GENERALIZED GREEN'S MATRICES FOR TWO-POINT BOUNDARY PROBLEMS*

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1. Introduction. To this author it has been of long interest that the initial introduction of the general reciprocal or generalized inverse of a linear operator was not in the algebraic case considered by E. H. Moore [10], [11], but in the case of the generalized Green's function (Greensche Funktionen im erweiterten Sinne) for ordinary and partial differential equations by Hilbert [7, pp. 44–45, 233], and the pseudoresolution for integral equations by Hurwitz [8]. For various historical references to literature on the generalized Green's function and Green's matrix for ordinary differential systems in which the number of independent boundary conditions is equal to the order of the system, the reader is referred to an old paper of the author [12], which was written before he was aware of the E. H. Moore general reciprocal. Under classical hypotheses on the coefficients of the involved differential operator, Greub and Rheinboldt [5] and Wyler [20] have treated recently the generalized Green's function and Green's matrix for differential systems in which the number of independent boundary conditions is not required to be equal to the order of the system.

In the details of construction of a generalized Green's matrix considerable simplification is afforded by the use of the E. H. Moore general reciprocal in designating the solution of certain algebraic equations expressing the boundary conditions, and this procedure has been used by Bradley, both in his dissertation [2] on a class of quasi-differential equations, and in a subsequent paper [3] on general compatible differential systems involving two-point boundary conditions. This procedure has also been used by Loud [9], who was evidently unaware of the extensive literature dealing specifically with generalized Green's functions and Green's matrices.

Now one of the methods for constructing the generalized inverse in a finite-dimensional problem is that of "bordered matrices", or the "embedding of the given problem, which in general is singular, in a nonsingular problem", (see, for example, Blattner [1], Hestenes [6, §§13, 14, 15], and Reid [18, §VI]). The corresponding procedure is available for the construc-

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tion of the generalized inverse of a differential operator, and the purpose of the present paper is to present the details of such a procedure for a two-point boundary problem. In addition to affording a ready proof of the existence of a generalized Green's matrix, this procedure provides an explicit representation in terms of elements of an ordinary Green's matrix for an associated incompatible problem, and thus permits ready discussion of the analytic structure and functional properties of such a matrix.

The following treatment is for a two-point boundary problem, which in character is more general than previous treatments in that it includes as special instances problems involving interface conditions of the type which occur in the accessory problem for a calculus of variations problem with "discontinuous solutions" (see Reid [13]), and the problem considered by Stallard [19] in which the solutions may be merely of bounded variation. Prefatory results for the general differential system are presented in §2, and the existence of a generalized Green's matrix is established in §3. Equivalent differential operators are discussed in §4, with a particular application to the determination of a generalized Green's matrix \( G(t, s; \lambda_0) \) for a fully symmetrizable two-point boundary problem with proper value \( \lambda_0 \), and which is such that as a kernel of an associated integral equation this generalized Green's matrix retains the property of full symmetrizability possessed by \( G(t, s; \lambda) \) when \( \lambda \) is a real number that is not a proper value of the boundary problem.

Matrix notation is used throughout; in particular, matrices of one column are called vectors. The linear vector space of ordered \( n \)-tuples of complex numbers, with complex scalars, is denoted by \( \mathbb{C}_n \). The \( n \times n \) identity matrix is denoted by \( \mathbb{E}_n \), or merely \( \mathbb{E} \) when there is no ambiguity, and 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix \( M \) is designated by \( M^* \). A matrix function is called continuous, integrable, etc., when each element of the matrix possesses the specified property. If \( M(t) \) is a.c. (absolutely continuous) on \([a, b]\), then \( M'(t) \) signifies the matrix of derivatives at values for which these derivatives exist, and the zero matrix elsewhere; correspondingly, if \( M(t) \) is (Lebesgue) integrable on \([a, b]\), then \( M(t)\ dt \) denotes the matrix of integrals of respective elements. If \( M(t) \) and \( N(t) \) are equal a.e. (almost everywhere) on their domain of existence, we write simply \( M(t) = N(t) \).

For a given interval \([a, b]\) on the real line, the symbols \( \mathcal{C}_{hk} \), \( \mathcal{L}_{hk} \), \( \mathcal{L}_{hk}^\infty \), and \( \mathcal{A}_{hk} \) are used to denote the class of \( h \times k \) matrix functions which are respectively continuous, (Lebesgue) integrable, (Lebesgue) measurable and essentially bounded, and a.e. on \([a, b]\), respectively. For brevity, \( \mathcal{C}_h \), \( \mathcal{L}_h \), \( \mathcal{L}_h^\infty \) and \( \mathcal{A}_h \) are written for the corresponding classes specified \( h, k \).
with \( k = 1 \). Moreover, if \( y(t) \) and \( z(t) \) are \( n \)-dimensional vector functions such that \( z^*(t)y(t) \) is integrable on \([a, b]\), we write \( \langle y, z \rangle \) for the integral
\[
\int_a^b z^*(t)y(t) \, dt.
\]

2. Properties of differential systems. Suppose that \( A_\alpha(t), \alpha = 0, 1, 2, \)
are \( n \times n \) matrix functions on \([a, b]\) such that \( A_1(t), A_2(t) \) are nonsingular,
and \( A_1, A_2, A_1^{-1}, A_2^{-1} \) all belong to \( \mathbb{F}_{\infty} \), while \( A_0 \in \mathbb{F}_n \). Let \( \mathcal{D} \) denote
the class of \( n \)-dimensional vector functions \( y \) in \( \mathbb{F}_n \) such that \( y = A_2^{-1}u_y \)
with \( u_y \in \mathbb{H}_n \). If \( y \in \mathcal{D} \) then clearly \( y \in \mathbb{F}_n \subset \mathbb{F}_n \), and the differential operator
\( L \) with domain \( D(L) \subset \mathcal{D} \) and value
\[
L[y](t) = A_1(t)[A_2(t)y(t)]' + A_0(t)y(t)
\]
(2.1)
is such that \( L[y] \in \mathbb{F}_n \) whenever \( y \in D(L) \). We shall restrict attention to such
operators \( L \) with \( D(L) \) a linear manifold in \( \mathbb{F}_n \) satisfying
\[
\mathcal{D}_0 \subset D(L) \subset \mathcal{D}, \quad \text{where} \quad \mathcal{D}_0 = \{ y \mid y \in \mathcal{D}, u_y(a) = 0 = u_y(b) \}.
\]
For brevity, if \( U = [U_{\alpha\beta}], \alpha = 1, \ldots, n, \beta = 1, \ldots, r, \) is an \( n \times r \) matrix
function defined on \([a, b]\), we shall denote by \( \vec{U} \) the corresponding \( 2n \times r \)
matrix \( [W_{\alpha\beta}], \sigma = 1, \ldots, 2n, \beta = 1, \ldots, r, \) such that \( W_{\alpha\beta} = U_{\alpha\beta}(a), W_{n+\alpha,\beta} = U_{\alpha\beta}(b), \alpha = 1, \ldots, n. \) In particular, \( \mathcal{D}_0 = \{ y \mid y \in \mathcal{D}, u_y = 0 \} \).
If \( D(L; a, b) \) denotes the set of \( 2n \)-dimensional vectors \( \xi \) such that there
exists a \( y \in D(L) \) with \( \dot{u}_y = \xi \), then \( D(L; a, b) \) is a linear manifold in \( \mathbb{C}_{2n} \), and since \( \mathcal{D}_0 \subset D(L) \), it follows that
\[
D(L) = \{ y \mid y \in \mathcal{D}, \dot{u}_y \in D(L; a, b) \}.
\]
That is, since the matrices \( A_1 \) and \( A_2 \) are nonsingular and the interval
\([a, b]\) is compact, the differential operator \( L \) is regular and its domain \( D(L) \)
is specified by the condition that \( y \) is a member of the above defined class \( \mathcal{D}_1 \), together with the condition that the associated absolutely continuous
vector function \( u_y \) satisfies the two-point linear homogeneous boundary
conditions expressing the fact that the end-values \( u_y(a), u_y(b) \) belong
to the linear manifold \( D(L; a, b) \) in \( \mathbb{C}_{2n} \). For the given operator \( L \), as
well as its adjoint which is defined in the next paragraph, specific forms
of these two-point boundary conditions are presented below in Lemma 2.1.

Let \( \mathcal{D}_n \) denote the totality of \( n \)-dimensional vector functions \( z \) such that
\( z, A_1z, (A_2^*)^{-1}A_0z \) are in \( \mathbb{F}_n \), \( z^*L[y] \in \mathbb{F}_1 \) for arbitrary \( y \in D(L) \), and for
which there exists a corresponding \( f_z \in \mathbb{F}_n \) such that \( \langle L[y], z \rangle = \langle y, f_z \rangle \)
for all \( y \in D(L) \). The operator \( T_a \) with domain \( \mathcal{D}_n \) and value \( T_a z = f_z \)
is called the adjoint of \( L \). Now let \( \Gamma_0 \) denote the subset of \( D(L) \) on which
\( u_y \in \mathbb{C}_{n0} \), where \( \mathbb{C}_{n0} \) signifies the class of continuously differentiable
n-dimensional vector functions \( w \) on \([a, b]\) satisfying \( w(a) = 0 = w(b) \).

Then \( \langle L[y], z \rangle = \langle y, f_z \rangle \) for \( z \in \mathcal{D}_a \), \( y \in \Gamma_0 \) implies

\[
0 = \int_a^b [z^*A_1u' + (z^*A_0 - f_z^*)A_z^{-1}u] \, dt
= \int_a^b \left[ z^*A_1 - \int_a^t (z^*A_0 - f_z^*)A_z^{-1} \right] u' \, dt
\]

for all \( u \in C^1_0 \), and from a moderate extension of the classical fundamental lemma of the calculus of variations (see Reid [16, §2]), it follows that there exists a constant \( \gamma \in C_a \) such that, a.e. on \([a, b]\),

\[
A_1^*(t)z(t) - \int_a^t (A_2^*)^{-1}A_0^*(z - f_z) \, ds = \gamma.
\]

Consequently, \( v_z(t) = \gamma + \int_a^t (A_2^*)^{-1}(A_0^*z - f_z) \, ds \) is such that \( v_z \in \mathcal{R}_a \), \( z = (A_1^*)^{-1}v_z \in \mathcal{R}_{\mathcal{E}_a} \subset \mathcal{E}_a \), and \( v'_z = (A_2^*)^{-1}(A_0^*z - f_z) \), so that \( f_z = -A_2^*v'_z + A_0^*z = -A_2^*(A_1^*)' + A_0^*z \). Consequently, if \( \mathcal{D}_z \) is the set of \( z \) in \( \mathcal{E}_a \) with \( v_z = A_1^*z \in \mathcal{R}_a \), and \( L^* \) is the differential operator with domain

\[
D(L^*) = \{ z \mid z \in \mathcal{D}_z, v_z u_y |^b_a = 0 \text{ for } y \in D(L) \},
\]

and functional value

\[
L^*[z](t) = -A_2^*(t)[A_1^*(t)z(t)]' + A_0^*(t)z(t)
= -A_2^*(t)v'_z(t) + A_0^*(t)(A_1^*)^{-1}(t)v_z(t),
\]

then \( T_a \) is a restriction of \( L^* \). On the other hand, if \( z \in D(L^*) \) then \( \langle L[y], z \rangle - \langle y, L^*[z] \rangle = v_z u_y |^b_a = 0 \) for \( y \in D(L) \), so that \( L^* \) is a restriction of \( T_a \), and hence \( T_a = L^* \). In general,

\[
z^*L[y] - (L^*[z])^*y = (v_z u_y)' \text{ for } y \in \mathcal{D}, z \in \mathcal{D}_z^*;
\]

in particular, if \( L[y] = 0 \) and \( L^*[z] = 0 \), then \( v_z(t)u_y(t) \) is constant. Also, if \( \mathcal{D}_0^* = \{ z \mid z \in \mathcal{D}_z^*, v_z = 0 \} \), then \( D(L^*) \) is a linear manifold in \( \mathcal{E}_a \) and

\[
\mathcal{D}_0^* \subset D(L^*) \subset \mathcal{D}_z^*.\]

Moreover, if \( D(L^*; a, b) \) is the set of \( 2n \)-dimensional vectors \( \xi \) such that there exists a \( z \in D(L^*) \) with \( \theta_z = \xi \), then \( D(L^*; a, b) \) is a linear manifold in \( \mathcal{C}_{2n} \), and

\[
D(L^*) = \{ z \mid z \in \mathcal{D}_z^*, \theta_z \in D(L^*; a, b) \}.\]

The results of the following three lemmas are consequences of relations
(2.3), (2.4), (2.7) and, as their counterparts for related problems are well known (see, for example, [3], [4, Chap. 11], [5], [17], [20]), they will be stated without proof.

**Lemma 2.1.** If \( \dim D(L; a, b) = 2n - m \), then \( \dim D(L^*; a, b) = m \). Moreover, (a) if \( m = 0 \), then \( D(L) = \mathcal{D}, D(L^*) = \mathcal{D}^* \); (b) if \( m = 2n \), then \( D(L) = \mathcal{D}_0, D(L^*) = \mathcal{D}^* \); (c) if \( 0 < m < 2n \), then there exist an \( m \times 2n \) matrix \( M \) and a \( (2n - m) \times 2n \) matrix \( P \) such that \( M \) has rank \( m \), \( P \) has rank \( 2n - m \), and

\[
\begin{align*}
(i) & \quad MKP^* = 0, \quad \text{where } K = \text{diag} \{E_n, -E_n\}, \\
(ii) & \quad D(L) = \{ y \mid y \in \mathcal{D}, M \hat{\omega}_y = 0 \}, \\
D(L^*) = \{ z \mid z \in \mathcal{D}^*, P\hat{\eta}_z = 0 \}.
\end{align*}
\]

The range and null space of \( L \) are denoted by \( \mathcal{R}(L) \) and \( \mathcal{N}(L) \), respectively, with corresponding notations for the range and null space of \( L^* \).

**Lemma 2.2.** \( \mathcal{R}(L) = [\mathcal{R}(L^*)]^+, \mathcal{R}(L^*) = [\mathcal{R}(L)]^+ \), i.e., if \( f \in \mathcal{L}_n \), then

\[
\begin{align*}
(2.9) & \quad L[y] = f, \quad y \in D(L) \\
\{ \text{or} \} & \quad L^*[z] = f, \quad z \in D(L^*) \}.
\end{align*}
\]

has a solution if and only if \( \langle z, f \rangle = 0 \) or \( \langle y, f \rangle = 0 \) for all solutions \( z \) or \( y \) of

\[
\begin{align*}
(2.10) & \quad L^*[z] = 0, \quad z \in D(L^*) \\
\{ \text{or} \} & \quad L[y] = 0, \quad y \in D(L) \}.
\end{align*}
\]

We shall denote by \( k \) and \( k^* \) the indices of compatibility of (2.9') and (2.10'), respectively, i.e., \( k = \dim \mathcal{R}(L), k^* = \dim \mathcal{R}(L^*) \).

**Lemma 2.3.** \( n + k^* = m + k \).

The following theorem is the classic result on the existence and form of the Green's matrix for a first order vector differential system, amended to provide for the generality of \( L[y] \) and \( L^*[z] \).

**Theorem 2.1.** If \( \dim D(L; a, b) = n \) and \( k = 0 \), then \( k^* = 0 \); and for \( f \in \mathcal{L}_n \) the system (2.9) has a unique solution given by

\[
y(t) = \int_a^t G(t, s)f(s) \, ds,
\]

where \( G(t, s) \) is the Green's matrix of explicit form

\[
G(t, s) = \frac{1}{2} Y(t) [E_n \text{ sgn } (t - s) + [M\hat{U}_y]^{-1}[MK\hat{U}_y]Z^*(s),
\]

where \( Y \) and \( Z \) are fundamental solutions of \( L[Y] = 0, L^*[Z] = 0 \), such that \( V_z U^* = E_n \).
3. Generalized Green's matrices. Independent of the restriction that \( \dim D(L; a, b) = n \) or \( k = 0 \), a matrix \( G(t, s) \) is called a generalized Green's matrix for the differential operator \( L \), if as a function of \( (t, s) \) it is of class \( \mathcal{C}^n \) on \( [a, b] \times [a, b] \), and (2.11) provides a linear mapping of \( \mathcal{R}(L) \) into \( D(L) \); that is, if \( f \in \mathcal{C}_n \) and is such that (2.9) has a solution, then a particular solution of (2.9) is given by (2.11). If \( m \neq n \), or if \( m = n \) and \( k \neq 0 \), then the generalized Green's matrix is not unique. If \( G_0(t, s) \) is one generalized Green's matrix, while \( Y_0(t) \) is an \( n \times k \) matrix whose column vectors form a basis for \( \mathcal{R}(L) \), and \( Z_0(t) \) is an \( n \times k^* \) matrix whose column vectors form a basis for \( \mathcal{R}(L^*) \), then for \( H(t) \in \mathcal{C}^\infty_\kappa \) and \( K(t) \in \mathcal{C}^\infty_{\kappa^*} \) the matrix

\[
G(t, s) = G_0(t, s) + Y_0(t)H^*(s) + K(t)Z_0^*(s)
\]

is also a generalized Green's matrix for \( L \). Indeed, it may be shown (see, for example, Reid [12], Bradley [3]) that an arbitrary generalized Green's matrix for \( L \) is of the form (3.1).

The existence of a generalized Green's matrix, and indeed its specific form which permits ready treatment of its structure and properties, is provided by the following theorem.

**Theorem 3.1.** If \( \Theta(t) \in \mathcal{C}^\infty_\kappa \), \( \Psi(t) \in \mathcal{C}^\infty_{\kappa^*} \) are such that the matrices

\[
\begin{align*}
(a) & \quad \int_a^b \Theta^*(t)Y_0(t) \, dt, \\
(b) & \quad \int_a^b Z_0^*(t)\Psi(t) \, dt
\end{align*}
\]

are nonsingular, where the column vectors of \( Y_0(t) \) and \( Z_0(t) \) form a basis for \( \mathcal{R}(L) \) and \( \mathcal{R}(L^*) \), respectively, then there is a unique generalized Green's matrix \( G(t, s) = G_{\Theta, \Psi}(t, s | L) \) of \( L \) such that

\[
\int_a^b \Theta^*(t)G(t, s) \, ds = 0 \quad \text{for} \quad s \in [a, b],
\]

\[
\int_a^b G(t, s)\Psi(s) \, ds = 0 \quad \text{for} \quad t \in [a, b].
\]

Moreover,

\[
G(t, s) = \frac{1}{2}Y(t)\{E_n \operatorname{sgn} (t - s) + Q(t, s)\}Z^*(s),
\]

where \( Y \) and \( Z \) are fundamental solutions of \( L[Y] = 0 \), \( L^*[Z] = 0 \), respectively, such that \( V^*_Z U_Y \equiv E_n \). Moreover,

(i) if \( 0 < m < 2n \), then

\[
Q(t, s) = Q_1 - 2S_2T(s) - \Omega(t)[Q_2 - 2S_4T(s)],
\]

where \( M, K, P \) are as in (2.8), \( T(s) = \int_a^s \Theta^*(t)Y(t) \, dt \), \( Q_1 = S_1\Delta + S_2T(b) \), \( Q_2 = S_3\Delta + S_4T(b) \), \( \Delta = MK^\epsilon \), \( Y_1(t) = -Y(t)\Omega(t) \) with
\[ \Omega(t) = \int_a^t Z^*(s) \Psi(s) \, ds, \quad T_1(s) = \int_a^s \Theta^*(t) Y_1(t) \, dt, \text{ and} \]

\[
\begin{bmatrix}
S_1 & S_2 \\
S_3 & S_4
\end{bmatrix} = \begin{bmatrix}
M\hat{\mathcal{O}}_r & M\hat{\mathcal{O}}_{r1} \\
-T(b) & -T_1(b)
\end{bmatrix}^{-1};
\]

(ii) if \( m = 0 \), then \( T(b) = \int_a^b \Theta^*(t) Y(t) \, dt \) is nonsingular, and

\[ Q(t, s) = -E + 2T^{-1}(b)T(s). \]

(iii) if \( m = 2n \), then \( Q(t, s) = [S_1 - \Omega(t)S_2]K\hat{\mathcal{O}}_r \), where

\[ \begin{bmatrix}
S_1 \\
S_3
\end{bmatrix} = [\hat{\mathcal{O}}_r \hat{\mathcal{O}}_{r1}]^{-1}. \]

In order to establish the above theorem, consider the following "bordered" or "enlarged" differential system in the vector functions \( y(t) \) of dimension \( n \), \( \rho(t) \) of dimension \( k^* \), and \( \mu(t) \) of dimension \( k \):

\begin{align*}
(a) & \quad L[y] + \Psi(t)\rho = f(t), \\
(b) & \quad \rho' = 0, \\
(c) & \quad \mu' + \Theta^*(t)y = 0,
\end{align*}

with the two-point boundary conditions

\begin{align*}
(3.6) (a) & \quad y \in D(L), \quad (b) \quad \mu(a) = 0, \quad \mu(b) = 0.
\end{align*}

In (3.5), (3.6) it is to be understood that the term in (3.5a) involving \( \rho \), and the differential equation (3.5b), do not occur if \( k^* = 0 \), while (3.5c) and (3.6b) do not occur if \( k = 0 \). With corresponding conventions on the nonpresence of certain matrices, if \( y(t) \) is the vector function of dimension \( N = n + k^* + k = m + 2k \), with \( y_\alpha = y_{\alpha}, \alpha = 1, \cdots, n, \quad y_{n+\beta} = \rho_\beta, \beta = 1, \cdots, k^*, \quad y_{n+k+\gamma} = \mu_\gamma, \gamma = 1, \cdots, k \), the differential equations (3.5) may be written in vector form as

\begin{align*}
(3.7) & \quad \mathcal{L}[y] = \mathcal{A}_1(t)[\mathcal{A}_2(t)y]' + \mathcal{A}_0(t)y = f, \\
\end{align*}

where

\[ \mathcal{A}_1 = \text{diag} \{ A_1(t), E_{k^*}, E_k \}, \]

\[ \mathcal{A}_2 = \text{diag} \{ A_2(t), E_{k^*}, E_k \}, \]

\[ \mathcal{A}_0 = \begin{bmatrix}
A_0(t) & \Psi(t) & 0 \\
0 & 0 & 0 \\
\Theta^*(t) & 0 & 0
\end{bmatrix}, \quad f = \begin{bmatrix}
f \\
0 \\
0
\end{bmatrix}. \]
The boundary conditions (3.6) may be written as $\mathfrak{N} \hat{u}_y = 0$, where $\mathfrak{N}$ is the $N \times 2N$ matrix such that:

(i) if $0 < m < 2n$ and the $M$ of (2.8ii) is of the form $[M_1 \ M_2]$, where $M_1, M_2$ are $m \times n$ matrices, then

$$\mathfrak{N} = \begin{bmatrix} M_1 & 0 & 0 & M_2 & 0 & 0 \\ 0 & 0 & E_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_k \end{bmatrix};$$

(ii) if $m = 0$, then $k = n$, $k^* = 0$, $N = n + k = 2n$, and

$$\mathfrak{N}' = \begin{bmatrix} 0 & E_n & 0 \\ 0 & 0 & E_n \end{bmatrix};$$

(iii) if $m = 2n$, then $k = 0$, $k^* = n$, and

$$\mathfrak{N}'' = \begin{bmatrix} E_n & 0 & 0 \\ 0 & 0 & E_n \end{bmatrix}.$$

Now, if for a given $f \in \mathcal{G}$ the system (3.5), (3.6) has a solution $y, \rho, \mu$, then

$$\int_a^b Z_0 \Psi dt = \int_a^b Z_0^* L[y] dt + \left( \int_a^b Z_0^* \Psi dt \right) \rho = \left( \int_a^b Z_0^* \Psi dt \right) \rho,$$

since the fact that the column vectors of $Z_0$ form a basis for $\mathfrak{N}(L^*)$, and $y \in D(L)$, imply

$$\int_a^b Z_0^* L[y] dt = \int_a^b \{Z_0^* L[y] - (L^*[Z_0])^* y\} dt = V_{Z_0} u_b = 0.$$

Therefore, if $f \in \mathcal{G}(L)$, so that $\int_a^b Z_0^* f dt = 0$ by Lemma 2.2, it follows from the nonsingularity of the matrix (3.2b) and the condition (3.10) that $\rho = 0$; hence $y(t) = y_0(t) + Y_0(t)\eta$, where $y_0(t)$ is a particular solution of (2.9) and $\eta$ is a $k$-dimensional constant vector which is uniquely determined by the relation $\int_a^b \Theta^*(t)y(t) dt = 0$, implied by the boundary conditions (3.6b). In particular, the homogeneous system given by $f = 0$ has only the null solution so that the differential operator $L[y]$ defined in (3.7), and with domain $\{(y, \rho, \mu) \mid y \in D(L), \rho \in \mathfrak{N}, \mu \in \mathfrak{N}, \mu(a) = 0 = \mu(b)\}$, has a Green's matrix $\mathcal{G}(t, s)$ as in Theorem 2.1. If $\mathcal{G}(t, s) = [g_{\alpha\beta}(t, s)], \alpha, \beta = 1, 2, 3$, where $g_{\alpha\beta}(t, s)$ is an $r_1 \times r_2$ matrix with $r_1 = n, r_2 = k^*, r_3 = k$, then for arbitrary $f \in \mathcal{G}$ the solution $(y, \rho, \mu)$ of
(3.5), (3.6) is such that
\[ y(t) = \int_a^b \mathcal{G}_H(t, s)f(s) \, ds, \quad \rho = \int_a^b \mathcal{G}_H(t, s)f(s) \, ds, \]
\[ \mu(b) - \mu(a) = \int_a^b \left[ \int_a^b \Theta^*(t)\mathcal{G}_H(t, s) \, dt \right] f(s) \, ds = 0. \]

Also, if \( f(t) = \Psi(t)\lambda \), where \( \lambda \) is a constant \( k \)-dimensional vector, then (3.5), (3.6) has the solution \( y(t) = 0, \rho(t) = \lambda, \mu(t) = 0 \), and in view of the uniqueness of solution of this system it follows that
\[ \int_a^b \mathcal{G}_H(t, s)\Psi(s) \, ds = 0 \quad \text{for} \quad t \in [a, b]. \]

Consequently, \( G(t, s) = \mathcal{G}_H(t, s) \) is a generalized Green's matrix for \( L \) satisfying the conditions (3.3). Now, for \( f \in \mathcal{L}_n \) we have
\[ f_1(t) = f(t) - \Psi(t) \left[ \int_a^b Z_0^*\Psi \, ds \right]^{-1} \int_a^b Z_0^*f \, ds \in [\mathfrak{R}(L^*)]^k = \mathfrak{R}(L); \]
and therefore, if \( G_1(t, s) \) is any generalized Green's matrix for \( L \), then
\[ y(t) = \int_a^b G_1(t, s)f_1(s) \, ds \]
is a particular solution of \( L[y] = f_1, y \in D(L) \).

If, in addition, \( G_1(t, s) \) satisfies the conditions (3.3), then
\[ y(t) = \int_a^b G_1(t, s)f(s) \, ds \quad \text{and} \quad \int_a^b \Theta^*(t)y(t) \, dt = 0, \]
so that
\[ y(t) = \int_a^b G_1(t, s)f(s) \, ds, \]
\[ \rho = \left[ \int_a^b Z_0^*\Psi \, ds \right]^{-1} \int_a^b Z_0^*f \, ds, \]
\[ \mu(t) = -\int_a^t \Theta^*(s)y(s) \, ds \]
is a solution of (3.5), (3.6). Therefore, \( \int_a^b G_1(t, s)f(s) \, ds = \int_a^b \mathcal{G}_H(t, s)f(s) \, ds \)
for arbitrary \( f \in \mathcal{L}_n \), and hence \( G_1(t, s) = \mathcal{G}_H(t, s) \) a.e. on \([a, b] \times [a, b]\).

This completes the proof that there is a unique generalized Green's matrix satisfying the conditions (3.3). The explicit form (3.4) for \( G_0, \Psi(t, s | L) \) follows from (2.12) and the two facts: (i) if \( Y \) and \( Z \) are fundamental matrix solutions of \( L[Y] = 0 \) and \( L[Z] = 0 \) such that \( E_n \equiv V_Z^*U_Y = Z^*A_1A_2Y \), then \( Y_1(t) = -Y(t) \int_a^t Z^*(s)\Psi(s) \, ds \) is the solution of \( L[Y_1] + \Psi = 0, Y_1(a) = 0 \); (ii) for \( T(t) = \int_a^t \Theta^*(s)Y(s) \, ds \) and \( T_1(t) \)
= \int_a^t \Theta^*(s) Y_1(s) \, ds, \text{ a fundamental matrix } y(t) \text{ of } \mathcal{L}[y] = 0 \text{ and the fundamental matrix solution } Z(t) = [\Theta_1 \Theta_2 y]^* \text{ of the corresponding adjoint system are such that}

\[ y = \begin{bmatrix} Y & Y_1 & 0 \\ 0 & E_k & 0 \\ -T & -T_1 & E_k \end{bmatrix}, \quad Z^* = \begin{bmatrix} Y^{-1} A_2^{-1} A_1^{-1} & -Y^{-1} Y_1 A_2^{-1} A_1^{-1} & 0 \\ 0 & E_k & 0 \\ TY^{-1} & -TY^{-1} Y_1 + T_1 & E_k \end{bmatrix}, \]

where, as mentioned above, it is understood that certain of the matrices do not occur if \( k^* = 0 \) or \( k = 0 \). By direct computation it follows that the \( N \times N \) matrix \([\Theta u y]^{-1} \Theta u y\) of \( \Theta(t, s) \) is equal to:

\[
\begin{bmatrix}
S_1 \Delta + S_2 T(b) & S_1 \Delta_1 + S_2 T_1(b) & -2S_2 \\
S_3 \Delta + S_4 T(b) & S_3 \Delta_1 + S_4 T_1(b) & -2S_4 \\
0 & 0 & E_k
\end{bmatrix}
\]

if \( 0 < m < 2n \),

\[
\begin{bmatrix}
-E_n & 2T^{-1}(b) \\
0 & E_n
\end{bmatrix}
\]

if \( m = 0 \),

\[
\begin{bmatrix}
S_7 K U_\tau & S_7 K U_{\tau_1} \\
S_8 K U_\tau & S_8 K U_{\tau_1}
\end{bmatrix}
\]

if \( m = 2n \).

It is to be commented that in a previous paper [14, §9] the author has used the “bordering” procedure in the induction proof of the existence of proper values for certain types of definitely self-adjoint boundary problems, and, as there noted, this method had been used earlier in the consideration of accessory boundary problems of the calculus of variations.

4. Equivalent differential operators. Consider a second differential operator

\[
L_0[y^0] = A_1^0[A_2^0 y^0]' + A_0^0 y^0,
\]

where \( A_0^0, A_1^0, A_2^0 \) satisfy the conditions specified for \( A_0, A_1, A_2 \) in §2, with correspondingly \( \mathcal{D}^0 = \{ y^0 : y^0 \in \mathcal{L}_{\delta}, y^0 = (A_2^0)^{-1} u_{\phi^0} \text{ with } u_{\phi^0} \in \mathcal{K}_{\eta}\}, \mathcal{D}_0^0 = \{ y^0 : y^0 \in \mathcal{D}^0, u_{\phi^0}(a) = 0 = u_{\phi^0}(b)\}, \) and \( D(L_0) \) a linear manifold in \( \mathcal{L}_{\delta} \) satisfying \( \mathcal{D}_0^0 \subset D(L_0) \subset \mathcal{D}^0 \). The system (2.9') is said to be equivalent to the system

\[
L_0[y^0] = 0, \quad y^0 \in D(L_0),
\]

under the transformation

\[
y^0(t) = H(t)y(t), \quad a \leq t \leq b,
\]
provided (4.3) defines a one-to-one mapping of \( D \) onto \( D^0 \) such that \( y(t) \) belongs to \( D(L) \), or satisfies the differential equation of \((2.9')\), if and only if \( y^0 \) belongs to \( D(L_0) \), or satisfies the differential equation of \((4.2)\). Now the mapping \((4.3)\) is equivalent to the mapping

\[
(4.3') \quad u^0_0(t) = H_1(t)u^0_y(t), \quad \text{where} \quad H_1(t) = A^0_2(t)H(t)A_2^{-1}(t),
\]

and by the methods of Reid [17, §3] (see also Bradley [3, §5]), one obtains the following result.

**Theorem 4.1.** The system \((2.9')\) is equivalent to \((4.2)\) under the transformation \((4.3)\) if and only if

(i) the \( H_1(t) \) of \((4.3')\) is a nonsingular matrix in \( \mathfrak{A}_{nn} \) satisfying

\[
H_1' + A^0_y H_1 - H_1 A^0_y = 0, \quad \text{where} \quad A^0_y = A_1^{-1}A_0A_2^{-1},
\]

(ii) \( D(L_0) = \mathfrak{D}^0 \) if \( D(L) = \mathfrak{D} \), and \( D(L_0) = \mathfrak{D}^0_0 \) if \( D(L) = \mathfrak{D}_0 \);

(iii) if \( m = 2n - \dim D(L; a, b) \) satisfies \( 0 < m < 2n \), then \( D(L_0) = \{ y^0 \mid M^0u^0_y = 0 \} \), where \( M^0 \) is an \( m \times 2n \) matrix of rank \( m \) satisfying

\[
M^0[\text{diag} \{ H(a), -H(b) \}]P^* = 0,
\]

where \( P \) is as in Lemma 2.1.

Moreover, the general solution of \((4.4)\) is

\[
H_1(t) = A^0_2(t)Y^0(t)CY^{-1}(t)A_2^{-1}(t),
\]

where \( Y, Y^0 \) are fundamental matrix solutions of \( L[Y] = 0 \) and \( L_0[Y^0] = 0 \), respectively, and \( C \) is a constant matrix, so that the most general form of the transformation matrix \( H \) is \( H(t) = Y^0(t)CY^{-1}(t) \), with \( C \) a nonsingular constant matrix.

Corresponding to the results of Lemma 2.2 and Theorem 3.3 of Reid [17], one has the following result.

**Theorem 4.2.** Suppose that \( \dim D(L; a, b) = n \), and \((2.9')\) is incompatible. Then \( \dim D(L^*; a, b) = n \), \((2.10')\) is incompatible also, and if \( G, H \) are the respective Green’s matrices for \( L, L^* \), then \( H(t, s) = [G(s, t)]^* \).

Moreover, if \((2.9')\) is equivalent to \((4.2)\) under the transformation \((4.3)\), then \( \dim D(L^*; a, b) = n \), \((4.2)\) is incompatible, and the respective Green’s matrices \( G, G^0 \) of \( L, L_0 \) are such that

\[
(4.5) \quad G^0(t, s)A^0_1(s)A^0_2(s)H(s) = H(t)G(t, s)A_1(s)A_2(s).
\]

For brevity, let \( S(L; \Theta, \Psi) \) denote the system \((3.5)\) and \((3.6)\), determined by the differential operator \( L \) and matrix functions \( \Theta, \Psi \) such that the corresponding matrices \((3.2)\) are nonsingular. Now the system adjoint to \( S(L; \Theta, \Psi) \) is not \( S(L^*; \Psi, \Theta) \), but is related to it by a transformation \((4.3)\),
in which $H(t)$ is the $N \times N$ constant matrix $H(t) = \text{diag} \{ E_n, J_{k,k^*} \}$, where $J_{k,k^*}$ is the $(k + k^*) \times (k + k^*)$ matrix

$$J_{k,k^*} = \begin{bmatrix} 0 & -E_{k^*} \\ E_k & 0 \end{bmatrix}.$$ 

Consequently, upon applying Theorem 4.2 to the adjoint of $S(L; \Theta, \Psi)$ and $S(L^*; \Psi, \Theta)$, one has the following result.

**Theorem 4.3.** $G_{\Theta,\Theta}(t, s | L^*) = [G_{\Theta,\Psi}(s, t | L)]^*.$

Of particular interest is an application of the preceding results to a boundary problem

$$L[y; \lambda] = L[y] - \lambda B(t)y = 0, \quad y \in D(L),$$

which is symmetrizable under the transformation

$$z(t) = T(t)y(t),$$

that is, for each value of $\lambda$ the boundary problem (4.7) is equivalent to the adjoint boundary problem

$$L^*[z; \lambda] = L^*[z] - \lambda B^*(t)z = 0, \quad z \in D(L^*),$$

under (4.8), and the associated matrix

$$S(t) = T^*(t)B(t)$$

is Hermitian on $[a, b]$. It is supposed that the coefficient matrices of $L$ satisfy the conditions as specified in §2, and $B(t) \in \mathbb{R}_{nn}$ and is not equal a.e. to the zero matrix. In view of Theorem 4.1, and similar to the results of §§4, 5 of Reid [17], it follows that (4.7) is equivalent to (4.9) under (4.8) if and only if $\dim D(L; a, b) = n$ and $T_1(t) = A_1^*(t)T(t)A_2^{-1}(t)$ is such that

(a) $T_1' - A_\#^*T_1 - T_1A_\# = 0, \quad B_\#^*T_1 + T_1B_\# = 0,$

where $A_\# = A_1^{-1}A_0A_2^{-1}$, $B_\# = A_1^{-1}BA_2^{-1}$.

(b) $P[\text{diag} \{ T_1(a), -T_1(b) \}]P^* = 0,$

where $P$ is as in Lemma 2.1.

Moreover, the general solution of (4.11a) is $T_1 = (A_2^*)^{-1}(Y^*)^{-1}CY^{-1}A_2^{-1}$, where $Y$ is a fundamental matrix solution of $L[Y] = 0$, and $C$ is a constant matrix. If $T_1 = T_1(t)$ satisfies conditions (4.11), so also does $T_1 = T_1^*(t)$. Consequently, if (4.7) is equivalent to (4.9) under (4.8), then (4.7) is also equivalent to (4.9) under $z(t) = T_a(t)y(t)$, where $T_a = c_1T + c_2(A_1^*)^{-1}(A_2^*)^{-1}T^*A_1A_2$, with $c_1, c_2$ constants such that $T_a$ is nonsingular for some value $t \in [a, b]$; moreover, since (4.10) is Hermitian, it follows that $T_a^*(t)B(t) = (\tilde{c}_1 - \tilde{c}_2)T^*(t)B(t).$
If (4.7) is symmetrizable under (4.8), then, corresponding to the result (i) of [17, Theorem 5.1], it may be established that if \( \lambda \) is not a proper value of (4.7) then the Green's matrix \( G(t; s; \lambda) \) for \( L[\cdot; \lambda] \) is such that \( K_1(t; s; \lambda) = S(t)G(t; s; \lambda)B(s) \) satisfies \( K_1(t; s; \lambda) = [K_1(s, t; \lambda)]^* \). Moreover, corresponding to the result (ii) of [17, Theorem 5.1], if \( \Lambda \) denotes the set of \( y \in D(L) \) such that \( L[y] = B(t)g(t) \) with \( g \in \mathfrak{R}_a^\infty \), then for arbitrary real constants \( c_1, c_2 \) the functional \( L[y; c_1, c_2; T] = T^*(t)[c_1L[y] + c_2B(t)y] \) is Hermitian on \( \Lambda \) in the sense that

\[
\langle L[y_1; c_1, c_2; T], y_2 \rangle = \langle y_1, L[y_2; c_1, c_2; T] \rangle \quad \text{for} \quad y_\alpha \in \Lambda, \alpha = 1, 2.
\]

Now suppose that (4.7) is fully symmetrizable under (4.8); that is, (4.7) is symmetrizable under this transformation, and if \( y \) is a proper function for (4.7) then \( \langle Sy, y \rangle \neq 0 \). In particular, (4.7) is fully symmetrizable under (4.8) if this boundary problem is normal in the sense that \( y(t) = 0 \) is the only solution of \( L[y] = 0 \), \( y \in D(L) \) for which \( B(t)y(t) = 0 \) throughout \([a, b]\), and definite in the sense that there exist real constants \( c_1, c_2 \), not both zero, such that \( \langle L[y; c_1, c_2; T], y \rangle \) is positive for all \( y \in \Lambda \) with \( B(t)y(t) \neq 0 \) throughout \([a, b]\). For a discussion of these concepts, and proofs of results for a system more restrictive in nature than the one here considered, the reader is referred to [17, §§5, 6, 7]. In particular, for a system (4.7) that is fully symmetrizable under (4.8) all proper values are real; and if for a given proper value \( \lambda_0 \) the column vectors of the \( n \times k \) matrix \( Y_0(t) \) form a basis for the set of solutions of (4.7) for \( \lambda = \lambda_0 \), then the \( k \times k \) matrix

\[
\int_a^b Y_0^*(t)S(t)Y_0(t) \, dt
\]

is nonsingular. Moreover, \( Z_0(t) = T(t)Y_0(t) \) is an \( n \times k \) matrix whose column vectors form a basis for the set of solutions of the adjoint system \( L^*[z; \lambda_0] = 0 \), \( z \in D(L^*) \), and

\[
\begin{align*}
\Theta(t) &= S(t)Y_0(t), \\
\Psi(t) &= B(t)Y_0(t),
\end{align*}
\]

are \( n \times k \) matrix functions such that the \( k \times k \) matrices of (3.2) are nonsingular. Also, the incompatible system \( S(L[\cdot; \lambda_0]; SY_0, BY_0) \) is fully symmetrizable under the transformation

\[
\gamma(t) = 3(t)\gamma(t),
\]

where \( \gamma(t) \) is the \( (n + 2k) \)-dimensional vector function with \( y_\alpha = y_\alpha, \lambda \alpha = 1, \ldots, n, y_{n+\beta} = \rho_\beta, y_{n+k+\beta} = \mu_\beta, \beta = 1, \ldots, k, \) and \( 3(t) = \text{diag} \{ T(t), J_{k,k} \} \), and \( J_{k,k} \) is defined by (4.6). The corresponding \( (n + 2k) \times (n + 2k) \) matrices \( \delta(t) \) and \( \delta(t) \) for \( S(L[\cdot; \lambda_0]; SY_0, BY_0) \) are

\[
\delta(t) = \text{diag} \{ B(t), 0 \} \quad \text{and} \quad \delta(t) = \text{diag} \{ S(t), 0 \}.
\]

If \( [G_{\alpha\beta}(t, s)] \), \( \alpha, \beta = 1, 2, 3, \) is the Green's matrix for the differential operator \( L[\cdot; \lambda_0] \) of \( S(L[\cdot; \lambda_0]; SY_0, BY_0) \) as in (3.7), then \( \Xi(t, s) = S(t)G(t, s)\delta(s) \) is such that \( \Xi(t, s) = [\Xi(s, t)]^* \). Consequently, since
$G_{01}(t, s)$ is the corresponding generalized Green's matrix for $L[\cdot; \lambda_0]$, we have the following result.

**Theorem 4.4.** Suppose that (4.7) is fully symmetrizable under the transformation (4.8), that $\lambda_0$ is a proper value of (4.7) of index $k$, and $Y_0(t)$ is an $n \times k$ matrix whose column vectors form a basis for the set of solutions of (4.7) for $\lambda = \lambda_0$. Then the generalized Green’s matrix $G_{\Theta, \Psi}(t, s; \lambda_0)$ for $L[\cdot; \lambda_0]$ with $\Theta, \Psi$ given by (4.12) is such that $K(t, s) = S(t)G_{\Theta, \Psi}(t, s; \lambda_0)B(s)$ satisfies $K(t, s) \leq [K(s, t)]^*$. In particular, if (4.7) is fully symmetrizable under (4.8) and $G_{\Theta, \Psi}(t, s; \lambda_0)$ is as in Theorem 4.4, then in the Hilbert space $\mathcal{S}$ of $n$-dimensional vector functions of integrable square on $[a, b]$ the transformations $\mathcal{R}: k = \mathcal{R}g$ and $\mathcal{S}: s = \mathcal{S}g$ on $\mathcal{S}$ to $\mathcal{S}$ with corresponding functional values

$$k(t) = \int_a^b G_{\Theta, \Psi}(t, s; \lambda_0)B(s)g(s) \, ds, \quad s(t) = S(t)g(t),$$

are such that $\mathcal{R}$ is completely continuous and fully symmetrizable by each of the transformations $\mathcal{R}^p$, $p = 0, 1, \cdots$, and $\mu$ is a proper value of the vector integral equation

$$y(t) = \mu \int_a^b G_{\Theta, \Psi}(t, s; \lambda_0)B(s)y(s) \, ds, \quad t \in [a, b],$$

with corresponding proper vector $y(t)$ if and only if $\lambda = \lambda_0 + \mu$ is a proper value of (4.7) with $y(t)$ a corresponding proper vector. That is, $G_{\Theta, \Psi}(t, s; \lambda_0)$ is a “deletion operator” for (4.7) in the sense that (4.13) has for its proper vector functions precisely those which are proper functions for (4.7) corresponding to proper values $\lambda$ which are distinct from $\lambda_0$, and also as a kernel of the associated integral equation (4.13) this generalized Green’s matrix retains the property of full symmetrizability possessed by the ordinary Green’s matrix $G(t, s; \lambda)$ when $\lambda$ is a real number that is not a proper value of (4.7). For a discussion of results for (4.7) which are deducible from the theory of symmetrizable transformations in Hilbert space the reader is referred to Reid [15], [17, §7] and Zaanen [21, Chaps. 11, 12, 15].

**REFERENCES**