A Recourse Certainty Equivalent for Decisions Under Uncertainty*†

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August 1, 1989
Revised May 25, 1990

Abstract

We propose a new criterion for **decision-making under uncertainty**. The criterion is based on a **certainty equivalent** (CE) of a (monetary valued) random variable Z,

$$S_v(Z) = \sup_z \{z + E_Z v(Z - z)\}$$

where $v(\cdot)$ is the decision maker's **value-risk function**. This CE is derived from considerations of **stochastic optimization with recourse**, and is called **recourse certainty equivalent** (RCE). We study (i) the properties of the RCE, (ii) the recoverability of $v(\cdot)$ from $S_v(\cdot)$ (in terms of the rate of change in risk), (iii) comparison with the "classical CE" $u^{-1}Eu(\cdot)$ in **expected utility** (EU) theory, (iv) relation to risk-aversion, (v) connection with Machina's **generalized expected utility** theory, and its use to explain the **Allais paradox** and other decision theoretic paradoxes, and (vi) applications to models of **production under price uncertainty**, **investment in risky and safe assets** and **insurance**. In these models the RCE gives intuitively appealing answers for all risk-averse decision makers, unlike the EU model which gives only partial answers, and requires, in addition to risk-aversion, also assumptions on the so-called **Arrow-Pratt indices**.

Key words: Stochastic optimization with recourse. Decision-making under uncertainty. Expected utility. Certainty equivalents. The Allais paradox and other decision theoretic paradoxes. Risk aversion. Production under price uncertainty. Investment in risky and safe assets. Insurance.

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*Supported by National Science Foundation Grant DDM-8996112.
†Helpful comments by the referees are gratefully acknowledged.
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1 Introduction

We propose a new criterion for decision-making under uncertainty using one of the fundamental concepts of stochastic programming, namely that of **recourse**, [5], [14], [15] and [47]. Recourse refers to corrective action, after realization of the random variables, which is taken into account when making the actual decision (before realization). For the role of recourse in stochastic programming see [16], [25] and [49].

Decision making under uncertainty presupposes the ability to rank random variables, i.e. a **complete order** $\succeq$ on the space of RV's, with $X \succeq Y$ denotes X **preferred** to Y. If the **preference order** $\succeq$ is given in terms of a real valued function CE$(\cdot)$ on the space of RV's,

$$X \succeq Y \iff \text{CE}(X) \geq \text{CE}(Y) \quad \text{for all RV's } X,\ Y$$

we call CE(Z) a **certainty equivalent** (CE) of Z, corresponding to the preference $\succeq$. In particular, a **decision maker** (DM for short) is indifferent between a RV Z and a constant[1] z iff $z = \text{CE}(Z)$.

In the **expected utility** (EU) model, the DM is assumed to have a **utility function** $u(\cdot)$ which typically is **strictly increasing** (more is better) and **concave**. The DM's preference is then given by

$$\begin{aligned} X \succeq Y &\iff E\,u(X) \geq E\,u(Y) \\ &\iff u^{-1}E\,u(X) \geq u^{-1}E\,u(Y), \end{aligned} \tag{1.1}$$

Accordingly we define the **classical certainty equivalent** (CCE) by

$$C_u(Z) = u^{-1}Eu(Z) \tag{1.2}$$

Another CE, suggested by expected utility, is the u-**mean CE** $M_u(\cdot)$ defined, for any RV Z, by

$$E\,u(Z - M_u(Z)) = 0 \tag{1.3}$$

see e.g. [11, p. 86], under "principle of zero utility".

Still another CE is based on the "dual theory" of Yaari, [50]. Given a monotone function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, **Yaari's certainty equivalent** $Y_f(\cdot)$ is

$$Y_f(Z) = \int f(1 - F_Z(t))dt \tag{1.4}$$

where F_Z is the cumulative distribution function of the RV Z. Unlike the CCE (1.2), both Yaari's CE (1.4) and the u-mean CE (1.3) are **shift additive** in the sense that

$$\text{CE}(Z + c) = \text{CE}(Z) + c \quad \text{for all RV } Z \text{ and constant } c \tag{1.5}$$

[1]Regarded as a degenerate RV.

In the EU model a **risk-averse decision maker**, i.e. one for whom a RV X is less desirable than a sure reward of EX, is characterized by a concave utility function u. The concavity of u also expresses the attitude towards wealth (decreasing marginal utility). Thus the DM's **attitude towards wealth** and his **attitude towards risk** are "forever bonded together", [50, p. 95]. Certain difficulties with the EU model are due to this fact. In Yaari's dual theory [50], and in the RCE model proposed below, the attitude towards wealth and the attitude towards risk are effectively separated. In particular, Yaari's risk aversion is compatible with **linearity in payments**[2].

The CE advocated here is the **recourse certainty equivalent** (RCE)

$$S_v(Z) := \sup_z \{z + E_Z v(Z - z)\} \tag{1.6}$$

where $v(\cdot)$ is the **value-risk function** of the DM. The RCE was first introduced in [9]. We propose here the RCE S_v as a criterion for decision making under uncertainty i.e. for ranking monetary valued RV's. The given value-risk function v induces a complete order $\succeq$ on RV's,

$$X \succeq Y \quad \Longleftrightarrow \quad S_v(X) \geq S_v(Y) \tag{1.7}$$

in which case X is preferred over Y by a DM with a value-risk function v.

To place the RCE in the framework of stochastic programming with recourse, consider a mathematical program with stochastic RHS,

$$\max \{f(z) : \; g(z) \leq Z \} \tag{1.8}$$

Here:

z - decision variable,

Z - budget (\$),

$g(z)$ - the budget consumed by z,

$f(z)$ - the profit resulting from z.

In **Stochastic Programming with Recourse** (SPwR), the optimal decision z^* is determined by considering for each realization of Z a **second stage decision** y, consuming $h(y)$ budget units and contributing $v(y)$ to the profit. Thus z^* is the optimal solution of

$$\max_z \{f(z) + E_Z \left(\max_y \{v(y) : \; g(z) + h(y) \leq Z\} \right)\} \tag{1.9}$$

The success of SPwR stems from the fact that it takes into account the trade off between **greed** (profit maximization) and **caution** (aversion to insolvency).

[2] "In studying the behavior of *firms*, linearity in payments may in fact be an appealing feature", [50, p. 96]. Indeed, a firm which divides the last dollar of its income as dividends, cannot be equated with the proverbial rich who value the marginal dollar at less than that. Yet both the firm, and the rich, can be risk averse.

In those cases where $z,\ y$ are scalars (e.g. levels of production), $h(\cdot)$ is monotone increasing ("more costs more") and $v(\cdot)$ is monotone increasing ("more is better"), we can rewrite (1.9) as

$$\max_z \{f(z) + E_Z \left(\max_y \{v(y) :\ g(z) + y \leq Z\}\right)\} \tag{1.10}$$

where $y,\ v$ correspond in (1.9) to $h(y),\ v \circ h^{-1}$ respectively. If v is monotonely increasing then (1.10) is equivalent to:

$$\max_z \{f(z) + E_Z\, v(Z - g(z))\} \tag{1.11}$$

in which y has been eliminated. The optimal value in (1.11) is the "SPwR value" of the SP (1.8).

In this paper we use the SPwR paradigm to "evaluate" RV's. Our thesis is that **assigning a value to a RV** is in itself a **decision problem**. Thus, the "value" of a RV Z to a DM is the "most that he can make of it", i.e.

$$\text{value of } Z = \max\ \{z : z \leq Z\} \tag{1.12}$$

and we interpret (1.12) as the "SPwR value" which, by analogy with (1.11), is the RCE (1.6)

$$\sup_z\ \{z + E_Z v(Z - z)\}$$

We call $v(\cdot)$ the **value-risk function**. Its meaning is explained in § 3.

We apply the RCE to study three classical models of economic decisions under uncertainty

- **competitive firm under price uncertainty**, [41],[29]
- **investment in safe and in risky assets**, [3],[12],[22]
- **optimal insurance coverage**, [18]

In these models the classical EU theory usually requires conditions on the risk-aversion indices associated with the utility u: the **Arrow-Pratt absolute risk-aversion index**

$$r(z) = -\frac{u''(z)}{u'(z)} \tag{1.13}$$

and the **Arrow-Pratt relative risk-aversion index**

$$R(z) = zr(z) \tag{1.14}$$

In contrast, the RCE theory gives unambigious predictions for <u>all</u> risk-averse DM's, without further restrictive assumptions. Moreover, the RCE is mathematically tractable, comparable in simplicity and elegance to the EU model.

2 Properties of the Recourse Certainty Equivalent

We begin by listing several reasonable assumptions on $v(\cdot)$. Important properties of the RCE follow from combinations of these assumptions.

Assumption 2.1
(v1) $v(0) = 0$
(v2) $v(\cdot)$ *is strictly increasing*
(v3) $v(x) \leq x$ *for all* x
(v4) $v(\cdot)$ *is strictly concave*
(v5) v *is continuously differentiable*

Remark 2.1 By Assumptions 2.1(v1),(v2)

$$v(x) < 0 \quad \text{for} \quad x < 0$$

thus $v(\cdot)$ can also be viewed as a **penalty function**, penalizing violations of the constraint

$$z \leq Z$$

We can think of the variable z in (1.6) as a sure amount diverted from (or a loan taken against) a RV Z before realization, where insolvency (resulting from negative realizations of $Z - z$) is penalized by the function $v(\cdot)$. The RCE (1.6) thus represents maximization of present value of a lottery subject to strong aversion to insolvency.

Of particular interest is the following class of value-risk functions

$$\mathcal{U} = \left\{ v : \begin{array}{c} v \text{ strictly increasing, strictly concave, continuously} \\ \text{differentiable,} \quad v(0) = 0, \quad v'(0) = 1 \end{array} \right\} \tag{2.1}$$

which, for the purpose of comparison with the EU model, can be thought of as **normalized utility functions** [3].

The attainment of supremum in (1.6) is settled, for any $v \in \mathcal{U}$, as follows.

[3] For concave v the gradient inequality

$$v(x) \leq v(0) + v'(0)x$$

shows that all $v \in \mathcal{U}$ satisfy (v3) of Assumption 2.1.

Lemma 2.1 *Let the RV Z have support $[z_{\min},\ z_{\max}]$, with finite $z_{\min}$ and $z_{\max}$. Then for any $v \in \mathcal{U}$ the supremum in (1.6) is attained uniquely at some z_S,*

$$z_{\min} \le z_S \le z_{\max}, \tag{2.2}$$

which is the solution of

$$E\, v'(Z - z_S) = 1, \tag{2.3}$$

so that

$$S_v(Z) = z_S + E\, v(Z - z_S) \tag{2.4}$$

Proof. Note that $Z - z_{\min} \ge 0$ with probability 1. Also $v'(\cdot)$ is decreasing since v is concave. Therefore

$$E\, v'(Z - z_{\min}) \le E\, v'(0) = 1$$

Similarly

$$E\, v'(Z - z_{\max}) \ge E\, v'(0) = 1$$

Since v' is continuous, the equation [4]

$$E\, v'(Z - z) = 1$$

has a solution z_S in $[z_{\min}, z_{\max}]$, which is unique by the strict monotonicity of v'. Now z_S is a stationary point of the function

$$f(z) = z + E\, v(Z - z) \tag{2.5}$$

which is concave since $v \in \mathcal{U}$, see (2.1). Therefore the supremum of (2.5) is attained at z_S. □

Remark 2.2 The assumption $v \in \mathcal{U}$ in Lemma 2.1 can be relaxed: Differentiability is not necessary for the attainment of supremum in (2.5). Since a concave function v has derivatives from the left v'_- and from the right v'_+ at any point in the interior of its effective domain, we can replace

$$v'(0) = 1$$

by the weaker condition that v is finite in a neighborhood of 0, and that

$$v'_-(0) \ge 1 \ge v'_+(0) \tag{2.6}$$

i.e. the subgradient of v at 0 contains 1. Attainment (perhaps nonunique) of supremum in (2.5) can then be shown. See Example 2.4 for a nonsmooth value-risk function.

[4] This equation is the necessary condition for maximum in (1.6). Differentiation "inside the expectation" is valid if e.g. v' is continuous and $E\, v'(\cdot) < \infty$, see [10, p. 99].

Theorem 2.1 (Properties of the RCE)

(a) **Shift additivity**. *For any $v : \mathbb{R} \to \mathbb{R}$, any RV Z and any constant c*

$$S_v(Z + c) = S_v(Z) + c \tag{2.7}$$

(b) **Consistency**. *If v satisfies (v1), (v3) then, for any constant c* [5]*,*

$$S_v(c) = c \tag{2.8}$$

(c) **Subhomogeneity**. *If v satisfies (v1) and (v4) then, for any RV Z,*

$$\frac{1}{\lambda} S_v(\lambda Z) \quad \textit{is decreasing in } \lambda, \quad \lambda > 0$$

(d) **Monotonicity**. *If v satisfies (v2) then, for any RV X and any nonnegative RV Y,*

$$S_v(X + Y) \geq S_v(X)$$

(e) **Risk aversion**. *v satisfies (v3) if and only if*

$$S_v(Z) \leq EZ \qquad \textit{for all RV's } Z \tag{2.9}$$

(f) **Concavity**. *If $v \in \mathcal{U}$ then for any RV's X_0 , X_1 and $0 < \alpha < 1$,*

$$S_v(\alpha X_1 + (1 - \alpha) X_0) \geq \alpha S_v(X_1) + (1 - \alpha) S_v(X_0) \tag{2.10}$$

(g) **2nd order stochastic dominance**. *Let X, Y be RV's with compact supports. Then*

$$S_v(X) \geq S_v(Y) \quad \textit{for all } v \in \mathcal{U} \tag{2.11}$$

if and only if

$$E\, v(X) \geq E\, v(Y) \quad \textit{for all } v \in \mathcal{U} \tag{2.12}$$

Proof. (a) For any function $v : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} S_v(Z + c) &= \sup_z \{z + E\, v(Z + c - z)\} \\ &= c + \sup_z \{(z - c) + E\, v(Z - (z - c))\} = c + S_v(Z) \end{aligned}$$

(b) For any constant c,

$$\begin{aligned} S_v(c) &= \sup_z \{z + v(c - z)\} \\ &\leq \sup_z \{z + (c - z)\} \quad \text{by (v3)} \\ &= c \end{aligned}$$

[5] Considered as a degenerate RV.

Conversely,

$$\begin{aligned} S_v(c) &\geq \{c + v(c-c)\} \\ &= c \quad \text{by (v1)} \end{aligned}$$

(c) For any $v : \mathbb{R} \to \mathbb{R}$ and $\lambda > 0$ define v_λ by

$$v_\lambda(x) := \frac{1}{\lambda} v(\lambda x), \quad \forall\ x \tag{2.13}$$

Then

$$S_{v_\lambda}(Z) = \frac{1}{\lambda} S_v(\lambda Z), \quad \text{for all RV } Z, \tag{2.14}$$

as follows from,

$$\begin{aligned} S_{v_\lambda}(Z) &= \sup_z \{z + \frac{1}{\lambda} E_Z v(\lambda(Z-z)\} \\ &= \frac{1}{\lambda} \sup_{\bar{z}} \{\bar{z} + E_Z v(\lambda Z - \bar{z})\} \quad (\bar{z} = \lambda z) \\ &= \frac{1}{\lambda} S_v(\lambda Z) \end{aligned}$$

It therefore suffices to show that

$$v_\lambda(z) \quad \text{is decreasing in } \lambda, \quad \lambda > 0$$

Indeed, let

$$0 < \lambda_1 < \lambda_2$$

By the concavity of v it follows, for all z,

$$\frac{v(\lambda_2 z) - v(\lambda_1 z)}{\lambda_2 - \lambda_1} \leq \frac{v(\lambda_1 z) - v(0)}{\lambda_1}$$

and by (v1)

$$\frac{v(\lambda_2 z)}{\lambda_2} \leq \frac{v(\lambda_1 z)}{\lambda_1}$$

(d)

$$\begin{aligned} S_v(X+Y) &= \sup_z \{z + Ev(X+Y-z)\} \\ &\geq \sup_z \{z + Ev(X-z)\} \quad \text{by (v2)} \end{aligned}$$

(e) If v satisfies (v3) then for any RV Z,

$$\begin{aligned} S_v(Z) &= \sup_z \{z + Ev(Z-z)\} \\ &\leq \sup_z \{z + E(Z-z)\} = EZ \end{aligned}$$

Conversely, if for all RV's Z

$$S_v(Z) \leq EZ$$

then, for any RV Z and any constant z,

$$\begin{aligned} z + Ev(Z-z) &\leq EZ \\ \therefore\ Ev(Z-z) &\leq E(Z-z) \\ \therefore\ Ev(Z) &\leq EZ \end{aligned}$$

proving (v3).

(f) Let $0 < \alpha < 1$, and $X_\alpha = \alpha X_1 + (1-\alpha)X_0$. Then by the concavity of v, for all $z_0,\ z_1$,

$$Ev(X_\alpha - \alpha z_1 - (1-\alpha)z_0) \geq \alpha Ev(X_1 - z_1) + (1-\alpha)Ev(X_0 - z_0)$$

Adding $\alpha z_1 + (1-\alpha)z_0$ to both sides, and supremizing jointly with respect to $z_1,\ z_0$, we get

$$\begin{aligned} S_v(X_\alpha) &\geq \sup_{z_1,\ z_0} \{\alpha\,[z_1 + Ev(X_1 - z_1)] + (1-\alpha)\,[z_0 + Ev(X_0 - z_0)]\} \\ &= \alpha S_v(X_1) + (1-\alpha)S_v(X_0) \end{aligned}$$

(g) (2.12) $\Longrightarrow$ (2.11). Since each $v \in \mathcal{U}$ is increasing, (2.12) implies

$$z + Ev(X-z) \geq z + Ev(Y-z) \quad \forall\ z, \text{ and } \forall\ v \in \mathcal{U}$$

and (2.11) follows by taking suprema.

(2.11) $\Longrightarrow$ (2.12). Let $z_X,\ z_Y$ be points where the suprema defining $S_v(X)$ and $S_v(Y)$ are attained, see Lemma 2.1. Then, for any $v \in \mathcal{U}$,

$$\begin{aligned} S_v(X) &= z_X + Ev(X - z_X) \geq z_Y + Ev(Y - z_Y), \quad \text{by (2.11)} \\ &\geq z_X + Ev(Y - z_X) \end{aligned}$$

Therefore

$$Ev(X - z_X) \geq Ev(Y - z_X) \ \text{ for all } v \in \mathcal{U}, \ \text{ implying (2.12).} \quad \square$$

Remark 2.3 Theorem 2.1 lists properties which seem reasonable for any certainty equivalent. Property (b) is natural and requires no justification. The remaining properties will now be discussed one by one.
(a) Note that shift additivity holds for all functions v, i.e. it is a **generic property** of the RCE. To explain shift additivity consider a decision-maker indifferent between a lottery Z and a sure amount S. If 1 Dollar is added to all the possible outcomes of the lottery, then an addition of 1 Dollar to S will keep the decision maker indifferent. Recall that shift additivity holds also for the Yaari CE (1.4), and for the u-mean (1.3). For the classical CE (1.2), shift additivity holds iff the utility u is linear or exponential, see Example 2.1 below.
(c) An important consequence (and the reason for the name "subhomogeneity") is

$$S_v(\lambda Z) \leq \lambda S_v(Z), \quad \text{for all RV } Z \text{ and } \lambda > 1$$

Thus indifference between the RV Z and its CE $S_v(Z)$ goes together with preference for $\lambda S_v(Z)$ over the RV λZ, for $\lambda > 1$. This is explained by

$$E(\lambda Z) = \lambda E Z$$

$$\mathrm{Var}\,(\lambda Z) = \lambda^2 \mathrm{Var}\,(Z) > \lambda \mathrm{Var}\,(Z) \text{ if } \lambda > 1$$

An interesting result, in view of (c) and (e), is that for $v \in \mathcal{U}$,

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} S_v(\lambda Z) = EZ$$

(d) If v satisfies (v1) and (v2), and if the RV Z satisfies $Z \geq z_{\min}$ with probability 1, then

$$S_v(Z) \geq z_{\min} \tag{2.15}$$

This follows from part (d) by taking $X = z_{\min}$ (degenerate RV) and $Y = Z - z_{\min}$.
(e) In the EU model, risk aversion is characterized by the concavity of the utility function. In the RCE model risk aversion is carried by the weaker property $v(x) \leq x$, $\forall x$. We show in § 4 that concavity of v corresponds to strong risk aversion in the sense of Rothschild and Stiglitz, [38].
(f) The concavity of $S_u(\cdot)$, for all $u \in \mathcal{U}$, expresses risk-aversion as aversion to variability. To gain insight consider the case of two independent RV's X_1 and X_0 with the same mean and variance. The mixed RV $X_\alpha = \alpha X_1 + (1-\alpha) X_0$ has the same mean, but a smaller variance. Concavity of S_u means that the more centered RV X_α is preferred.

The risk-aversion inequality (2.9) is implied by (f): Let $Z,\ Z_1,\ Z_2,\ \ldots$ be independent, identically distributed RV's. Then by (f),

$$\begin{aligned} S_u\Big(\frac{1}{n}\sum_{i=1}^n Z_i\Big) &\geq \frac{1}{n}\sum_{i=1}^n S_u(Z_i) \\ &= S_u(Z) \end{aligned}$$

As $n \to \infty$, (2.9) follows by the strong law of large numbers.

In contrast, the classical CE $u^{-1}Eu(\cdot)$ is not necessarily concave for all concave u.

(g) In general, for a given $u \in \mathcal{U}$,

$$E\,u(X) \geq E\,u(Y) \tag{2.16}$$

does not imply

$$S_u(X) \geq S_u(Y) \tag{2.17}$$

i.e. (2.16) and (2.17) may induce different orders on RV's, see [9]. Note however that in (2.11) and (2.12) the inequality holds for all $u \in \mathcal{U}$ [6]. This defines a partial order on RV's, the (2nd order) stochastic dominance, [23].

Example 2.1 (Exponential value-risk function) *Here*

$$u(z) = 1 - e^{-z} \;, \quad \forall\; z, \tag{2.18}$$

and equation (2.3) becomes $Ee^{-Z+z} = 1$, *giving* $z_S = -\log\; Ee^{-Z}$ *and the same value for the RCE*

$$S_u(Z) = -\log\; Ee^{-Z} \tag{2.19}$$

A special feature of the exponential utility function (2.18) is that the classical CE (1.2) becomes

$$u^{-1}Eu(Z) = -\log\; Ee^{-Z}$$

showing that for the exponential function, the certainty equivalents (1.6) and (1.2) coincide.

Example 2.2 (Quadratic value-risk function) *Here*[7]

$$u(z) = z - \frac{1}{2}z^2 \;, \qquad z \leq 1 \tag{2.20}$$

and for a RV Z *with* $z_{\max} \leq 1$, $EZ = \mu$ *and variance* σ^2, *equation (2.3) gives* $z_S = \mu$, *and by (2.4)*

$$S_u(Z) = \mu - \frac{1}{2}\sigma^2 \tag{2.21}$$

Corollary 2.1 *In both the exponential and quadratic value-risk functions*

$$S_u(\sum_{i=1}^{n} Z_i) = \sum_{i=1}^{n} S_u(Z_i) \tag{2.22}$$

for independent RV's $\{\; Z_1,\; Z_2,\; \ldots\;,\; Z_n\}$ [8] $\square$

[6] In which case Y is called **riskier** than X.

[7] The restriction $z \leq 1$ in (2.20) guarantees that u is increasing throughout its domain.

[8] The classical CE (1.2) is additive, for independent RV's, if u is exponential but not if u is quadratic.

Example 2.3 *For the so-called* **hybrid model** ([4],[42]) *with exponential utility u and a normally distributed RV $Z \sim N(\mu,\ \sigma^2)$,*

$$S_u(Z) = \mu - \frac{1}{2}\sigma^2$$

Example 2.4 (Piecewise linear value-risk function) *Let*

$$v(t) = \begin{cases} \beta t, & t \leq 0 \\ \alpha t, & t > 0 \end{cases} \quad 0 < \alpha < 1 < \beta \tag{2.23}$$

If F is the cumulative distribution function of the RV Z, then the maximizing z in (1.6) is the $\frac{1-\alpha}{\beta-\alpha}$-percentile of the distribution F of Z:

$$z^* = F^{-1}(\frac{1-\alpha}{\beta-\alpha})$$

and the RCE associated with (2.23) is

$$S_v(Z) = \beta \int^{z^*} t\, dF(t) + \alpha \int_{z^*} t\, dF(t).$$

The following result is stated for discrete RV's. Let X be a RV assuming finitely many values,

$$\text{Prob}\,\{X = x_i\} = p_i \tag{2.24}$$

We denote X by

$$X = [\mathbf{x}, \mathbf{p}]\,,\ \mathbf{x} = (x_1, x_2, \ldots\ , x_n),\ \mathbf{p} = (p_1, p_2, \ldots\ , p_n) \tag{2.25}$$

The RCE of $[\mathbf{x}, \mathbf{p}]$ is

$$S_v([\mathbf{x}, \mathbf{p}]) = \max_z\ \{z + \sum_{i=1}^{n} p_i\, v(x_i - z)\} \tag{2.26}$$

We consider $S_v([\mathbf{x}, \mathbf{p}])$ as a function of the arguments $\mathbf{x}$ and $\mathbf{p}$.

Theorem 2.2

(a) For any function $v : I\!R \to I\!R$, and any $\mathbf{x} = (x_1, x_2, \ldots\ , x_n)$, the RCE $S_v([\mathbf{x}, \mathbf{p}])$ is convex in $\mathbf{p}$.
(b) For v concave, and any probability vector $\mathbf{p}$, *the RCE $S_v([\mathbf{x}, \mathbf{p}])$ is concave in* $\mathbf{x}$.

Proof. (a) A pointwise supremum of affine functions, see (2.26), is convex.
(b) The supremand

$$z + \sum_{i=1}^{n} p_i \, v(x_i - z)$$

is **jointly concave** in z and x. The supremum over z is concave in x, [37]. □

We summarize, for a RV $[\mathbf{x}, \mathbf{p}]$, the dependence on $\mathbf{p}$ and $\mathbf{x}$, of the expected utility $Eu(\cdot)$ and 3 certainty equivalents.

	As a function of $\mathbf{p}$	As a function of $\mathbf{x}$
Eu, u concave	linear	concave
$u^{-1}Eu$, u concave	convex	?
Y_f (1.4)	convex	linear
S_v	convex	concave (if v is concave)

3 Recoverability and the Meaning of $v(\cdot)$

In § 2 we studied properties of S_v induced by v. This section is devoted to the inverse problem, of **recovering** v from a **given** S_v.

The discussion is restricted to RCE's S_v defined by $v \in \mathcal{U}$. For these RCE's, we find $v \in \mathcal{U}$ satisfying (1.6).

Our results are stated in terms of an elementary RV X

$$X = \begin{cases} x, & \text{with probability } p \\ 0, & \text{with probability } \bar{p} = 1 - p \end{cases} \tag{3.1}$$

which we denote (x, p). For this RV,

$$S_v((x, p)) = \sup_z \{z + p\, v(x - z) + \bar{p}\, v(-z)\} \tag{3.2}$$

which we abbreviate $S_v(x, p)$.

Theorem 3.1 *If* $v \in \mathcal{U}$ *then*

$$v(x) = \frac{\partial}{\partial p} S_v(x, p) \mid_{p=0} \tag{3.3}$$

Proof. For $v \in \mathcal{U}$ the supremum in (3.2) is attained at $z = z(x, p)$ satisfying the optimality condition (2.3)

$$p\, v'(x - z(x, p)) + \bar{p}\, v'(-z(x, p)) = 1 \tag{3.4}$$

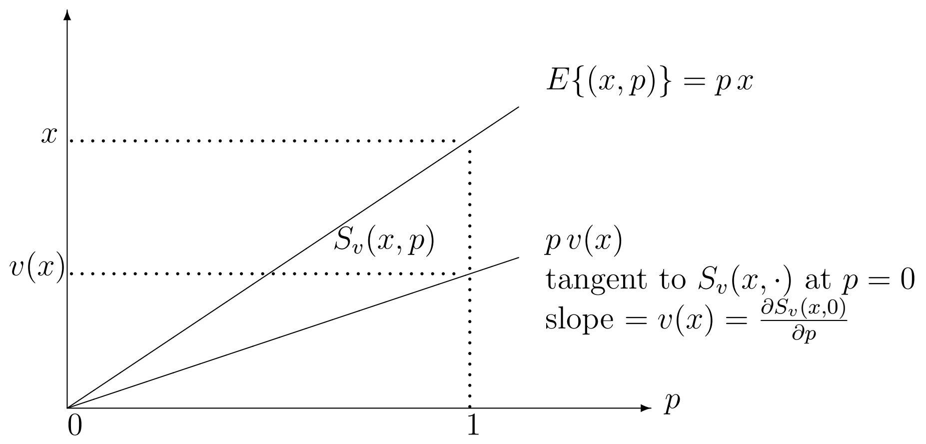

Figure 3.1: Recovering $v(x)$ from $S_v(x,p)$

which, for $p = 0$ gives

$$v'(-z(x,0)) = 1$$

and since $v \in \mathcal{U}$,

$$z(x,0) = 0 \tag{3.5}$$

Now, by the envelope theorem (appendix A),

$$\frac{\partial S_v(x,p)}{\partial p} = v(x - z(x,p)) - v(-z(x,p)) \tag{3.6}$$

and (3.3) follows by substituting (3.5) and $v(0) = 0$ in (3.6). □

To interpret this result consider an RCE maximizing individual who currently owns 0 \$, and is offered the sum x with probability p. The resulting change in his RCE is

$$\Delta(x,p) = S_v(x,p) - S_v(x,0)$$

and the rate of change is $\frac{\Delta(x,p)}{p}$. Theorem 3.1 says that this rate of change, for an infinitesimal change in risk ($p \to 0$) is precisely $v(x)$, the value-risk function evaluated at x.

Note that for a risk-neutral DM the added value Δ is $E\{(x,p)\} = px$. We illustrate this, for fixed x, in Fig. 3.1.

The following theorem is a companion of Theorem 3.1. It says that the limiting rate of change $\frac{\Delta(x,p)}{x}$ is exactly the probability p of obtaining x.

Theorem 3.2 *If $v \in \mathcal{U}$ then*

$$p = \frac{\partial}{\partial x} S_v(x,p) \mid_{x=0} \tag{3.7}$$

Proof. Substituting $x = 0$ in (3.4) gives

$$v'(-z(0,p)) = 1 \tag{3.8}$$

By the envelope theorem (Appendix A) we get

$$\frac{\partial S_v(x,p)}{\partial x} = p\, v'(x - z(x,p))$$

which, substituting $x = 0$ and (3.8) gives,

$$\frac{\partial S_v(0,p)}{\partial x} = p \quad \square$$

It is natural to ask, for any certainty equivalent $\mathrm{CE}(x,p)$, for the values

$\frac{\partial}{\partial p}\mathrm{CE}(x,0)$	the **value risk function** of CE
$\frac{\partial}{\partial x}\mathrm{CE}(0,p)$	the **probability risk function** of CE

We summarize the results, in the following table, for the classical CE $u^{-1}Eu$, the u-mean M_u (1.3) (in both cases we assume that the underlying utility u is normalized: $u(0) = 0,\ u'(0) = 1$) and the RCE S_v.

Certainty equivalent $\mathrm{CE}(x,p)$	$\frac{\partial}{\partial p}\mathrm{CE}(x,0)$	$\frac{\partial}{\partial x}\mathrm{CE}(0,p)$
$u^{-1}Eu$	$u(x)$	p
M_u	$u(x)$	p
S_v	$v(x)$	p

For the Yaari CE (1.4)

$$Y_f(x,p) = \begin{cases} x f(p), & x \geq 0 \\ x\left[1 - f(\bar{p})\right], & x \leq 0 \end{cases} \tag{3.9}$$

We get:

$$\lim_{p \to 0^+} \frac{Y_f(x,p) - Y_f(x,0)}{p} = \begin{cases} x f'(0), & x \geq 0 \\ x f'(1), & x \leq 0 \end{cases}$$

and the two-sided derivatives

$$\begin{aligned}
\lim_{x\to 0^+} \frac{Y_f(x,p) - Y_f(0,p)}{x} &= f(p) \\
\lim_{x\to 0^-} \frac{Y_f(x,p) - Y_f(0,p)}{x} &= 1 - f(\bar{p})
\end{aligned}$$

Remark 3.1

(a) The value-risk function for the EU model is thus precisely the (normalized) utility $u(x)$. This suggests a new way to recover the utility function in the EU theory.
(b) The probability-risk function (for a nonnegative RV) in Yaari's theory is thus precisely the function $f(p)$ in terms of which Y_f is uniquely defined. This is a new interpretation of f.
(c) Note that the value-risk function corresponding to Yaari's CE is of the form

$$v(x) = \begin{cases} \alpha x, & x \leq 0 \\ \beta x, & x \geq 0 \end{cases} \tag{3.10}$$

with $\alpha = f'(0),\ \beta = f'(1)$. The convexity of f plus its normalization $f(0) = 0,\ f(1) = 1$ imply

$$\alpha < 1 < \beta$$

The function v in(3.10) is the source of the piecewise linear value-risk function in Exmaple 2.4.

4 Strong Risk Aversion

In the EU model risk aversion is characterized by the concavity of the utility function, while in the RCE model it is equivalent to the weaker property (Theorem 2.1(e))

$$v(x) \leq x, \qquad \forall\ x. \tag{4.1}$$

It is natural to ask what corresponds, in the RCE model, to the **concavity** of v, i.e.

$$v \in \mathcal{U} \tag{4.2}$$

The answer is given here in terms of a classical notion of risk-aversion due to Rothschild and Stiglitz [38], see also [17].

Definition 4.1 *Let F_X, F_Y be the c.d.f. of the RV's X, Y with support $[a,\ b]$ and equal expected values.*
(a) If there is a $c \in [a,\ b]$ such that

$$\begin{aligned} F_Y(t) &\geq F_X(t), \quad a \leq t \leq c \\ F_X(t) &\geq F_Y(t), \quad c \leq t \leq b \end{aligned}$$

then F_Y is said to **differ** *from F_X by a* **mean preserving simple increase in risk** *(MPSIR).*
(b) F_Y is said to **differ** *from F_X by a* **mean preserving increase in risk** *(MPIR) if it differs from F_X by a sequence of MPSIR's.*

Definition 4.2 *An RCE maximizing DM with a value-risk function v exhibits* **strong risk-aversion** *if*

$$\left\{ \begin{array}{l} F_Y \text{ differs from } F_X \\ \text{by a MPIR} \end{array} \right\} \implies S_v(Y) \leq S_v(X)$$

This concept is best illustrated graphically as in [33]. Let

$$x_1 < x_2 < x_3$$

be fixed, and let $D\{x_1, x_2, x_3\}$ denote the **probability distributions** over the values x_1, x_2, x_3. Each $\mathbf{p} = (p_1, p_2, p_3) \in D\{x_1, x_2, x_3\}$ can be represented by a point in the unit triangle in the (p_1, p_3)-plane as in Fig. 4.1, where p_2 is determined by $p_2 = 1 - p_1 - p_3$. The dotted lines are loci of distributions with same expectation (**iso-mean lines**) i.e. points (p_1, p_3) such that

$$p_1\, x_1 + (1 - p_1 - p_3)\, x_2 + p_3\, x_3 = \text{constant} \tag{4.3}$$

As one moves in the unit triangle across the iso-mean lines, from the southeast (SE) corner to the northwest (NW) corner, the values of the mean (4.3) increase. Thus movement from the SE to the NW is in the preferred direction.

The iso-mean lines are parallel with slope (i.e. $\Delta p_3 / \Delta p_1$)

$$\text{slope of iso-mean lines } = \frac{x_2 - x_1}{x_3 - x_2} > 0 \tag{4.4}$$

A movement along the iso-mean lines, in the NE direction corresponds to an MPIR as in Definition 4.1(b).

Similarly, the solid lines in Fig. 4.1 represent **iso expected utility curves** which are parallel straight lines (due to the "linearity in probabilities" of the EU functional) with

$$\text{slope of iso-}EU\text{ lines } = \frac{u(x_2) - u(x_1)}{u(x_3) - u(x_2)} > 0 \tag{4.5}$$

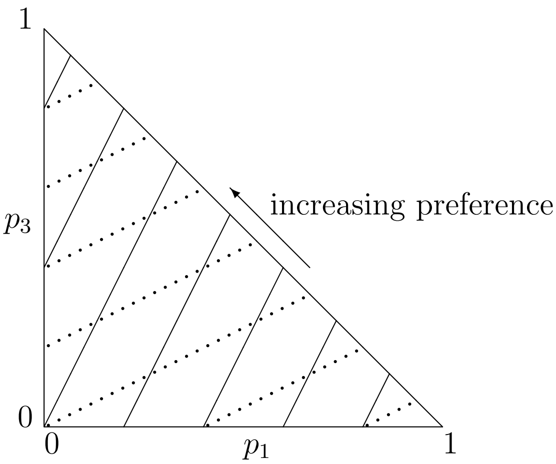

Figure 4.1: Iso-EU and iso-mean lines in $D\{x_1, x_2, x_3\}$

The slope (4.5) is positive because of the monotonicity of u.

For EU-maximizers **strong risk-aversion** corresponds to the iso-EU lines being steeper than the iso-mean lines, i.e.

$$\frac{u(x_2) - u(x_1)}{u(x_3) - u(x_2)} > \frac{x_2 - x_1}{x_3 - x_2} \tag{4.6}$$

which holds for all $x_1 < x_2 < x_3$ iff u is concave.

Turning to the RCE functional, the iso-RCE curves are not straight lines (since the RCE functional is convex in the probabilities). For RCE-maximizers **strong risk-aversion** means that at each point (p_1, p_3), **the slope of the iso-RCE curve** (through that point) **is steeper than the slope of the iso-mean line** (given by (4.4), see Fig. 4.2. Let now $p_3 = p_3(p_1)$ be the representation of an iso-RCE curve. By the definition (1.6), p_3 is solved from

$$\sup_z \{z + p_1 v(x_1 - z) + (1 - p_1 - p_3) v(x_3 - z) + p_3 v(x_3 - z)\} = \text{constant} \tag{4.7}$$

Then

$$\text{strong risk-aversion} \iff \frac{dp_3}{dp_1} > \frac{x_2 - x_1}{x_3 - x_2} \tag{4.8}$$

Let the left side of (4.7) be written as a function

$$s(p_1, p_2, p_3)$$

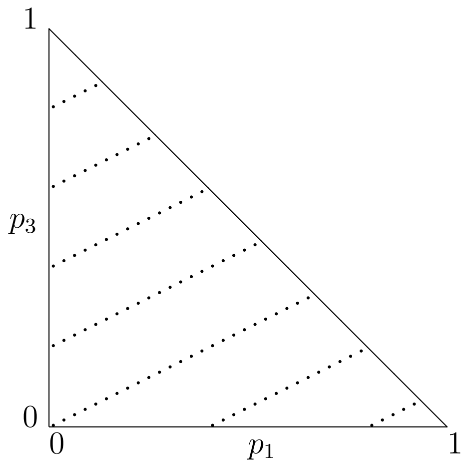

Figure 4.2: Iso-RCE curves and iso-mean lines in $D\{x_1, x_2, x_3\}$

of the probabilities p_i. Differentiating (4.7) with respect to p_1 we get

$$s_1 - s_2 - (s_2 - s_3)p_3' = 0, \quad \text{where} \quad s_i = \frac{\partial s}{\partial p_i} \tag{4.9}$$

By the envelope theorem (Appendix A)

$$s_i = \frac{\partial s}{\partial p_i} = u(x_i - z^*) \tag{4.10}$$

where $z^* = z^*(p_1, p_3)$ is uniquely determined by (2.3)

$$p_1\, v'(x_1 - z^*) + (1 - p_1 - p_3)\, v'(x_2 - z^*) + p_3\, v'(x_3 - z^*) = 1$$

Combining (4.9) and (4.10) we thus get

$$p_3'(p_1) = \frac{v(x_2 - z^*) - v(x_1 - z^*)}{v(x_3 - z^*) - v(x_2 - z^*)}$$

and the Diamond-Stiglitz risk-aversion is, by (4.8)

$$\frac{v(x_2 - z^*) - v(x_1 - z^*)}{v(x_3 - z^*) - v(x_2 - z^*)} > \frac{x_2 - x_1}{x_3 - x_2} = \frac{(x_2 - z^*) - (x_1 - z^*)}{(x_3 - z^*) - (x_2 - z^*)} \tag{4.11}$$

which holds for all $x_1 < x_2 < x_3$ iff v is concave.

The above discussion can be generalized to a general RV X with distribution function $F \in D(T)$, where

$$D(T) := \{\text{distribution functions with compact support } T\} \tag{4.12}$$

We note that the RCE $S_v(X)$ can be written as

$$S_v(X) = \int U(x, F)\, dF(x) \tag{4.13}$$

where

$$U(x, F) = z(F) + v(x - z(F)) \tag{4.14}$$

and the maximizing $z(F)$ is obtained implicitly from (2.3)

$$\int v'(x - z(F))\, dF(x) = 1 \tag{4.15}$$

Thus S_v, regarded as a function of F,

$$S_v(X) = V(F) \tag{4.16}$$

is a **generalized expected utility preference functional** in the sense of Machina, [30]. By (4.13), $U(x, F)$ is then the **local utility function** of Machina. We recall [30, Theorem 2] that for $V(F)$ Fréchet differentiable on $D(T)$, the preference order induced by V is strongly risk-averse iff $U(x, F)$ is concave in x for all $F \in D(T)$. Finally, by (4.14), the local utility $U(\cdot, F)$ is concave for all F iff the risk-value function $v(\cdot)$ is concave.

Remark 4.1 In the EU theory concavity of the utility u characterizes both **risk aversion** ($\mathrm{CE}(X) \leq E\,X$) and **strong risk aversion**, hence the two are equivalent in EU theory. In the RCE theory, risk aversion requires that $v(x) \leq x$ while strong risk aversion requires the stronger property that v is concave.

For non-EU theories this divergence between the two notions of risk-aversion is not surprising. We note that in Yaari's dual theory

$$Y_f(X) \leq E\,X \quad \text{requires} \quad f(t) \leq t,\ \forall t$$

whereas strong risk-aversion requires the convexity of f, [50, Theorem 2][9].

[9] The convexity of f, plus Yaari's normalization $f(0) = 0,\ f(1) = 1$ implies $f(t) \leq t$.

5 The RCE and Decision Theoretic Paradoxes

The recent interest in "non-EU" theories [10] is motivated mainly by empirical evidence of systematic violations of the EU axioms, notably the **Independence Axiom** underlying the "linearity in probabilities", [46]. Deviations from the behavior prescribed by EU Theory have become known as ("decision theoretic") **paradoxes**, in particular, the **Allais paradox** ([1], [2]), the **common consequence effect**, the **common ratio effect**, **over sensitivity to small probabilities** and the **utility evaluation effect**, e.g. [26], [34] and [31].

The RCE is nonlinear (in fact convex, see § 2) in probabilities. Also, equations (4.13,4.14) show the RCE S_v to be a Machina's generalized expected utility preference functional, if the value-risk function v belongs to the special class (2.1) of **normalized utility functions** $\mathcal{U}$. We show now that a further restriction of v to a subclass $\mathcal{U}_{\rm AP} \in \mathcal{U}$ "immunizes" the RCE S_v against the above-mentioned decision theoretic paradoxes.

The subclass $\mathcal{U}_{\rm AP}$ is defined in terms of a construct similar to the **Arrow-Pratt absolute risk-aversion index**. Let $v \in \mathcal{U}$ be twice continuously differentiable. Then its **risk index** r_v is defined as

$$r_v(x) = -\frac{v''(x)}{v'(x)} \tag{5.1}$$

For Machina's **local utility function** $U(x, F)$ we similarly define

$$r_{U,F}(x) = -\frac{U_{xx}(x, F)}{U_x(x, F)} \tag{5.2}$$

The class $\mathcal{U}_{\rm AP}$ is then

$$\mathcal{U}_{\rm AP} = \left\{ v \in \mathcal{U} \text{ of (2.1)} : \begin{array}{c} v \text{ twice continuously differentiable} \\ r_v(x) \text{ is nonincreasing} \end{array} \right\} \tag{5.3}$$

In Machina's **Generalized Expected Utility** (GEU) Theory, the preference order $V(F)$ (over distributions F) is an integral of the **local utility function** $U(x, F)$

$$V(F) = \int U(x, F)\, dF(x) \tag{5.4}$$

and the decision theoretic paradoxes can be explained if $U(x, F)$ satisfies two hypotheses on the risk index (5.2), see [30, § 4.1]

Hypothesis I: For any distribution $F \in D(T)$, the risk index $r_{U,F}(x)$ is nonincreasing in $x \in T$.

Hypothesis II: If $F_1,\ F_2\ \in\ D(T)$, and if F_1 stochastically dominates F_2 (1st order), then

$$r_{U,F_1}(x) \geq r_{U,F_2}(x), \forall\ x \in\ T \tag{5.5}$$

[10] In particular [6], [7], [8], [13], [20], [21], [24], [26], [27], [30], [36], [48], [50].

Since, for $v \in \mathcal{U}$, the RCE S_v is a GEU preference functional (see (4.13,4.14)), we may use Machina's results of [30]. For the local utility function $U(x, F)$ of (4.14), Hypothesis I reduces to the following statement: For every $F \in D(T)$,

$$-\frac{v''(x - z(F))}{v'(x - z(F))} \text{ is nonincreasing in } x$$

which clearly holds for $v \in \mathcal{U}_{\text{AP}}$. We show now that Hypothesis II also holds in $\mathcal{U}_{\text{AP}}$.

Proposition 5.1 *Let $v \in \mathcal{U}_{\text{AP}}$, and let*

$$U(x, F) = z(F) + v(x - z(F))$$

be the underlying local utility (in the sense of (4.13, 4.14)). Then Hypothesis II holds.

Proof. Let F_1 dominate F_2 stochastically (1st order), i.e.

$$E_{F_1}(h(X)) \; \geq \; E_{F_2}(h(X)) \; \forall \text{ nondecreasing functions } h \tag{5.6}$$

For the above $u(x, F)$, Hypothesis II reduces to

$$r_v(x - z(F_1)) \geq r_v(x - z(F_2)) \; \forall \; x$$

which holds, for $v \in \mathcal{U}_{\text{AP}}$, if and only if,

$$z(F_1) \geq z(F_2) \tag{5.7}$$

We have to show that (5.6) implies (5.7). Suppose (5.6) holds, but

$$z(F_1) < z(F_2) \tag{5.8}$$

Then, from the definition (4.15) of $z(F)$, the fact that $-v'$ is strictly increasing (since $v \in \mathcal{U}_{\text{AP}} \subset \mathcal{U}$) and the assumed inequality (5.8),

$$\begin{aligned} -1 = -E_{F_1} v'(x - z(F_1)) \;\; &> \;\; -E_{F_1} v'(x - z(F_2)) \\ &\geq \;\; -E_{F_2} v'(x - z(F_2)) \text{ by (5.6) with } h = -v' \\ &= \;\; -1 \text{ by (4.15), contradicting (5.8).} \; \Box \end{aligned}$$

To summarize: When confined to risk-value functions in the class $\mathcal{U}_{\text{AP}}$ (5.3) (in particular, strictly concave and nonincreasing risk indices), the RCE theory is compatible with the observed behavior (e.g. Allais paradox, common consequence effect, common ratio effect) which violates the Independence Axiom of EU theory.

6 Functionals and Approximations

Let $\mathbf{Z} = (Z_i)$ be a RV in $\mathbb{R}^n$, with expectation $\boldsymbol{\mu}$(vector) and covariance matrix Σ (if $n = 1$ then as above $\Sigma = \sigma^2$). For any vector $\mathbf{y} \in \mathbb{R}^n$, the **inner product**,

$$\mathbf{y} \cdot \mathbf{Z} = \sum_{i=1}^{n} y_i Z_i$$

is a scalar RV. Given $u \in \mathcal{U}$, the corresponding RCE of $\mathbf{y} \cdot \mathbf{Z}$ are taken as functionals in $\mathbf{y}$, the **RCE functional**

$$s_u(\mathbf{y}) := S_u(\mathbf{y} \cdot \mathbf{Z}), \tag{6.1}$$

We collect properties of the RCE functional in the following theorem, whose proof appears in Appendix B.

Theorem 6.1 *Let $u \in \mathcal{U}$ be twice continuously differentiable, and let $\mathbf{Z}$ and $s_u(\cdot)$ be as above. Then:*
(a) The functional s_u is concave, and given by

$$s_u(\mathbf{y}) = z_S(\mathbf{y}) + Eu(\mathbf{y} \cdot \mathbf{Z} - z_S(\mathbf{y})) \tag{6.2}$$

where $z_S(\mathbf{y})$ is the unique solution z of

$$E\, u'(\mathbf{y} \cdot \mathbf{Z} - z) = 1 \tag{6.3}$$

(b) Moreover,

$$s_u(\mathbf{0}) = 0,\ \nabla s_u(\mathbf{0}) = \boldsymbol{\mu},\ \ \nabla^2 s_u(\mathbf{0}) = u''(0)\Sigma \tag{6.4}$$

$$z_S(\mathbf{0}) = 0,\ \nabla z_S(\mathbf{0}) = \boldsymbol{\mu} \tag{6.5}$$

and if u is three times continuously differentiable,

$$\nabla^2 z_S(\mathbf{0}) = \frac{u'''(\mathbf{0})}{u''(\mathbf{0})}\Sigma \qquad \square \tag{6.6}$$

Theorem 6.1 can be used to obtain the following approximation of the functional $s_u(\cdot)$ based on its Taylor expansion around $\mathbf{y} = \mathbf{0}$.

Corollary 6.1 *If u is three times continuously differentiable then*

$$s_u(\mathbf{y}) = \boldsymbol{\mu} \cdot \mathbf{y} + \frac{1}{2}\, u''(\mathbf{0})\mathbf{y} \cdot \Sigma\mathbf{y} + \circ(\| \mathbf{y} \|^2) \quad \square \tag{6.7}$$

Remark 6.1

(a) In particular, for $n = 1$ and $y = 1$, it follows from (6.7) that the RCE has the following second-order approximation

$$\begin{aligned} S_u(Z) &\approx \mu + \frac{1}{2}u''(0)\sigma^2 \\ &= \mu - \frac{1}{2}r(0)\sigma^2 \end{aligned} \tag{6.8}$$

where $r(\cdot)$ is the Arrow-Pratt risk-aversion index (1.13).
(b) We also note that the approximation (6.7) is exact if

(i) u is quadratic, or
(ii) u is exponential, Z is normal.

(c) By differentiating, and calculating the Taylor expansion of the classical CE (1.2) of $y \cdot Z$,

$$c_u(y) = u^{-1}Eu(y \cdot Z) \tag{6.9}$$

it follows that $c_u(y)$ is approximated by the right-hand side of (6.7). Thus we have

$$c_u(y) - s_u(y) = \circ(\| y \|^2) \tag{6.10}$$

showing that the CE functionals (6.1) and (6.9) are close for small y.

7 Competitive Firm under Uncertainty

The first application of the RCE is to the classical model studied by Sandmo [41], see also [29, §5.2]. A firm sells its **output** q at a **price** P, which is a RV with a known distribution function and expected value $EP = \mu$. Let $C(q)$ be the **total cost** of producing q, which consists of a **fixed cost** B and a **variable cost** $c(q)$,

$$C(q) = c(q) + B$$

The function $c(\cdot)$ is assumed normalized, increasing and strictly convex,

$$c(0) = 0, \quad c'(q) > 0, \quad c''(q) > 0 \quad \forall q \geq 0 \tag{7.1}$$

The firm has a strictly concave utility function u, i.e.

$$u' > 0, \quad u'' < 0$$

which is normalized so that $u(0) = 0,\ u'(0) = 1$. The objective is to maximize **profit**

$$\pi(q) = qP - c(q) - B$$

which is a RV. The classical CE (1.2) is used is Sandmo's analysis, so that the model studied is

$$\max_{q \geq 0} u^{-1} E u(\pi(q))$$

or equivalently,

$$\max_{q \geq 0} E u(\pi(q)) \tag{7.2}$$

Here we analyze the same model using the RCE. For the sake of comparison with the EU model, we assume that the firm's value-risk function is $u \in \mathcal{U}$, i.e. is a utility. The objective of the firm is therefore

$$\max_{q \geq 0} S_u(\pi(q)) \tag{7.3}$$

Now

$$\begin{aligned} \max_{q \geq 0} S_u(\pi(q)) &= \max_{q \geq 0} S_u(qP - c(q) - B) \\ &= \max_{q \geq 0} \{S_u(qP) - c(q)\} - B \end{aligned}$$

by (2.7). We conclude:

Proposition 7.1 *The optimal production output q^* is independent of the fixed cost B.* □

This result is in sharp contrast to the expected utility model (7.2) where the optimal output $\bar{q}$ depends on the fixed cost B: $\bar{q}$ increases [decreases] with B if the Arrow-Pratt index $r(\cdot)$ is an increasing [decreasing] function; the dependence is ambigious for utilities for which $r(\cdot)$ is not monotone.

Note that the objective function in (7.3) is

$$f(q) = s_u(q) - c(q) \tag{7.4}$$

where $s_u(\cdot)$ is the RCE functional (6.1). The function f is concave by Theorem 6.1 and the assumptions on c. Therefore, the optimal solution q^* of (7.3) is positive if and only if $f'(0) > 0$. By (6.4) $s'(0) = \mu$, so

$$q^* > 0 \quad \text{if and only if} \quad \mu > c'(0) \tag{7.5}$$

in agreement with the expected utility model (7.2). We assume from now on that

$$\mu > c'(0)$$

A central result in the theory of production under uncertainty is that, for the risk-averse firm (i.e. concave utility function), the optimal production under uncertainty is less than the corresponding optimal production q_{cer} under certainty, that is for P a degenerate RV with value μ. We will prove now that the same result holds for the model (7.3). First recall that the optimality condition for q_{cer} is that marginal cost equals marginal revenue

$$c'(q_{\text{cer}}) = \mu \tag{7.6}$$

Proposition 7.2 *$q^* < q_{cer}$ for all $u \in \mathcal{U}$.*

Proof. The optimality condition for q^* is

$$0 = f'(q^*) = s'_u(q^*) - c'(q^*) \tag{7.7}$$

By Theorem 6.1

$$s_u(q) = z(q) + Eu(qP - z(q)) \tag{7.8}$$

where $z(q)$ is a differentiable function, uniquely determined by the equation

$$Eu'(qP - z(q)) = 1 \tag{7.9}$$

By the envelope theorem (Appendix A),

$$s'_u(q) = E\{Pu'(qP - z(q))\} \tag{7.10}$$

and the optimality condition (7.7) becomes

$$EPu'(q^*P - z(q^*)) = c'(q^*) \tag{7.11}$$

Multiplying (7.9) by μ and subtracting from (7.11) we get

$$E(P - \mu)u'(q^*P - z(q^*)) = c'(q^*) - \mu \tag{7.12}$$

or

$$E\{Zh(Z)\} = c'(q^*) - \mu \tag{7.13}$$

where we denote

$$Z := P - \mu \ , \ h(Z) := u'(q^*Z + q^*\mu - z(q^*))$$

Since $u \in \mathcal{U}$, it follows that h is positive and decreasing, and it can then be shown (see e.g. [29, p. 249]) that

$$E\{Zh(Z)\} < h(0)EZ$$

but $EZ = E\{P - \mu\} = 0$, and so, by (7.13),

$$c'(q^*) < \mu$$

and by using (7.6)

$$c'(q^*) < c'(q_{\text{cer}})$$

and since c' is increasing,

$$q^* < q_{\text{cer}} \quad \square$$

7.1 Effect of Profits Tax

Suppose there is a proportional **profits tax** at rate $0 < t < 1$, so that the profit after tax is

$$\pi(q) = (1-t)(qP - C(q))$$

As before, the firm seeks the optimal solution q^* of (7.3), which here becomes

$$\begin{aligned} \max_{q\geq 0} \; S_u(\pi(q)) &= \max_{q\geq 0} \; S_u((1-t)(qP - c(q) - B)) \\ &= \max_{q\geq 0} \; \{S_u((1-t)qP) - (1-t)c(q)\} - (1-t)B \end{aligned}$$

which can be rewritten, using the RCE functional $s_u(\cdot)$ and omitting the constant $(1-t)B$,

$$\max_{q\geq 0} s_u((1-t)q) - (1-t)c(q)$$

Let the optimal solution be $\bar{q} = \bar{q}(t)$. The optimality condition here is

$$(1-t)s_u'((1-t)q) - (1-t)c'(q) = 0$$

giving the identity (in t),

$$s_u'((1-t)\bar{q}(t)) \equiv c'(\bar{q}(t))$$

which, after differentiating (with respect to t),

$$[(1-t)\bar{q}'(t) - \bar{q}(t)]\, s''((1-t)\bar{q}) = \bar{q}'(t)c''(\bar{q})$$

and rearranging terms, gives

$$\bar{q}'(t)\{c''(\bar{q}) - (1-t)s_u''((1-t)\bar{q})\} = -\bar{q}(t)s_u''((1-t)\bar{q}) \tag{7.14}$$

The coefficient of $\bar{q}'(t)$ is positive since $c'' > 0$ and $s_u(\cdot)$ is concave (Theorem 6.1(a)). The right-hand side of (7.14) is positive since $\bar{q} > 0,\ s'' < 0$. Therefore, by (7.14),

$$\bar{q}'(t) > 0$$

and we proved:

Proposition 7.3 *A marginal increase in profit tax causes the firm to increase production.*
□

In the classical expected utility case the effect of taxation depends on third-derivative assumptions; it can be predicted unambigiously [11] only in one of the following cases:

(a) r constant and R increasing,
(b) r decreasing and R increasing,
(c) r decreasing and R constant.

In all these cases, the EU prediction agrees with our prediction in Proposition 7.3.

[11] See Katz's correction [28] to [41].

7.2 Effect of Price Increase

If price were to increase from P to $P + \epsilon$ (ϵ fixed), then the corresponding optimal output $\bar{q}(\epsilon)$ is the solution of

$$\max_{q \geq 0} \{ \ S_u((P + \epsilon)q) - c(q)\} = \max_{q \geq 0} \{ \ s_u(q) + \epsilon q - c(q)\}$$

The optimality condition for $\bar{q}(\epsilon)$ is

$$s'_u(\bar{q}(\epsilon)) + \epsilon = c'(\bar{q}(\epsilon))$$

Differentiating with respect to ϵ we get

$$\bar{q}'(\epsilon)s''_u(\bar{q}(\epsilon)) + 1 = \bar{q}'(\epsilon)c''(\bar{q}(\epsilon))$$

hence

$$\bar{q}'(\epsilon) = \frac{1}{c''(\bar{q}) - s''(\bar{q})} > 0$$

by the convexity of c and the concavity of s_u. We have so proved:

Proposition 7.4 *A marginal increase in selling price causes the firm to increase production.*
□

This highly intuitive result is proved in the expected utility case only under the assumption that $r(\cdot)$ is non-increasing.

7.3 Effect of Futures Price Increase

The RCE criterion was also applied to an extension [19] of Sandmo's model [41], dealing with a firm under price uncertainty and where a futures market exists for the firm's product. In [19, Proposition 5] it is shown that an increase in the current futures price causes a **speculator** or a **hedger** to increase sales, but not so for a **partial hedger**, unless constant absolute risk-aversion is assumed. This pathology is avoided in the RCE model, where the above three types of producers will all increase sales, [43].

8 Investment in One Risky and in One Safe Assets: The Arrow Model

Recall the classical model [3] of investment in a risky/safe pair of assets, concerning an individual with utility $u \in \mathcal{U}$ and **initial wealth** A. The **decision variable** is the amount

a to be invested in the risky asset, so that $m = A - a$ is the amount invested in the safe asset (cash).

The **rate of return** in the risky asset is a RV X.

The **final wealth** of the individual is then

$$Y = A - a + (1 + X)a = A + aX$$

In [3] the model is analyzed via the maximal EU principle, so the optimal investment a^* is the solution of

$$\max_{0 \le a \le A} Eu(A + aX) \tag{8.1}$$

or equivalently

$$\max_{0 \le a \le A} u^{-1} Eu(A + aX)$$

Some of the important results in [3] are:

(I1) $a^* > 0$ if and only if $EX > 0$.

(I2) a^* increases with wealth (i.e. $\frac{da^*}{dA} \ge 0$) if the absolute risk aversion index $r(\cdot)$ is decreasing.

(I3) The wealth elasticity of the demand for cash balance (investment in the safe asset)

$$\frac{Em}{EA} := \frac{dm/dA}{m/A} \quad \text{is at least one} \tag{8.2}$$

if the relative risk-aversion index

$$R(z) = -z \frac{u''(z)}{u'(z)} \quad \text{is increasing} \tag{8.3}$$

Arrow [3] postulated that reasonable utility functions **should** satisfy (8.3), since the empirical evidence for (8.2) is strong, see the references in [3, p. 103].

We analyze this investment problem using the RCE criterion, i.e.

$$\max_{0 \le a \le A} S_u(A + aX) \tag{8.4}$$

where again we assume that the investor's value-risk function is $u \in \mathcal{U}$. The optimization problem (8.4) is, by (2.7), equivalent to

$$\max_{0 \le a \le A} S_u(aX) + A$$

Let a^* be the optimal solution. Using the RCE functional $s_u(\cdot)$, a^* is in fact the solution of

$$\max_{0 \leq a \leq A} s_u(a) \tag{8.5}$$

Now, since $s_u(\cdot)$ is concave

$$a^* > 0 \quad \text{if and only if} \quad s'_u(0) > 0$$

but by (6.4) $s'(0) = EX$, and we recover the result (**I1**).

Assuming (as in [3]) an inner optimal solution (**diversification**)

$$0 < a^* < A \tag{8.6}$$

we conclude here, in contrast to (**I2**), that

$$\frac{da^*}{dA} = 0 \tag{8.7}$$

i.e. the optimal investment is independent of wealth [12].

An immediate consequence of (8.7) is

$$\frac{Em}{EA} > 1 \quad \forall\, u \in \mathcal{U}$$

indeed

$$\frac{Em}{EA} = \frac{A}{m}\frac{dm}{dA} = \frac{A}{A - a^*}\frac{d(A - a^*)}{dA} = \frac{A}{A - a^*}(1 - \frac{da^*}{dA}) = \frac{A}{A - a^*} > 1$$

proving (8.2) for **all** risk-averse investors. Thus, in the RCE model, there is no need for the controversial postulate (8.3).

The quadratic utility (2.33)

$$u(z) = z - \frac{1}{2}z^2 \quad z \leq 1$$

violates both of Arrow's postulates (r decreasing, R increasing), and is consequently "banned" from the EU model. In the RCE model, on the other hand, a quadratic value-risk function is acceptable[13]. For this function the optimal investment a^* is the optimal solution of

$$\max_{0 \leq a \leq A} \{s_u(a) = \mu a - \frac{1}{2}\sigma^2 a^2\}$$

[12] However, initial wealth will in general determine when divesification will be optimal, i.e. when (8.6) will hold.

[13] Assuming $0 \leq X \leq 1$.

where $\mu = EX,\ \sigma^2 = \mathrm{Var}(X)$. Therefore

$$a^* = \begin{cases} \mu/\sigma^2 & \text{if } 0 < \mu/\sigma^2 < A \\ A & \text{if } \mu/\sigma^2 \geq A \end{cases}$$

showing that, for the full range of A values, $a^*(A)$ is non-decreasing, in agreement with (**I2**). Moreover, if diversification is optimal, then

$$\frac{Em}{EA} = \frac{A}{A - \mu/\sigma^2} > 1$$

Following [3] we consider the effects on optimal investment, of shifts in the RV X. Let h be the **shift parameter**, and assume that the **shifted** RV $X(h)$ is a differentiable function of h, with $X(0) = X$. Examples are:

$$\begin{array}{ll} X(h) = X + h & (\textbf{additive shift}), \\ X(h) = (1+h)X & (\textbf{multiplicative shift}). \end{array}$$

For the shifted problem, the objective is

$$\max_{0 \leq a \leq A} S_u(aX(h)) \tag{8.8}$$

Let $a(h)$ be the optimal solution of (8.8), in particular $a(0) = a^*$. Now

$$S_u(aX(h)) = \xi(a) + Eu(aX(h) - \xi(a)) \tag{8.9}$$

where $\xi(a)$ is the unique solution of

$$Eu'(aX(h) - \xi(a)) = 1 \tag{8.10}$$

The optimality condition for $a(h)$ is

$$\frac{d}{da}\{\xi(a) + Eu(aX(h) - \xi(a))\} = 0$$

which gives (using (8.10)) the following identities in h

$$E\{X(h)u'(a(h)X(h) - \xi(a(h)))\} \equiv 0 \tag{8.11}$$

$$E\{u'(a(h)X(h) - \xi(a(h)))\} \equiv 1 \tag{8.12}$$

Differentiating (8.11) with respect to h we get, denoting $Z = aX(h) - \xi(a(h))$,

$$\dot{a}(h)E\{u''(Z)X(X - \xi'(a(h))\} = E\{\dot{X}(h)\,[u'(Z) + a(h)X(h)u''(Z)]\} \tag{8.13}$$

where $\dot{a}(h) = \frac{d}{dh}a(h)$ and similarly for $\dot{X}(h)$.

The second order optimality condition for $a(h)$, $\frac{d^2}{da^2}S_u(aX(h)) \le 0$, is here

$$Eu''(Z)X(X - \xi'(a(h))) \ge 0$$

hence, by (8.13),

$$\text{sign of } \dot{a}(h) = \text{sign of } E\{\dot{X}(h)\,[u'(Z) + aXu''(Z)]\}$$

exactly the same condition for the sign of $\frac{d}{dh}a(h)$ as in [3, p. 105, eq. (18)]. Therefore, the conclusions of the EU model are also valid for the RCE model. In particular:

Proposition 8.1 *As a function of the shift parameter h,*
$a(h)$ increases for additive shift,
$a(h)$ decreases for multiplicative shift.

These results are illustrated for the quadratic value-risk function. There

$$a^* = \frac{EX}{\mathrm{Var}(X)}$$

and

$$\begin{aligned} a(h) &= a^* + \frac{h}{\mathrm{Var}(X)} \quad \text{for an additive shift} \\ a(h) &= \frac{1}{1+h}a^* \quad \text{for a multiplicative shift} \end{aligned} \tag{8.14}$$

In fact, (8.14) holds for arbitrary $u \in \mathcal{U}$, a result proved in [45] for the EU model.

Proposition 8.2 *If a^* is the demand for the risky asset when the return is the RV X, then $a(h) = a^*/1 + h$ is the demand when the return is $(1 + h)X$.*

Proof. The optimality condition for a^* is

$$E\{u'(a^*X - \xi^*)X\} = 0 \tag{8.15}$$

where ξ^* is the unique solution of

$$Eu'(a^*X - \xi^*) = 1 \tag{8.16}$$

The optimality conditions for $a(h)$ are given by (8.11), (8.12). Now, for $a(h) = \frac{1}{1+h}a^*$,

$$a(h)X(h) = a^*X \tag{8.17}$$

and it follows, by comparing (8.12) with (8.16), that

$$\xi(a(h)) = \xi^*$$

Substituting this in (8.11) and using (8.17), we see that (8.11) is equivalent to (8.16), and that $a(h) = a^*/1 + h$ indeed satisfies the optimality conditions (8.11), (8.12). □

9 Investment in a Risky/Safe Pair of Assets: An Extension

We study the model discussed in [12] and [22], which is an extension of the model in Section 8. The analysis applies to a fixed time interval, say a year. An investor allocates a proportion $0 \leq k \leq 1$ of his investment capital W_0 to a risky asset, and proportion $1-k$ of W_0 to a safe asset where the total annual return per dollar invested is $\tau \geq 1$. The total annual return t per dollar invested in the risky asset, is a nonnegative RV. The investor's **total annual return** is

$$kW_0 t + (1-k)W_0 \tau$$

and for a utility function u, the optimal allocation k^* is the solution of

$$\max_{0 \leq k \leq 1} Eu(kW_0 t + (1-k)W_0 \tau) \tag{9.1}$$

The model of §5, is a special case with $W_0 = A,\ t = 1 + X,\ kW_0 = a,\ \tau = 1$.

It is assumed in [12], [22] that $u' > 0$ and $u'' < 0$, thus we assume without loss of generality that $u \in \mathcal{U}$.

One of the main issues in [22] is the effect of an increase in the safe asset return τ on the optimal allocation. The following are proved:

(**F1**) An investor maximizing expected utility will diversify (invest a positive amount in each of the assets) if and only if

$$\frac{Etu'(W_0 t)}{Eu'(W_0 t)} < \tau < E(t) \tag{9.2}$$

(**F2**) Given (9.2) he will **increase** the proportion invested in the safe asset when τ increases if either

(a) the absolute risk aversion index $r(\cdot)$ is non-decreasing, or

(b) the relative risk aversion index $R(\cdot)$ is at most 1.

The same model is now analyzed using the RCE approach, i.e. with the objective

$$\max_{0 \leq k \leq 1} S_u(kW_0 t + (1-k)W_0 \tau)$$

where u denotes the investor's value risk function, assumed in $\mathcal{U}$. Using (2.7) and the definition (6.1), the objective becomes

$$\max_{0 \leq k \leq 1} \{(1-k)W_0 \tau + s_u(W_0 k)\}) \tag{9.3}$$

The following proposition, proved in Appendix C, gives the analogs of results (**F1**, (**F2**) in the RCE model.

Proposition 9.1 *(a) The RCE maximizing investor will diversify if and only if*

$$Etu'(W_0t - \eta) < \tau < E(t) \tag{9.4}$$

where η is the unique solution of

$$Eu'(W_0t - \eta) = 1 \tag{9.5}$$

(b) Given (9.4), he will **increase** *the proportion invested in the safe asset when τ* **increases**. □

Comparing part (b) with (**F2**), we see that plausible behavior (k^* increases with τ) holds in the RCE model for all $u \in \mathcal{U}$, but in the EU model only for a restricted class of utilities.

We illustrate Proposition 9.1 in the case of the quadratic value-risk function (2.20). Here the optimal proportion invested in the risky asset is:

$$k^* = \begin{cases} 0 & \text{if } \tau > E(t) \\ \frac{E(t)-\tau}{W_0\sigma^2} & \text{if } E(t) - W_0\sigma^2 \le \tau \le E(t) \\ 1 & \text{if } E(t) - W_0\sigma^2 > \tau \end{cases} \tag{9.6}$$

where σ^2 is the variance of t. Thus k^* is increasing in $E(t)$, decreasing with σ^2 and decreasing with τ (so that, the proportion $1 - k^*$ invested in the safe asset is increasing with safe asset return τ). These are reasonable reactions of a risk-averse investor.

We also see from (9.6) that k^* **decreases** when the investment capital W_0 **increases**. This result holds for arbitrary $u \in \mathcal{U}$, see the next proposition (proved in Appendix B). In the EU model, the effect of W_0 on k^* depends on the relative risk-aversion index, see [12].

Proposition 9.2 *If the investment capital increases, then the RCE-maximizing investor will increase the proportion invested in the safe asset.* □.

Following the analysis in [3] and § 8, we consider now the elasticity of cash-balance (with respect to W_0). Here the cash balance (the amount invested in the safe asset) is

$$m = (1 - k^*)W_0$$

and the elasticity in question is $\frac{Em}{EW_0}$.

Proposition 9.3 *For every RCE-maximizing investor with $u \in \mathcal{U}$,*

$$\frac{Em}{EW_0} \ge 1$$

Proof.

$$\frac{Em}{EW_0} = \frac{dm/dW_0}{m/W_0} = \frac{1 - k^*(W_0) - W_0 \frac{dk^*(W_0)}{dW_0}}{1 - k^*(W_0)}$$

hence

$$\frac{Em}{EW_0} \geq 1 \quad \text{if and only if} \quad \frac{dk^*(W_0)}{dW_0} \leq 0 \tag{9.7}$$

and the proof is completed by Proposition 9.2. □

The equivalence in (9.7) shows that the empirically observed fact that $Em/EW_0 \geq 1$ can be explained **only** by the result established in Proposition 9.2 that $dk^*/dW_0 \leq 0$, a result which is not necessarily true for many utilities in the EU analysis.

10 Optimal Insurance Coverage

Insurance models with two states of nature were studied in [18], [29] and the references therein. In this section we solve an insurance model with n states of nature, and give an **explicit formula for the optimal allocation of the insurance budget**, thus illustrating the analytic power of the RCE theory.

10.1 Description of the Model

The elements of the model are:

n	**states of nature**
$\mathbf{p}$	$= (p_1, \ldots, p_n)$ their **probabilities**
$\bar{q}_i$	= **premium** for 1$ coverage in state i, $\bar{q}_i > 0$
$\bar{B}$	= **insurance budget**
q_i	$= \bar{q}_i / \sum_{j=1}^n \bar{q}_j$ = **normalized premium**
B	$= \bar{B} / \sum_{j=1}^n \bar{q}_j$ = **normalized budget**
x_i	= **income** in state i
$\mathbf{x}$	$= (x_1, \ldots, x_n)$ the **decision variable**

The **budget constraint** is

$$\sum_{i=1}^n q_i \, x_i = B \tag{10.1}$$

We allow negative values for some x_i's, i.e. we allow a person to "insure" and "gamble" at the same time, e.g. [18, p. 627].

For the RCE maximizer with value-risk function v, the **optimal value of the insurance plan** is

$$\begin{aligned} I^* &= \max_{\mathbf{x}} \{S_v([\mathbf{x},\mathbf{p}]) : \sum_{i=1}^n q_i\, x_i = B\} \\ &= \max_{\mathbf{x},\ \sum q_i x_i = B} \ \max_z \{z + \sum_{i=1}^n p_i\, v(x_i - z)\} \qquad (10.2) \\ &= S_v([\mathbf{x}^*,\mathbf{p}]) \end{aligned}$$

where $\mathbf{x}^* = (x_i^*)$ is the **optimal insurance coverage**.

10.2 The Solution

Theorem 10.1 *The optimal insurance coverage is*

$$x_i^* = B + \phi(\frac{q_i}{p_i}) - \sum_{j=1}^n q_j\, \phi(\frac{q_j}{p_j}) \qquad (10.3)$$

where

$$\phi = (v')^{-1} \qquad (10.4)$$

Moreover, the optimal value of the insurance plan is

$$I^* = B - \sum q_i\, \phi(\frac{q_i}{p_i}) + \sum p_i\, v(\phi(\frac{q_i}{p_i})) \qquad (10.5)$$

Proof. The problem (10.2) is maximizing a concave function subject to linear constraints. Since the Kuhn-Tucker conditions are necessary and sufficient

$$I^* = \min_\lambda \ \max_{\mathbf{x}} \ L(\mathbf{x}, z, \lambda) \qquad (10.6)$$

where L is the **Lagrangian**

$$L(\mathbf{x}, z, \lambda) = z + \sum p_i\, v(x_i - z) + \lambda(B - \sum q_i\, x_i) \qquad (10.7)$$

The optimal $\mathbf{x}^*, z^*, \lambda^*$ satisfy

$$\begin{aligned} \frac{\partial L}{\partial z} &= 1 - \sum p_i\, v'(x_i^* - z^*) = 0 \qquad (10.8) \\ \frac{\partial L}{\partial x_i} &= p_i\, v'(x_i^* - z^*) - \lambda^* q_i = 0, \quad (i = 1, \ldots, n) \qquad (10.9) \\ \frac{\partial L}{\partial \lambda} &= B - \sum q_i\, x_i^* = 0 \qquad (10.10) \end{aligned}$$

From (10.9) and (10.8) we get

$$\lambda^* = \frac{1}{\sum q_i} = 1$$

and consequently

$$v'(x_i^* - z^*) = \frac{q_i}{p_i}$$

Since v' is monotone decreasing (v is strictly concave) we write, using (10.4)

$$x_i^* - z^* = \phi(\frac{q_i}{p_i}), \quad (i = 1, \ldots, n) \tag{10.11}$$

Multiplying (10.11) by q_i and summing we get

$$\begin{aligned} \sum q_i\,(x_i^* - z^*) &= \sum q_i\,\phi(\frac{q_i}{p_i}) \\ \therefore\ B - z^* &= \sum q_i\,\phi(\frac{q_i}{p_i}) \\ \therefore\ z^* &= B - \sum q_i\,\phi(\frac{q_i}{p_i})) \end{aligned} \tag{10.12}$$

which is compared with (10.11) to give (10.3). Finally,

$$I^* = z^* + \sum p_i\,v(x_i^* - z^*)$$

and (10.5) follows by (10.11) and (10.12). □ In the above model, the price of insurance is **actuarially fair** if

$$q_i = p_i \qquad (i = 1, \ldots, n)$$

i.e. if the normalized premiums agree with the probabilities.

For actuarially fair premiums we get from (10.3), using that $v'(0) = 1$ implies $\phi(1) = 0$,

$$x_i^* = B \qquad (i = 1, \ldots, n)$$

i.e. the individual is indifferent between the occurrence of states $i = 1, \ldots, n$.

10.3 Special Case: Two States of Nature

We translate the results of Theorem 10.1 to the special case of two states, as given in [18], [29, §3].

Consider insurance against a single disaster. Specifically, let there be two states of nature:

State	Disaster	Probability
1	occurs	p
2	does not occur	$1 - p$

The **final wealth** is a RV

$$X(s) = \begin{cases} y + s & \text{with probability} \quad p \qquad \text{(State 1)} \\ W - \pi s & \text{with probability} \quad 1-p \quad \text{(State 2)} \end{cases} \tag{10.13}$$

where

W	**initial wealth**
s	**insurance coverage**
π	**premium**
y	**income in disaster state**

In [29, §3] this model is treated using the EU model

$$\max_s \; E\, u(X(s))$$

obtaining first order optimality conditions, comparative statics, and, in the case of exponential utility

$$u_\lambda(x) = \frac{1}{\lambda}\,(1 - e^{-\lambda x}), \tag{10.14}$$

the explicit solution

$$s^* = \frac{W - y}{\pi + 1} - \frac{1}{\lambda(\pi + 1)} \;\log\left(\frac{\pi}{\frac{p}{1-p}}\right) \tag{10.15}$$

To apply Theorem 10.1 here we write the incomes in the two states and their probabilities

$$\begin{aligned} x_1 &= y + s, \quad & p_1 &= p \\ x_2 &= W - \pi s, \quad & p_2 &= 1 - p \end{aligned}$$

We define the normalized premiums

$$q_1 \;:=\; \frac{\pi}{1 + \pi} \tag{10.16}$$

$$q_2 \;:=\; 1 - q_1 = \frac{1}{1 + \pi} \tag{10.17}$$

The insurance budget (10.1) is implicit in this model. The budget B can be computed by

$$\begin{aligned} q_1 x_1 + q_2 x_2 &= q_1\,(y + s) + q_2\,(W - \pi s) \\ &= q_1\, y + q_2\, W + s(q_1 - q_2 \pi) \end{aligned}$$

but $q_1 - q_2\pi = 0$ by (10.16) and (10.17), and therefore the budget is

$$B = q_1\, y + q_2\, W \tag{10.18}$$

Now, from (10.3),

$$\begin{aligned} x_1^* &= B + (1-q_1)\phi(\frac{q_1}{p_1}) - q_2\phi(\frac{q_2}{p_2}) \\ &= q_1 y + q_2 W + q_2\left[\phi(\frac{q_1}{p_1}) - \phi(\frac{q_2}{p_2})\right] \end{aligned}$$

and therefore the optimal coverage is

$$\begin{aligned} s^* &= x_1^* - y \\ &= q_2\left[W - y + \phi(\frac{q_1}{p_1}) - \phi(\frac{q_2}{p_2})\right] \\ &= \frac{1}{1+\pi}\left[W - y + \phi\left(\frac{\pi}{(1+\pi)p}\right) - \phi\left(\frac{1}{(1+\pi)(1-p)}\right)\right] \qquad (10.19) \end{aligned}$$

Note that in this two state model, actuarlially fair insurance means $\pi = \frac{p}{1-p}$, in which case $s^* = \frac{1}{1+\pi}(W-y)$

For the utility u_λ of (10.14), we get by (10.4)

$$\phi(t) = (u_\lambda')^{-1}(t) = -\frac{1}{\lambda}\ \log t$$

which, substituted in (10.19), gives the formula (10.15) of s^*.

10.4 Related Work

The RCE criterion was applied in [43] for studying the existence of optimal insurance contracts. Two fundamental results of Arrow [3] concerning

- the optimality of 100% coverage (above deductibles) for a risk-averse buyer of insurance, and
- the Pareto optimality of **coinsurance** for risk-averse insurer and buyer of insurance,

were shown to hold as well in the RCE model.

11 Why Does the RCE Work ?

The models discussed above (§§ 7-10), give sufficient data for comparing the predictive powers of the RCE theory and the EU theory. We saw that the plausible predictions of EU are shared by RCE, and that the RCE criterion is a simpler and a more powerful analytical tool, e.g. § 10.2 where it gives an explicit solution for all risk-averse DM's, while in general the EU model can only provide comparative statics. Also the RCE predictions hold for all risk-averse DM's, while in the EU model risk-aversion does not suffice and, in order to avoid implausible predictions, restrictions (occassionally severe) must be imposed on the DM's subjective preference.

The simplicity of the RCE criterion can be explained at the technical level. Shift additivity makes risky choices independent of constant factors (fixed costs, initial wealth), and by using the envelope theorem, comparative statics are free of certain ungainly derivatives. Such conveniences are in general unavailable to the EU maximizer.

This however is not the whole story. The main advantage of the RCE theory, at the fundamental level of modelling choice under risk, is that its risk aversion is of the "right kind" from the start, without a need for qualifiers such as the Arrow-Pratt indices. Indeed, in the EU theory, behavior under uncertainty is analyzed in terms of the Arrow-Pratt indices $r(\cdot)$ and $R(\cdot)$. The typical postulates are

(A1) $r(w) = -\frac{u''(w)}{u'(w)}$ is a non-increasing function of w

(A2) $R(w) = -\frac{w\,u''(w)}{u'(w)}$ is a non-decreasing function of w

The economic literature contains several alternative formulations. In particular ([17, pp. 352-354] and [32, pp. 20-21]) (**A1**) is equivalent to

(B1) If $u(w_1 + c_1) = E\,u(w_1 + X)$ and $u(w_2 + c_2) = E\,u(w_2 + X)$ for $w_1 < w_2$, then $c_1 \leq c_2$

and (**A2**) is equivalent to

(B2) If $u(w_1 c_1) = E\,u(w_1 X)$ and $u(w_2 c_2) = E\,u(w_2 X)$ for $w_1 < w_2$, then $c_1 \geq c_2$

Properties (**B1**), (**B2**) can be expressed directly in terms of the classical CE

$$C_u(X) = u^{-1} E\,u(X)$$

Indeed, (**B1**) is equivalent to

(C1) $C_u(X + w) - w$ is a non-decreasing function of w

and (**B2**) is equivalent to

(C2) $\frac{1}{w}\,C_u(wX)$ is a non-increasing function of w

Consider now the RCE $S_v(X)$. The properties corresponding to (**C1**), (**C2**) are

(S1) $S_v(X + w) - w$ is a non-decreasing function of w

(S2) $\frac{1}{w} S_v(wX)$ is a non-increasing function of w

Now (**S1**) holds trivially, for any function $v : \mathbb{R} \to \mathbb{R}$, by the shift additivity of the RCE, Theorem 2.1(a). In fact, $S_v(X + w) - w$ is $S_v(X)$, a constant in w. Moreover, (**S2**) is the subhomogeneity property, proved in Theorem 2.1(c) for all $v \in \mathcal{U}$ [14].

Therefore, in the RCE theory the properties (**S1**) and (**S2**) hold for all value-risk function $v \in \mathcal{U}$, i.e. for all strongly risk-averse DM's. In the EU theory, risk-aversion coincides with strong risk-aversion (see § 4), but the properties (**A1**) and (**A2**) (which correspond to (**S1**) and (**S2**)) hold only for a restricted class of utilities.

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Appendix A. The Envelope Theorem

This result is used repeatedly in this paper. For convenience we cite an elementary version here. See [44] and [40] for details and examples.

Theorem A.1 (The Envelope Theorem). *Consider the unconstrained maximization*

$$\text{maximize}_z \; y = f(z,\ q)$$

Let $z^*(q)$ *be the maximizer, for given* q*, and let*

$$y^* = f(z^*(q),\ q) = \phi(q)$$

Then

$$\phi'(q) = \frac{\partial f(z^*(q),\ q)}{\partial q} \quad \square$$

Appendix B. Proof of Theorem 6.1

(a) By (6.1) and (1.6), $s_u(\cdot)$ is the pointwise supremum of concave functionals, hence concave. The rest of (a) is proved as in Lemma 2.1.

(b) For $\mathbf{y} = \mathbf{0}$, (6.3) gives

$$Eu'(-z_S(\mathbf{0})) = 1$$

or $u'(-z_S(\mathbf{0})) = 1$, proving that $z_S(\mathbf{0}) = \mathbf{0}$. From (6.2) it follows then that $s_u(\mathbf{0}) = 0$.

Differentiating (6.3) with respect to $\mathbf{y}$ gives

$$Eu''(\mathbf{y} \cdot \mathbf{Z} - z_S(\mathbf{y}))(\mathbf{Z} - \nabla z_S(\mathbf{y})) = 0$$

which at $\mathbf{y} = \mathbf{0}$ becomes

$$u''(\mathbf{0})(E\mathbf{Z} - \nabla z_S(\mathbf{0})) = 0$$

proving that $\nabla z_S(\mathbf{0}) = \mu$. Then, by differentiating (6.2) at $\mathbf{y} = \mathbf{0}$ we get $\nabla s_u(\mathbf{0}) = \mathbf{0}$.

The expressions for $\nabla^2 z_S(\mathbf{0})$ and $\nabla^2 s_u(\mathbf{0})$ follow similarly by differentiating (6.3) and (6.2) twice at $\mathbf{y} = \mathbf{0}$. $\square$

Appendix C. Results from Section 9

Proof of Proposition 9.1.

(a) The objective function in (9.3)

$$h(k) = (1-k)W_0\tau + s_u(W_0 k)$$

is concave, by Theorem 6.1(a). Hence, the optimal solution x^* is an inner solution, i.e. $0 < k^* < 1$ if and only if

$$h'(0) > 0 \quad \text{and} \quad h'(1) < 0 \tag{C.1}$$

Now

$$h'(k) = -W_0\tau + W_0 s'_u(W_0 k) \tag{C.2}$$

which becomes, upon substitution of the computed expression for $s'_u(\cdot)$,

$$h'(k) = -W_0\tau + W_0 Etu'(W_0 kt - \eta(W_0 k)) \tag{C.3}$$

where $\eta(q)$ is the unique solution of

$$Eu'(qt - \eta) = 1 \tag{C.4}$$

Therefore

$$\begin{aligned} h'(0) &= -W_0\tau + W_0 E(t) \\ h'(1) &= -W_0\tau + W_0 Etu'(W_0 - \eta(W_0)) \end{aligned}$$

and (C.1) is equivalent to (9.4).
(b) Let $k(\tau)$ be the optimal solution of (9.3) for given τ, i.e. $h'(k(\tau)) = 0$, or using (C.3),

$$-\tau + E\{tu'(W_0 k(\tau)t - \eta(W_0 k(\tau))\} \equiv 0$$

Differentiating this identity (in τ) with respect to τ, we obtain

$$-1 + E\{tW_0(k'(\tau)t - k'(\tau)\eta'(W_0 k(\tau))u''\} = 0$$

or

$$k'(\tau)W_0 Et(t - \eta')u'' = 1 \tag{C.5}$$

Now, the second order condition for the maximality of $k(\tau)$ is

$$0 > h''(k) = W_0 E\{tW_0(t - \eta')u''\} \tag{C.6}$$

Therefore, $k'(\tau)$ is multiplied in (C.5) by a negative number, and consequently

$$k'(\tau) < 0$$

proving that $k(\tau)$ $[1 - k(\tau)]$, the proportion invested in the risky [safe] asset, is a decreasing [increasing] function of τ, the safe asset return. □

Proof of Proposition 9.2. Let $k = k(W_0)$ be the optimal solution of (9.3), i.e. $h'(k(W_0)) = 0$, or using (C.3)

$$-\tau + E\{tu'(W_0 k(W_0)t - \eta(W_0 k(W_0))\} \equiv 0 \tag{C.7}$$

Differentiating this identity (in W_0) we get

$$Et\left[k(W_0) + W_0 k'(W_0)\right]\left[t - \eta'(W_0 k(W_0))\right] u'' = 0$$

or

$$k' W_0 Et(t - \eta')u'' = -EtkU'' \tag{C.8}$$

By the second order optimality condition (C.6) it follows that, in (C.8), k' is multiplied by a negative number. Since the right hand side of (C.8) is positive ($t, k > 0,\ u'' < 0$), it follows that

$$k'(W_0) < 0 \quad \square$$