \int_{\mathbb{R}^{n-1}} f_{\mathbf{X}} \left(x_1, \cdots, x_{n-1}, \frac{y}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1}$$

Check condition (C) for $h(x_1,\cdots,x_n) := \sum_{i=1}^{n} v_i x_i,\ v_n \neq 0$

$$x_n = h^{-1}(y|x_1,\cdots,x_{n-1}) := \frac{y}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i$$

with $$\frac{\partial h^{-1}}{\partial y} = \frac{1}{v_n}\,, \quad \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\partial h^{-1}}{\partial x_i}\right)^2} = \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{v_i}{v_n}\right)^2} = \frac{\|\mathbf{v}\|}{|v_n|}$$

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$$= \frac{1}{|v_n|}\int_{\mathbb{R}^{n-1}} f_{\mathbf{X}}\left(x_1,\cdots,x_{n-1},\frac{y}{v_n}-\sum_{i=1}^{n-1}\frac{v_i}{v_n}x_i\right)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}$$

Check condition (C) for $h(x_1, \cdots, x_n) := \sum_{i=1}^{n} x_i^2$

Two solutions of $y = h(x_1, \cdots, x_n) := \sum_{i=1}^{n} x_i^2$ for x_n,

$$x_n = h^{-1}(y|x_1, \cdots, x_{n-1}) := \pm\sqrt{y - \sum_{i=1}^{n-1} x_i^2}$$

$$\text{with} \quad \frac{\partial h^{-1}}{\partial y} = \pm\frac{1}{2\sqrt{y - \sum_{i=1}^{n-1} x_i^2}}$$

$$\sqrt{1 + \sum_{i=1}^{n-1}\left(\frac{\partial h^{-1}}{\partial x_i}\right)^2} = \frac{\sqrt{y}}{\sqrt{y - \sum_{i=1}^{n-1} x_i^2}}$$

Therefore the density of $\sum \mathbf{X}_i^2$ is expressed in terms of the integral of $f_{\mathbf{X}}$ on the sphere $\mathbf{S}_n(\sqrt{y})$ of radius $\sqrt{y}$.

Corollary

Let $(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ *have joint density* $f_{\mathbf{X}}(x_1, \cdots, x_n)$*. The density of*

$$\mathbf{Y} = \sum_{i=1}^{n} \mathbf{X}_i^2 \quad \textit{is} \quad f_{\mathbf{Y}}(y) = \frac{1}{2\sqrt{y}} \int_{\mathbf{S}_n(\sqrt{y})} f_{\mathbf{X}}$$

the integral is over the sphere $\mathbf{S}_n(\sqrt{y})$ *of radius* $\sqrt{y}$*,*

$$\int_{\mathbf{S}_n(\sqrt{y})} f_{\mathbf{X}} = \int_{\mathbf{B}_{n-1}(\sqrt{y})} \left[f_{\mathbf{X}}\left(x_1, \cdots, x_{n-1}, \sqrt{y - \sum_{i=1}^{n-1} x_i^2}\right) + \right.$$

$$\left. + f_{\mathbf{X}}\left(x_1, \cdots, x_{n-1}, -\sqrt{y - \sum_{i=1}^{n-1} x_i^2}\right) \right] \frac{\sqrt{y}\, \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}}{\sqrt{y - \sum_{i=1}^{n-1} x_i^2}}$$

The factor $1/2\sqrt{y}$ is the width of the spherical shell bounded by the two spheres $\mathbf{S}_n(\sqrt{y})$ and $\mathbf{S}_n(\sqrt{y + \mathrm{d}y})$, i.e. the difference of radii $\sqrt{y + \mathrm{d}y} - \sqrt{y} \approx \frac{\mathrm{d}y}{2\sqrt{y}}$

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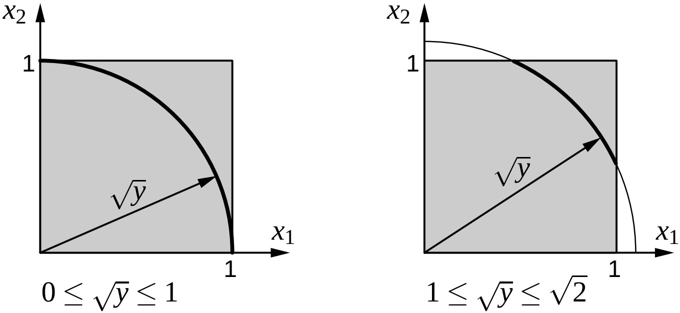
Example. Let $\mathbf{Y} = \mathbf{X}_1^2 + \mathbf{X}_2^2$, $\mathbf{X}_i$ i.i.d., $\mathbf{X}_i \sim \mathbf{U}[0,1]$. Then

$$f_{\mathbf{Y}}(y) = \begin{cases} \dfrac{\pi}{4}, & 0 \leq y \leq 1; \\ \dfrac{\pi}{4} - \arccos\left(\dfrac{1}{\sqrt{y}}\right), & 1 \leq y \leq 2. \end{cases}$$



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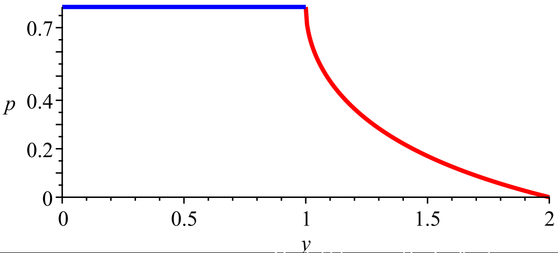
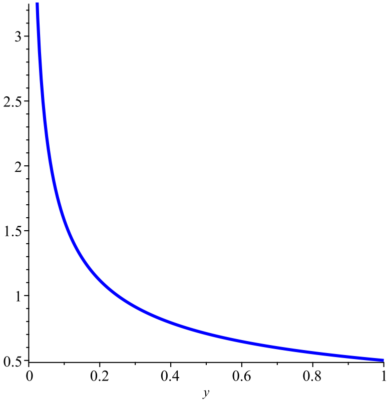
$$\mathbf{E}\{\mathbf{Y}\} = \frac{2}{3}, \quad \mathbf{Var}\{\mathbf{Y}\} = \frac{8}{45}$$



What makes this strange is that ⋯

the density of $\mathbf{Y} = \mathbf{X}^2, \ \mathbf{X} \sim \mathbf{U}[0,1]$ is

$$f_{\mathbf{Y}}(y) = \frac{1}{2\sqrt{y}}, \quad 0 < y \leq 1.$$



Random directions

Let $\mathbf{X} \sim \mathbf{U}(S_n)$. The probability of a surface element on S_n is,

$$\frac{\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}}{A_n\sqrt{1-\sum_{i=1}^{n-1}x_i^2}}$$

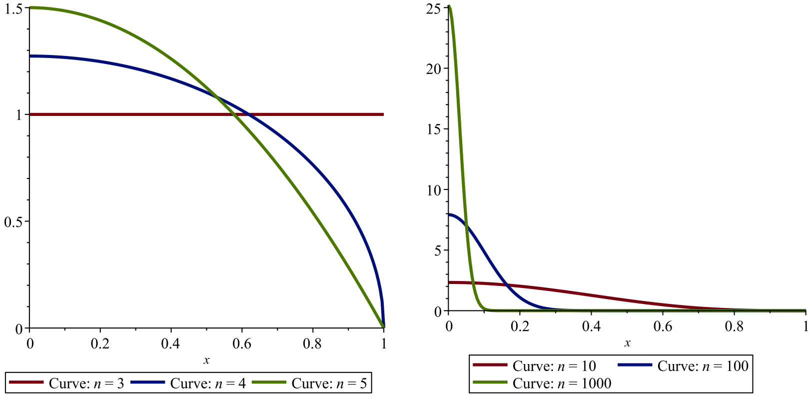
Let $\mathbf{L}_n$ be the length of the projection of $\mathbf{X}$ on a fixed line through the origin, say the x_n axis. Then $\mathbf{L}_n$ has the density

$$f_{\mathbf{L}_n}(x) = \frac{2}{B(\frac{1}{2},\frac{n-1}{2})}\left(1-x^2\right)^{\frac{n-3}{2}}$$

and expected value

$$\mathbf{E}\{\mathbf{L}_n\} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n+1}{2})}$$

Probability densities of $\mathbf{L}_n$ for $n = 3, 4, 5, 10, 100, 1000$



Outline

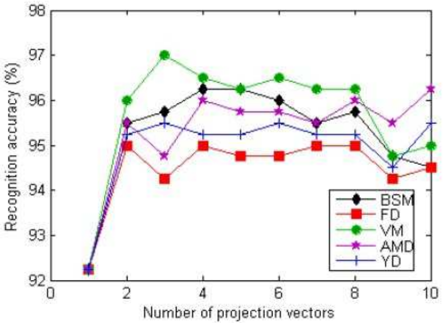
Face recognition [53], [83]

$\mathbf{Y} = \{Y_1, Y_2, \cdots, Y_N\}$ a set of known faces;
$Y =$ a face.

Question: $Y \in \mathbf{Y}$?

Answer: Yes, if $\min_{i \in 1:N} \text{vol}(Y - Y_i) < \epsilon$

No, otherwise.



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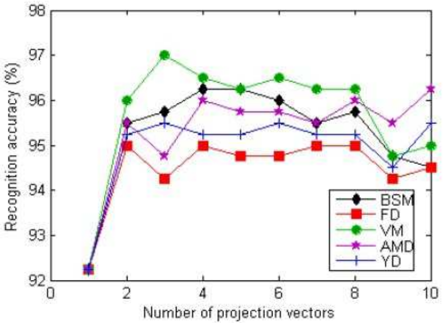

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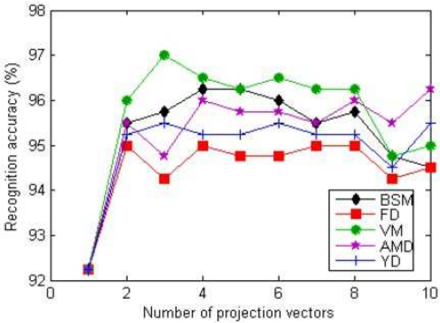

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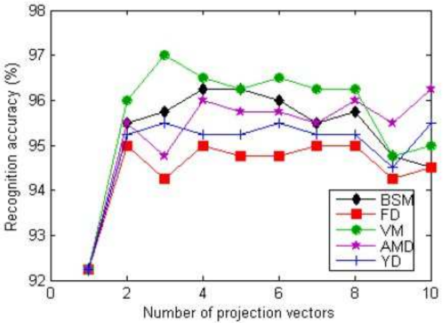

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Volume sampling [22]

Let $A \in \mathbb{R}^{m\times n}$,
S be a submatrix of k rows of A,
$\Delta(S)$ the k–simplex in $\mathbb{R}^n$ generated by S.

Theorem

If S is randomly chosen with probabilities

$$P_S = \frac{\mathrm{vol}^2(\Delta(S))}{\sum_T \mathrm{vol}^2(\Delta(T))}$$

then

$$\mathbf{E}_S\{\|A - \widehat{A}(S)\|_F^2\} \leq (k+1)\,\|A - A_k\|_F^2$$

where A_k is the best k–rank approximation of A,
$\widehat{A}(S)$ is the projection of A to the span of S.

$$\mathrm{vol}\,(\Delta(S)) = \frac{1}{k!}\,\mathrm{vol}\,(S)$$

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Applications to integration & integral operators

[2]: $\cdots = \int (1 + z_{x_1}^2 + \cdots + z_{x_n}^2)^{1/2}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$ (Theorem 2)

[7]: Rectangular Jacobian J_ϕ, replace $|\det(J_\phi)|$ by $\mathrm{vol}\,(J_\phi)$

[9]: Probability densities of $\sum v_i \mathbf{X}_i$ and $\sum \mathbf{X}_i^2$ for $\{\mathbf{X}_i\}$ i.i.d.

[13]: $\mathrm{vol}\,(J_\phi(\mathbf{x})) = \sqrt{1 + \|\nabla f(\mathbf{x})\|^2}$, etc. (P. 5105, bottom)

[14]: $L(S) \geq \iint \sqrt{J_1^2(u,v) + J_2^2(u,v) + J_3^2(u,v)}\, \mathrm{d}u\, \mathrm{d}v$ (3)

[21]: Fourier integral operators of Schrödinger type

[31]: Bayesian learning of kernel embeddings

[32]: Kernel embeddings & associated probability measures

[52]: $L(S) = \iint (X^2 + Y^2 + Z^2) \mathrm{d}u\, \mathrm{d}v = \iint (1 + p^2 + q^2)\, \mathrm{d}u\, \mathrm{d}v$ (17.3)

[57]: Integrals on n–sphere.

[68]: $L(S) = \iint \left\{ \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 \right\}^{1/2} \mathrm{d}u\, \mathrm{d}v$ (p. 76)

[71],[72]: Singular Jacobians in statistics

[74]: Rectangular Jacobian, $3N \times 6$, p. 401, section 2

[75]: Rectangular Jacobians, §A.1, Lemma 1

Outline

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