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DUALITY AND EQUILIBRIUM PRICES IN ECONOMICS OF UNCERTAINTY

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ABSTRACT. A random variable (RV) $\mathbf{X}$ is given a **minimum selling price**

(S) $$S_U(\mathbf{X}) \;:=\; \sup_x \{x + EU(\mathbf{X} - x)\}$$

and a **maximum buying price**

(B) $$B_P(\mathbf{X}) \;:=\; \inf_x \{x + EP(\mathbf{X} - x)\}$$

where $U(\cdot)$ and $P(\cdot)$ are appropriate functions. These prices are derived from considerations of **stochastic optimization with recourse**, and are called **recourse certainty equivalents** (RCE's) of $\mathbf{X}$. Both RCE's compute the "value" of a RV as an optimization problem, and both problems (S) and (B) have meaningful dual problems, stated in terms of the **Csiszár** ϕ**–divergence**

$$I_\phi(\mathbf{p}, \mathbf{q}) \;:=\; \sum_{i=1}^n q_i\, \phi\left(\frac{p_i}{q_i}\right)$$

a generalized entropy function, measuring the distance between RV's with probability vectors $\mathbf{p}$ and $\mathbf{q}$.

The RCE S_U was introduced in [5], studied further in [3], [4] and [6], and applied to production, investment and insurance problems. Here we study the RCE B_P, and apply it to problems of **inventory control** (where the attitude towards risk determines the stock levels and order sizes) and **optimal insurance coverage**, a problem stated as a game between the insurance company (setting the premiums) and the buyer of insurance, maximizing the RCE of his coverage.

1. Introduction

Decision making under uncertainty presupposes the ability to rank **random variables** (RV's for short), i.e. a **complete order** on the space of RV's. Let

$\mathbf{X} \succeq \mathbf{Y}$ denote $\mathbf{X}$ **preferred** to $\mathbf{Y}$,
$\mathbf{X} \sim \mathbf{Y}$ denote $\mathbf{X} \succeq \mathbf{Y}$ and $\mathbf{Y} \succeq \mathbf{X}$, i.e. **indifference** between $\mathbf{X}$ and $\mathbf{Y}$, and
$\mathbf{X} \succ \mathbf{Y}$ denote $\mathbf{X}$ **strictly preferred** to $\mathbf{Y}$, i.e. $\mathbf{X} \succeq \mathbf{Y}$ but not $\mathbf{Y} \succeq \mathbf{X}$.

A **decision maker** (DM for short), with a preference order $\succeq$, is

$$\textbf{risk} \left\{ \begin{array}{c} \textbf{averse} \\ \textbf{neutral} \\ \textbf{taker} \end{array} \right. \quad \text{if} \quad E\mathbf{X} \left\{ \begin{array}{c} \succ \\ \sim \\ \prec \end{array} \right. \mathbf{X}\,, \quad \text{for all RV's } \mathbf{X}\,, \tag{1}$$

where $E\mathbf{X}$ is the expected value[1] of $\mathbf{X}$. Risk–aversion means that no random variable is preferred to its expected value.

A **certainty equivalent** (CE for short) is defined here as a real–valued function (**criterion**) on the space of RV's, defining an order "$\succeq$" as follows

$$\mathbf{X} \succeq \mathbf{Y} \iff \mathrm{CE}(\mathbf{X}) \geq \mathrm{CE}(\mathbf{Y})\;, \text{ and} \tag{2a}$$

$$\mathbf{X} \sim \mathrm{CE}(\mathbf{X})\;, \quad \text{for all RV's } \mathbf{X}\,, \mathbf{Y}\;. \tag{2b}$$

The **expected utility** (EU) criterion of classical economics is a criterion inducing an order,

$$\mathbf{X} \succeq \mathbf{Y} \iff Eu(\mathbf{X}) \geq Eu(\mathbf{Y}) \tag{3}$$

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[1]Since $\succeq$ is a relation between RV's, we interpret $E\mathbf{X}$ as a degenerate RV.

where $u(\cdot)$ is the DM's **utility function**. The EU criterion does not in general satisfy (2b), and is therefore not a CE[2].

In view of (2b), a CE is a **value** of the RV. A value can arise in two different contexts, selling or buying, illustrated by the following two parables.

Parable 1 (Selling a RV)**.** You expect to inherit next year an unknown monetary amount $\mathbf{Z}$, considered as a RV. The unknown $\mathbf{Z}$ may even be negative[3]. You take now a loan of z against the anticipated random $\mathbf{Z}$. The bank will lend you any amount[4].

Next year, after the RV $\mathbf{Z}$ is realized, there is an adjustment, and you get from the bank an additional amount of $U(\mathbf{Z}-z)$, which is positive if $\mathbf{Z} > z$, and zero if $\mathbf{Z} = z$ (we ignore the interest rate). If $\mathbf{Z} < z$ (loan too large), the amount $U(\mathbf{Z}-z)$ is negative, i.e. you have to repay the bank. Because of this you may consider a negative loan ($z < 0$), if you expect $\mathbf{Z}$ to be negative, and you prefer to repay part of it in advance.

Following [3] we call $U(\cdot)$ the DM's **value–risk function**.

How much is the RV $\mathbf{Z}$ worth (for you) now? A natural answer is

$$\sup_z \{z + EU(\mathbf{Z}-z)\} \tag{4}$$

the maximum of the sum of this year's income z and the (expected) value of next year's income $U(\mathbf{Z}-z)$. We call (4) the **minimum selling price** of $\mathbf{Z}$: you should sell [not sell] $\mathbf{Z}$ to anyone offering you more [less] than (4). At the exact price (4) you should be indifferent between selling or not. □

Parable 2 (Buying a RV)**.** You are in business for yourself, and you have to pay tax, in advance, on next year's income[5]. That income is uncertain, and so is its tax liability, which we consider a RV $\mathbf{Z}$. You pay now the amount z against the liability $\mathbf{Z}$. When the RV $\mathbf{Z}$ is realized, you pay the government the amount $P(\mathbf{Z}-z)$ where $P(\cdot)$ is a **penalty function** (the function $P(\cdot)$ is known, in fact the government publishes tables of P). A negative value of $P(\mathbf{Z}-z)$ means refund for over–payment of tax.

By paying for the tax liability $\mathbf{Z}$ you are in fact buying it from the government, on an installment plan: pay z now, and $P(\mathbf{Z}-z)$ next year.

How much is the liability $\mathbf{Z}$ worth now? This question may arise if it is possible to buy $\mathbf{Z}$ by an alternative payment plan: a single payment p now. The answer analogous to (4) is

$$\inf_z \{z + EP(\mathbf{Z}-z)\} \tag{5}$$

the minimum of the sum of the current payment and the expected balance for next year. We call (5) the **maximum buying price** of $\mathbf{Z}$. If p is less than the value (5), you should pay p and settle the issue now. If p is greater than (5), stay with the installment plan. If p is equal to (5), there is no reason to prefer either plan. □

Both the values (4) and (5) can be justified using **stochastic programming with recourse**, ([11], [12]). The minimum selling price (4) is thus the optimal value of the stochastic program

$$\sup \{z : z \leq \mathbf{Z}\} \tag{6}$$

with a random RHS $\mathbf{Z}$, see [3]. Analogously, the maximum buying price is the optimal value of the stochastic program

$$\inf \{z : z \geq \mathbf{Z}\} \tag{7}$$

We denote these prices by

$$S_U(\mathbf{Z}) := \sup_z \{z + EU(\mathbf{Z}-z)\} \tag{8a}$$

$$B_P(\mathbf{Z}) := \inf_z \{z + EP(\mathbf{Z}-z)\} \tag{8b}$$

and use the name **recourse certainty equivalent** (RCE) for both, if the context (selling or buying) is clear.

The RCE (8a) was introduced in [5], studied further in [3], [4] and [6], and applied to production, investment and insurance problems. The paper [3] also makes a case for the RCE (8a) as an alternative to the EU model.

[2] If $\mathbf{X}$ and $u(\mathbf{X})$ have different units, (2b) is not even defined. A CE based on the EU criterion is $u^{-1}Eu(\mathbf{X})$.

[3] If the lawyers and government get there before you.

[4] This is how some countries, and banks, got into trouble in the 70's.

[5] If you are an employee, the tax is withheld from your pay, and this parable may be lost on you.

This paper is a sequel of [3] and [4]. It studies the RCE (8b), omitting proofs which are direct analogs of the corresponding proofs in [3] and [4].

In § 2 we study assumptions on the value–risk functions U and the penalty functions P used throughout this paper. Properties of the RCE B_P are collected in §§ 3–4. Section 5 studies a typical inventory control problem, where the optimal stock levels and order sizes are chosen so as to reflect the risk–attitude of the DM. Extremal principles for the RCE's and the Csiszár ϕ–divergence are given in §§ 6–8.

Section 9 highlights the role of S_U and B_P as equilibrium prices, against perturbations of the values of the RV's in question, and normalized prices. This interpretation is echoed in § 10 where the problem of optimal insurance coverage is solved for the insurance company, and for the coverage buyer, giving a game in which the company sets its optimal premiums and the buyer selects his optimal coverage.

2. Assumptions on the functions U and P

Parable 1 suggests the following properties of U:

Assumptions 1.

(u1)	$U(0) = 0$		(u2)	$U(\cdot)$ is strictly increasing
(u3)	$U(x) \leq x$ for all x	or alternatively:	(u3')	$U'(0) = 1$
(u4)	$U(\cdot)$ is strictly concave		(u5)	U is continuously differentiable

The class of value–risk functions satisfying (u1),(u2),(u3'), (u4) and (u5) is denoted:

$$\mathcal{U} = \left\{ U : \begin{array}{c} U \text{ strictly increasing, strictly concave, continuously} \\ \text{differentiable,} \quad U(0) = 0 \ , \ U'(0) = 1 \end{array} \right\} . \tag{9}$$

A function $U \in \mathcal{U}$ may be interpreted as a **utility function**, by assumptions (u2) and (u4), or as a **penalty function**, penalizing negative values of its argument, by assumptions (u1) and (u2).

In Parable 1, the bank's share $x - U(x)$ is nonnegative by Assumption (u3) and has an increasing rate by Assumption (u4).

The concavity of $U(\cdot)$ implies

$$U(x) \ \leq \ U(0) + U'(0)\, x$$

therefore (u3) follows from (u1), (u4), (u5) and (u3').
We list now assumptions on the penalty functions P of Parable 2.

Assumptions 2.

(p1)	$P(0) = 0$		(p2)	$P(\cdot)$ is strictly increasing
(p3)	$P(x) \geq x$ for all x	or alternatively:	(p3')	$P'(0) = 1$
(p4)	$P(\cdot)$ is strictly convex		(p5)	P is continuously differentiable

The class of penalty functions satisfying (p1),(p2),(p3'), (p4) and (p5) is denoted:

$$\mathcal{P} = \left\{ P : \begin{array}{c} P \text{ strictly increasing, strictly convex, continuously} \\ \text{differentiable,} \quad P(0) = 0 \ , \ P'(0) = 1 \end{array} \right\} . \tag{10}$$

These assumptions are justified by Parable 2. Let x stand for $\mathbf{Z} - z$, the amount still owed the government after paying z against the liability $\mathbf{Z}$. As above, let $P(x)$ denote the remaining liability (if $x > 0$), or refund (if $x < 0$). Assumptions (p1) and (p2) are then obvious, implying

$$P(x) > 0 \ , \quad \text{for } x > 0 \ ,$$

showing $P(\cdot)$ to be a **penalty function**, penalizing positive values of its argument (which result from underpayment of tax in Parable 2).

Assumptions (p4) and (p2) show the penalty rate $P'(x)$ to be positive and increasing.

Assumption (p3) says: the refund [penalty] is never greater [smaller] than the amount owed.

The convexity of $P(\cdot)$ implies that (p3) follows from (p1), (p4), (p5) and (p3').

Definition 1. A value–risk function $U \in \mathcal{U}$ and a penalty function $P \in \mathcal{P}$ are called **dual** if

$$P(x) \ = \ -U(-x) \ , \quad \forall \ x \ . \tag{11}$$

In this case there is no need to study (8a) and (8b) separately, since

$$B_P(\mathbf{Z}) \;=\; -S_U(-\mathbf{Z})\ , \quad \text{for all RV's } \mathbf{Z}\ . \tag{12}$$

Indeed,

$$\begin{aligned} S_U(-\mathbf{Z}) &= \sup_z \{z + EU(-\mathbf{Z}-z)\} \\ &= \sup_z \{z - EP(\mathbf{Z}+z)\}\ , \quad \text{by (11)}\ , \\ &= \sup_z \{-z - EP(\mathbf{Z}-z)\} \\ &= -\inf_z \{z + EP(\mathbf{Z}-z)\} \\ &= -B_P(\mathbf{Z})\ , \quad \text{proving (12)}\ . \end{aligned}$$

Duality is a 1:1 correspondence between the classes $\mathcal{U}$ and $\mathcal{P}$: if $U \in \mathcal{U}$ and if P satisfies (11) then $P \in \mathcal{P}$. Conversely, if $P \in \mathcal{P}$ and if U satisfies (11) then $U \in \mathcal{U}$.

Another 1:1 correspondence between the classes $\mathcal{U}$ and $\mathcal{P}$ is obtained by the inverse mapping $P = U^{-1}$.

3. Properties of the recourse certainty equivalent B_P

The properties of $B_P(\cdot)$ developed here are analogous to the properties of the RCE $S_U(\cdot)$ given in [3, § 2].

The question of the attainment of the infimum in (8b) is settled, for any $P \in \mathcal{P}$, in the following lemma, analogous to [3, Lemma 1].

Lemma 1. Let the RV $\mathbf{Z}$ have support $[z_{\min},\ z_{\max}]$, with finite $z_{\min}$ and $z_{\max}$. Then for any $P \in \mathcal{P}$ the infimum in (8b) is attained uniquely at some z^*,

$$z_{\min} \;\le\; z^* \;\le\; z_{\max}, \tag{13}$$

which is the solution of

$$EP'(\mathbf{Z} - z^*) \;=\; 1\ , \tag{14}$$

so that

$$B_P(\mathbf{Z}) \;=\; z^* + EP(\mathbf{Z} - z^*)\ . \tag{15}$$

Proof. Note that $\mathbf{Z} - z_{\min} \ge 0$ with probability 1. Also $P'(\cdot)$ is increasing since P is convex. Therefore

$$EP'(\mathbf{Z} - z_{\min}) \;\ge\; EP'(0) \;=\; 1\ .$$

Similarly

$$EP'(\mathbf{Z} - z_{\max}) \;\le\; EP'(0) \;=\; 1\ .$$

Since P' is continuous, the equation[6]

$$EP'(\mathbf{Z} - z) \;=\; 1$$

has a solution z^* in $[z_{\min}, z_{\max}]$, which is unique by the strict monotonicity of P'. Now z^* is a stationary point of the function

$$f(z) \;=\; z + EP(\mathbf{Z} - z) \tag{16}$$

which is convex since $P \in \mathcal{P}$, see (10). Therefore the infimum of (16) is attained at z^*. □

The following theorem lists properties of B_P. These are proved by a straightforward translation of the proofs of the corresponding properties of S_U in [3, Theorem 2.1]. The proof is omitted.

Theorem 1 (Properties of B_P)**.**

(a) **Shift additivity**. For any $P : \mathbb{R} \to \mathbb{R}$, any RV $\mathbf{Z}$ and any constant c

$$B_P(\mathbf{Z} + c) \;=\; B_P(\mathbf{Z}) + c\ . \tag{17}$$

(b) **Consistency**. If P satisfies (p1), (p3) then, for any constant c [7],

$$B_P(c) \;=\; c\ . \tag{18}$$

[6] This equation is the necessary condition for minimum in (8b). Differentiation "inside the expectation" is valid if e.g. P' is continuous and $EP'(\cdot) < \infty$, see [7, p. 99].

[7] Considered as a degenerate RV.

(c) **Super–homogeneity**. If P satisfies (p1) and (p4) then, for any RV $\mathbf{Z}$,

$$\frac{1}{\lambda}\, B_P(\lambda \mathbf{Z}) \text{ is increasing in } \lambda\ , \quad \lambda > 0\ .$$

(d) **Monotonicity**. If P satisfies (p2) then, for any RV $\mathbf{X}$ and any nonnegative RV $\mathbf{Y}$,

$$B_P(\mathbf{X}+\mathbf{Y})\ \geq\ B_P(\mathbf{X})\ .$$

(e) **Risk aversion**. P satisfies (p3) if and only if

$$B_P(\mathbf{Z})\ \geq\ E\mathbf{Z}\ , \quad \text{for all RV's } \mathbf{Z}\ . \tag{19}$$

(f) **Convexity**. If $P \in \mathcal{P}$ then for any RV's $\mathbf{X}_0$, $\mathbf{X}_1$ and $0 < \alpha < 1$,

$$B_P(\alpha \mathbf{X}_1 + (1-\alpha)\mathbf{X}_0)\ \leq\ \alpha B_P(\mathbf{X}_1) + (1-\alpha) B_P(\mathbf{X}_0)\ . \tag{20}$$

Proof. Analogous to the proof of [3, Theorem 2.1]. □

Remark 1. Theorem 1 lists properties which seem reasonable for any certainty equivalent. Property (b) is natural and requires no justification. The remaining properties will now be discussed one by one.
(a) Note that shift additivity holds for all functions P, i.e. it is a **generic property** of the RCE. Recall that shift additivity holds also for the Yaari criterion ([20]), the RCE S_U ([3]) and for the u-mean ([8]).
(c) An important consequence (and the reason for the name "super–homogeneity") is

$$B_P(\lambda \mathbf{Z})\ \geq\ \lambda B_P(\mathbf{Z}),\ , \quad \text{for all RV } \mathbf{Z} \text{ and } \lambda > 1$$

Thus indifference between the RV $\mathbf{Z}$ and its RCE $B_P(\mathbf{Z})$ goes together with preference for $\lambda B_P(\mathbf{Z})$ over the RV $\lambda \mathbf{Z}$, for $\lambda > 1$. This is explained by

$$\begin{aligned} E(\lambda \mathbf{Z}) &= \lambda\, E\mathbf{Z} \\ \mathrm{Var}\,(\lambda \mathbf{Z}) &= \lambda^2\, \mathrm{Var}\,(\mathbf{Z})\ >\ \lambda \mathrm{Var}\,(\mathbf{Z})\ , \quad \text{if } \lambda > 1 \end{aligned}$$

An interesting result, in view of (c) and (e), is that for $P \in \mathcal{P}$,

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda}\, B_P(\lambda \mathbf{Z})\ =\ E\mathbf{Z}$$

(d) If P satisfies (p1) and (p2), and if the RV $\mathbf{Z}$ satisfies $\mathbf{Z} \geq z_{\min}$ with probability 1, then

$$B_P(\mathbf{Z})\ \geq\ z_{\min} \tag{21}$$

This follows from part (d) by taking $\mathbf{X}\ :=\ z_{\min}$ (degenerate RV) and $\mathbf{Y}\ :=\ \mathbf{Z} - z_{\min}$.
(e) A DM indifferent between $\mathbf{Z}$ and $B_P(\mathbf{Z})$ will by (19) prefer $E\mathbf{Z}$ to $\mathbf{Z}$. If Assumption (p3) is replaced by a strict inequality

(p3")
$$P(x)\ >\ x\ , \quad \forall\, x \neq 0\ ,$$

then the inequality (19) also becomes strict,

$$B_P(\mathbf{Z})\ >\ E\mathbf{Z}\ , \quad \text{for all nondegenerate RV's } \mathbf{Z}$$

(f) The convexity of $B_P(\cdot)$, for all $P \in \mathcal{P}$, expresses risk-aversion as aversion to variability. To gain insight consider the case of two independent RV's $\mathbf{X}_1$ and $\mathbf{X}_0$ with the same mean and variance. The mixed RV $\mathbf{X}_\alpha\ =\ \alpha \mathbf{X}_1 + (1-\alpha)\mathbf{X}_0$ has the same mean, but a smaller variance. Convexity of B_P means that the more centered RV $\mathbf{X}_\alpha$ is preferred.

The risk-aversion inequality (19) is implied by (f): Let $\mathbf{Z}$, $\mathbf{Z}_1$, $\mathbf{Z}_2$, ... be independent, identically distributed RV's. Then by (f),

$$\begin{aligned} B_P\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) &\leq\ \frac{1}{n}\sum_{i=1}^{n} B_P(Z_i) \\ &=\ B_P(\mathbf{Z}) \end{aligned}$$

As $n \to \infty$, (19) follows by the strong law of large numbers.

Example 1 (Exponential penalty function)**.** Here

$$P(z) \;:=\; -1+e^z \;, \quad \forall\, z \;, \tag{22}$$

and equation (14) becomes $Ee^{\mathbf{Z}-z}=1$, giving $z^* \;=\; \log Ee^{\mathbf{Z}}$ and the same value for the RCE

$$B_P(\mathbf{Z}) \;=\; \log Ee^{\mathbf{Z}} \tag{23}$$

The penalty function (22) is the dual (see Definition 1) of the exponential value–risk function,

$$U(z)) \;:=\; 1-e^{-z} \tag{24}$$

for which, the minimum selling price is

$$S_U(\mathbf{Z}) \;=\; -\log Ee^{-\mathbf{Z}} \tag{25}$$

in agreement with (12). □

Example 2 (Quadratic penalty function)**.** Here[8]

$$P(z) \;:=\; z+\frac{1}{2}z^2 \;, \quad z \geq -1 \tag{26}$$

and for a RV $\mathbf{Z}$ with $z_{\min} \;\geq\; -1$, mean μ and variance σ^2, equation (14) gives $z^* \;=\; \mu$, and by (15)

$$B_P(\mathbf{Z}) = \mu + \frac{1}{2}\sigma^2 \tag{27}$$

The penalty (26) is the dual of the quadratic value-risk function

$$U(z) \;:=\; z-\frac{1}{2}z^2 \;, \quad z \leq 1 \tag{28}$$

for which, the minimum selling price of a RV $\mathbf{Z}$ with $z_{\max} \;\leq\; 1$, mean μ and variance σ^2, is

$$S_U(\mathbf{Z}) \;=\; \mu - \frac{1}{2}\sigma^2 \tag{29}$$

in agreement with (12).

A transaction (sale) between a potential seller and a potential buyer occurs only if the seller's minimum price $S_U(\mathbf{Z})$ is no greater than the buyer's maximum price $B_P(\mathbf{Z})$, in which case the interval $[S_U(\mathbf{Z}),\, B_P(\mathbf{Z})]$ represents all prices on which seller and buyer may agree. For the P and U of this example, sale is possible

$$B_P(\mathbf{Z}) - S_U(\mathbf{Z}) \;=\; \sigma^2 \;, \tag{30}$$

and the variance gives a whole interval of prices acceptable to both sides. □

Corollary 1. In both the exponential and quadratic penalty functions

$$B_P\Big(\sum_{i=1}^n \mathbf{Z}_i\Big) \;=\; \sum_{i=1}^n B_P(\mathbf{Z}_i) \tag{31}$$

for independent RV's $\{\,\mathbf{Z}_1,\ \mathbf{Z}_2,\ \cdots\ ,\ \mathbf{Z}_n\}$ □

Example 3. For the **hybrid model**, with exponential penalty P and a normally distributed RV $\mathbf{Z}$ with mean μ and variance σ^2,

$$B_P(\mathbf{Z}) \;=\; \mu + \frac{1}{2}\,\sigma^2 \;, \quad \text{see also [1],[18].}$$ □

Example 4 (Piecewise linear penalty function)**.** Let

$$P(t) = \begin{cases} \beta t, & t \leq 0 \\ \alpha t, & t > 0 \end{cases} \;, \quad 0 < \beta < 1 < \alpha \;. \tag{32}$$

If F is the cumulative distribution function of the RV $\mathbf{Z}$, then the maximizing z in (8a) is the $\left(\frac{1-\alpha}{\beta-\alpha}\right)$−percentile of the distribution F of $\mathbf{Z}$:

$$z^* \;=\; F^{-1}\left(\frac{1-\alpha}{\beta-\alpha}\right)$$

[8]The restriction $z \geq -1$ in (26) guarantees that P is increasing throughout its domain.

and the RCE associated with (32) is

$$B_P(\mathbf{Z}) \;=\; \beta \int^{z^*} t\,dF(t) \;+\; \alpha \int_{z^*} t\,dF(t)\;. \qquad \square$$

Consider now discrete RV's. Let $\mathbf{X}$ be a RV assuming finitely many values x_i with probabilities

$$\text{Prob}\,\{\mathbf{X} = x_i\} \;=\; p_i\;, \tag{33}$$

denoted by

$$\mathbf{X} \;:=\; [\mathbf{x}, \mathbf{p}]\;, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)\;,\; \mathbf{p} = (p_1, p_2, \ldots, p_n)\;. \tag{34}$$

The RCE of $[\mathbf{x}, \mathbf{p}]$ is

$$B_P([\mathbf{x}, \mathbf{p}]) \;=\; \min_z \left\{ z + \sum_{i=1}^n p_i\,P(x_i - z) \right\}\;. \tag{35}$$

The following result is stated for $B_P([\mathbf{x}, \mathbf{p}])$ as a function of the arguments $\mathbf{x}$ and $\mathbf{p}$.

Theorem 2.

(a) For any function $P : \mathbb{R} \to \mathbb{R}$, and any $\mathbf{x} \;=\; (x_1, x_2, \ldots, x_n)$, $B_P([\mathbf{x}, \mathbf{p}])$ is concave in $\mathbf{p}$.
(b) For P convex, and any probability vector $\mathbf{p}$, $B_P([\mathbf{x}, \mathbf{p}])$ is convex in $\mathbf{x}$.

Proof. Analogous to the proof of [3, Theorem 2.2]. $\square$

Consider now the inverse problem, of **recovering** P from a **given** B_P. The discussion is restricted to RCE's B_P defined by $P \in \mathcal{P}$. Our results are stated in terms of an elementary RV

$$\mathbf{X} \;:=\; \begin{cases} x, & \text{with probability } p \\ 0, & \text{with probability } \bar{p} = 1 - p \end{cases} \tag{36}$$

which we denote (x, p). For this RV,

$$B_P((x, p)) \;=\; \inf_z \{z + p\,P(x - z) + \bar{p}\,P(-z)\} \tag{37}$$

which we abbreviate $B_P(x, p)$.

Theorem 3. If $P \in \mathcal{P}$ then

$$P(x) \;=\; \frac{\partial}{\partial p} B_P(x, p)\;|_{p=0} \tag{38}$$

Proof. Analogous to the proof of [3, Theorem 3.1]. $\square$

To interpret this result consider an RCE minimizing individual who is free of any liability, and is offered an income with tax liability x occurring with probability p. The resulting change in his RCE is

$$\Delta(x, p) \;=\; B_P(x, p) - B_P(x, 0)$$

and the rate of change is $\dfrac{\Delta(x, p)}{p}$. Theorem 3 says that this rate of change, for an infinitesimal change in risk $(p \to 0)$ is precisely $P(x)$, the penalty function evaluated at x.

The following theorem is a companion of Theorem 3. It says that the limiting rate of change $\dfrac{\Delta(x, p)}{x}$ is exactly the probability p of obtaining x.

Theorem 4. If $P \in \mathcal{P}$ then

$$p \;=\; \frac{\partial}{\partial x} B_P(x, p)\;|_{x=0} \tag{39}$$

Proof. Analogous to the proof of [3, Theorem 3.2]. $\square$

4. Functionals and Approximations

Let $\mathbf{Z} = (\mathbf{Z}_i)$ be a RV in $\mathbb{R}^n$, with expectation $\boldsymbol{\mu}$ (vector) and covariance matrix Σ (if $n = 1$ then as above $\Sigma = \sigma^2$). For any vector $\mathbf{y} \in \mathbb{R}^n$, the **inner product**,

$$\mathbf{y} \cdot \mathbf{Z} = \sum_{i=1}^{n} y_i \mathbf{Z}_i$$

is a scalar RV. Given $P \in \mathcal{P}$, the corresponding RCE of $\mathbf{y} \cdot \mathbf{Z}$ is taken as a functional in $\mathbf{y}$, the **RCE functional**

$$b_P(\mathbf{y}) := B_P(\mathbf{y} \cdot \mathbf{Z}) . \tag{40}$$

We collect properties of the RCE functional in the following theorem.

Theorem 5. Let $P \in \mathcal{P}$ be twice continuously differentiable, and $\mathbf{Z}$ and $b_P(\cdot)$ as above. Then:
(a) The functional b_P is convex, and given by

$$b_P(\mathbf{y}) = z^*(\mathbf{y}) + EP(\mathbf{y} \cdot \mathbf{Z} - z^*(\mathbf{y})) \tag{41}$$

where $z^*(\mathbf{y})$ is the unique solution z of

$$E\, P'(\mathbf{y} \cdot \mathbf{Z} - z) = 1 \tag{42}$$

(b) Moreover,
$$\begin{array}{llll} b_P(\mathbf{0}) = 0 & , & \nabla b_P(\mathbf{0}) = \boldsymbol{\mu} & , \quad \nabla^2 b_P(\mathbf{0}) = P''(0)\Sigma \\ z^*(\mathbf{0}) = 0 & , & \nabla z^*(\mathbf{0}) = \boldsymbol{\mu} & \end{array} \tag{43}$$

and if P is three times continuously differentiable,

$$\nabla^2 z^*(\mathbf{0}) = \frac{P'''(\mathbf{0})}{P''(\mathbf{0})} \Sigma \tag{44}$$

Proof. Analogous to the proof of [3, Theorem 6.1]. □

Theorem 5 can be used to obtain the following approximation of the functional $b_P(\cdot)$ based on its Taylor expansion around $\mathbf{y} = \mathbf{0}$.

Corollary 2. If P is three times continuously differentiable then

$$b_P(\mathbf{y}) = \boldsymbol{\mu} \cdot \mathbf{y} + \frac{1}{2} P''(\mathbf{0}) \mathbf{y} \cdot \Sigma \mathbf{y} + \circ(\| \mathbf{y} \|^2) \tag{45}$$

□

Remark 2.

(a) In particular, for $n = 1$ and $y = 1$, it follows from (45) that the RCE has the following second-order approximation

$$B_P(\mathbf{Z}) \approx \mu + \frac{1}{2} P''(0)\, \sigma^2 \tag{46}$$

(b) We also note that the approximation (45) is exact if

(i) P is quadratic, or
(ii) P is exponential, $\mathbf{Z}$ is normal.

5. An inventory model

Consider a classical inventory model, e.g. [15, § 2.5], where **demand** (for the item in question) occurs at a rate of d units per day. Orders are received immediately after they are placed with the supplier. Unsatisfied demand is backlogged until it can be satisfied. **Holding cost** is h per unit per day, **shortage cost** (penalty for unsatisfied demand) is p per unit per day. **Material cost** is c per unit, and there is a fixed **transaction cost** of k per order (independent of the order size).

An (S, s) **policy** is used (whenever stock level falls below s, order up to S). The aim is to minimize **total cost** per period, given by

$$TC = \frac{hS^2}{2(S-s)} + \frac{ps^2}{2(S-s)} + \frac{kd}{S-s} + cd \tag{47}$$

and the optimal parameters of the policy are given by

$$S^* = \sqrt{\frac{2kd}{h}}\sqrt{\frac{p}{p+h}} \tag{48a}$$

$$s^* = -\frac{h}{p}\,S^* \tag{48b}$$

Note that s^* is negative (i.e. it is optimal to have a period of shortage, during which demand is not met) if the costs h, p are positive and finite. The limiting case $p \to \infty$ corresponds to the situation where shortage is not allowed, in which case (48b) gives $s^* = 0$ (i.e. order S^* when the stock is zero).

The least–cost order quantity $Q^* := S^* - s^*$ is, by (48),

$$Q^* = \sqrt{\frac{2kd}{h}}\sqrt{\frac{p+h}{p}} \tag{49}$$

We now analyze this model under the assumption that the demand $\mathbf{D}$ is a nonnegative RV. To compare our results with the deterministic case we assume

$$E\mathbf{D} = d\,. \tag{50}$$

We also denote the variance of $\mathbf{D}$ by σ^2. This being a cost minimization problem, we use the criterion B_P, and so minimize the RCE of the total cost (47),

$$\min_{s,S}\ B_P\left(\frac{hS^2}{2(S-s)} + \frac{ps^2}{2(S-s)} + \frac{k\mathbf{D}}{S-s} + c\mathbf{D}\right)\,, \tag{51}$$

which, by (17) and (40), reduces to

$$\min_{s,S}\ \left\{\frac{hS^2}{2(S-s)} + \frac{ps^2}{2(S-s)} + b_P\left(\frac{k}{S-s} + c\right)\right\}\,. \tag{52}$$

We denote the minimand of (52) by

$$f(S,s) := \frac{hS^2}{2(S-s)} + \frac{ps^2}{2(S-s)} + b_P\left(\frac{k}{S-s} + c\right)\,. \tag{53}$$

The first order necessary conditions for an optimal pair $(\hat{S},\hat{s})$ are

$$\frac{\partial f}{\partial S} = \frac{1}{2(S-s)^2}\left\{2(S-s)hS - (hS^2+ps^2) - 2kb'_P\left(\frac{k}{S-s}+c\right)\right\} = 0 \tag{54a}$$

$$\frac{\partial f}{\partial s} = \frac{1}{2(S-s)^2}\left\{2(S-s)ps + (hS^2+ps^2) + 2kb'_P\left(\frac{k}{S-s}+c\right)\right\} = 0 \tag{54b}$$

By adding (54a) and (54b) we obtain the relation

$$\hat{s} = -\frac{h}{p}\,\hat{S} \tag{55}$$

in analogy with (48b). Substituting (55) in (54a) we get an equation for $\hat{S}$:

$$T(\hat{S}) := \hat{S}^2 h\left(1+\frac{h}{p}\right) - 2kb'_P\left(\frac{k}{\hat{S}(1+\frac{h}{p})} + c\right) = 0\,, \tag{56}$$

The function $T(\cdot)$ is increasing for $\hat{S} \geq 0$, since the functional $b(\cdot)$ is convex, see Theorem 5(a). Moreover, $b'_P(y) > 0$ for all $y > 0$ since $b'_P(0) = E\mathbf{D} = d > 0$. Therefore,

$$T(0) = 0\,, \quad \text{and } T(\infty) = \infty$$

and equation (56) has a unique solution $\hat{S}$. Comparing the solution $(\hat{S},\hat{s})$ to the deterministic solution (S^*, s^*) (given in (48a)–(48b)), we get:

Proposition 1. Let $P \in \mathcal{P}$, let $\{\hat{S}, \hat{s}\}$ be the optimal parameters for a DM minimizing (53), and let $\{S^*, s^*\}$ be the optimal parameters in the deterministic model, corresponding by (50). Then

$$\hat{S} \;>\; S^* \tag{57a}$$
$$\hat{s} \;<\; s^* \tag{57b}$$
$$\text{and}\quad \hat{Q} \;:=\; \hat{S} - \hat{s} \;>\; S^* - s^* \;=\; Q^* \tag{57c}$$

Proof. By (56) with $\theta \;=\; 1 + \dfrac{h}{p}$,

$$\hat{S}^2 h\theta \;=\; 2kb'\left(\frac{k}{\hat{S}\theta} + c\right) \;>\; 2kb'(0) \;=\; 2kd$$
$$\therefore\quad \hat{S} \;>\; \sqrt{\frac{2kd}{h\theta}} \;=\; \sqrt{\frac{2kd}{h}}\sqrt{\frac{p}{p+h}} \;=\; S^*\ , \quad \text{proving (57a).}$$

Using (55) and (48b) we get a comparison of the other parameters

$$\hat{s} \;=\; -\frac{h}{p}\hat{S} \;<\; \frac{h}{p}S^* \;=\; s^*\ , \quad \text{proving (57b)}$$

and (57c) follows from (57a)–(57b). □

The effect of changes in the cost parameters on $\hat{S}$ can be determined from the optimality equation (56). These results are summarized in the table below, and compared to the analogous results in the deterministic case.

Increase in:	Effect on $\hat{S}$ (stochastic demand)	Effect on S^* (deterministic demand)
Shortage cost p	increase	increase
Holding cost h	decrease	decrease
Transaction cost k	increase	increase
Material cost c	increase	no effect

Thus we have the same effect of cost changes in the deterministic and stochastic cases, except for the material cost c: in the deterministic case the optimal policy is independent of c (see (48a), but in the stochastic case there is dependence, see (56).

Example 5. To illustrate the above results, consider the quadratic penalty (26). Then the functional $b_P(y)$, see (40), is

$$b_P(y) \;=\; dy + \frac{1}{2}\sigma^2 y^2$$

(for the quadratic penalty the approximation (45) is exact). The equation (56) determining the optimal $\hat{S}$ is here

$$S^2 h\theta - 2\sigma^2\frac{k^2}{\theta S} \;=\; 2k(d + \sigma^2 c) \tag{58}$$

where $\theta \;:=\; 1 + \frac{h}{\theta}$. From (58) it follows that $\hat{S}$ increases with the expected demand d, and with the demand variance σ^2. Since $\hat{Q} := \hat{S} - \hat{s} = \hat{S}\theta$ the same conclusions hold for the effects on the optimal order quantity $\hat{Q}$. The result that $\hat{Q}$ increases with σ^2 is intuitively clear: the risk–averse decision maker keeps a larger inventory to cope with increased fluctuations. □

6. An extremal principle for B_P

Let $\mathbb{P}^n$ denote the n**-dimensional probability vectors**,

$$\mathbb{P}^n \;:=\; \left\{\mathbf{p} = (p_i) : \mathbf{p} \in \mathbb{R}^n_+\,,\ \sum_{i=1}^n p_i = 1\right\}\ . \tag{59}$$

Let ϕ be a real valued function defined and convex on the nonnegative real line. The **Csiszár ϕ–divergence**, (see [9],[10]), is a function $I_\phi : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$, defined as:

$$I_\phi(\mathbf{p}, \mathbf{q}) \;:=\; \sum_{i=1}^{n} q_i\, \phi\left(\frac{p_i}{q_i}\right) \tag{60}$$

It has certain properties of a "distance" between $\mathbf{p}$ and $\mathbf{q}$, but is not a metric: The triangle inequality does not hold, and in general, for $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n_{++}$,

$$I_\phi(\mathbf{p}, \mathbf{q}) \;\neq\; I_\phi(\mathbf{q}, \mathbf{p})\ .$$

However, for any $\phi : \mathbb{R}_+ \to \mathbb{R}$ which is convex and **normalized** i.e.

$$\phi(1) \;=\; 0 \tag{61}$$

the **adjoint** function $\phi^\diamond$, defined by,

$$\phi^\diamond(t) \;:=\; t\,\phi\left(\frac{1}{t}\right)\ , \quad \forall t \in \mathbb{R}_+ \tag{62}$$

is convex, normalized, and satisfies

$$I_\phi(\mathbf{p}, \mathbf{q}) \;=\; I_{\phi^\diamond}(\mathbf{q}, \mathbf{p})\ , \quad \forall \mathbf{p}, \mathbf{q} \in \mathbb{R}^n_{++} \tag{63}$$

An extremal principle for the RCE S_U was given in [4, Theorem 4.1] as follows: Let $U : \mathbb{R} \to \mathbb{R}$ be strictly increasing, closed and concave, and let $\phi \;=\; -U_*$ (where U_* denotes the **concave conjugate** of U, see [4, §3], [16]). Then for any RV $\mathbf{X} = [\mathbf{x}, \mathbf{p}]$,

$$S_U([\mathbf{x}, \mathbf{p}]) \;=\; \inf_{\mathbf{q} \in \mathbb{P}^n} \left\{ I_\phi(\mathbf{q}, \mathbf{p}) + \sum_{i=1}^{n} q_i\, x_i \right\}\ . \tag{64}$$

The corresponding result for the RCE B_P follows.

Theorem 6. Let $P : \mathbb{R} \to \mathbb{R}$ be strictly increasing, closed and convex, and let ϕ be the **convex conjugate** of P, defined as

$$\phi(y) \;:=\; \sup_x \{yx - P(x)\} \tag{65}$$

Then for any RV $\mathbf{X} = [\mathbf{x}, \mathbf{p}]$,

$$B_P([\mathbf{x}, \mathbf{p}]) \;=\; \sup_{\mathbf{q} \in \mathbb{P}^n} \left\{ \sum_{i=1}^{n} q_i\, x_i - I_\phi(\mathbf{q}, \mathbf{p}) \right\}\ . \tag{66}$$

Proof. The RHS of (66) is the value of the optimization problem

$$\sup_{\substack{q_i \geq 0 \\ \sum q_i = 1}} \left\{ \sum_{i=1}^{n} q_i\, x_i - I_\phi(\mathbf{q}, \mathbf{p}) \right\}$$

with concave objective, and linear constraints. The optimal value is equal to the optimal value of the Lagrangian dual problem

$$\begin{aligned}
\sup_{\substack{q_i \ge 0 \\ \sum q_i = 1}} \left\{ \sum_{i=1}^n q_i\, x_i - I_{P^*}(\mathbf{q},\mathbf{p}) \right\} &= \inf_\lambda \sup_{q_i \ge 0} \left\{ \sum_{i=1}^n q_i\, x_i - I_{P^*}(\mathbf{q},\mathbf{p}) - \lambda \left(\sum_{i=1}^n q_i - 1 \right) \right\} \\
&\qquad \text{(where } \lambda \text{ is the Lagrange multiplier of } \textstyle\sum q_i = 1\text{)}, \\
&= \inf_\lambda \sup_{q_i \ge 0} \left\{ \lambda + \sum_{i=1}^n q_i\,(x_i - \lambda) - I_{P^*}(\mathbf{q},\mathbf{p}) \right\} \\
&= \inf_\lambda \left\{ \lambda + \sum_{i=1}^n \sup_{q_i \ge 0} \left\{ q_i(x_i - \lambda) - p_i P^*\left(\frac{q_i}{p_i}\right) \right\} \right\} \\
&= \inf_\lambda \left\{ \lambda + \sum_{i=1}^n p_i \sup_{q_i \ge 0} \left\{ \frac{q_i}{p_i}(x_i - \lambda) - P^*\left(\frac{q_i}{p_i}\right) \right\} \right\} \\
&= \inf_\lambda \left\{ \lambda + \sum_{i=1}^n p_i \sup_t \{ t(x_i - \lambda) - P^*(t) \} \right\} \\
&= \inf_\lambda \left\{ \lambda + \sum_{i=1}^n p_i P^{**}(x_i - \lambda) \right\} \\
&= \inf_\lambda \, \{ \lambda + E P^{**}(\mathbf{X} - \lambda) \} \\
&= \inf_\lambda \, \{ \lambda + E P(\mathbf{X} - \lambda) \} \; = \; B_P([\mathbf{x},\mathbf{p}]) \,.
\end{aligned}$$

□

Remark 3. The extremal principles (64) and (66) are equivalent. To see this we recall

$$\phi \text{ convex} \implies (-\phi)_*(x) \;=\; -\phi^*(-x) \;,\; \forall x \tag{67}$$

see e.g. [4, Lemma 3.1]. We prove now that (64) $\implies$ (66). let U be a strictly increasing, concave and closed function, let $\phi \;:=\; -U_*$, and let $P \;:=\; \phi^*$. Then $U \;=\; (-\phi)_*$ and, by (67),

$$P(x) \;=\; -U(-x) \;, \quad \forall\, x \tag{11}$$

i.e. P and U are dual. Then:

$$\begin{aligned}
B_P([\mathbf{x},\mathbf{p}]) &= -S_U([-\mathbf{x},\mathbf{p}]) \;, \quad \text{by (12)} \;, \\
&= -\inf_{\mathbf{q}\in\mathbb{P}^n} \left\{ I_\phi(\mathbf{q},\mathbf{p}) + \sum_{i=1}^n q_i\,(-x_i) \right\} \;, \quad \text{by (64)} \;, \\
&= \sup_{\mathbf{q}\in\mathbb{P}^n} \left\{ -I_\phi(\mathbf{q},\mathbf{p}) - \sum_{i=1}^n q_i\,(-x_i) \right\} \\
&= \sup_{\mathbf{q}\in\mathbb{P}^n} \left\{ \sum_{i=1}^n q_i\, x_i - I_\phi(\mathbf{q},\mathbf{p}) \right\} \;, \quad \text{proving (66)} \,.
\end{aligned}$$

The proof of (66) $\implies$ (64) is similar. □

Example 6. The conjugate of the penalty function

$$P(z) \;:=\; -1 + e^z \;, \quad \forall\, z \;, \tag{22}$$

is

$$P^*(t) \;=\; \sup_t \; \{ tz + 1 - e^z \} \;.$$

The supremum is attained at $t = e^z$, giving

$$P^*(t) \;=\; t\, \log t + 1 - t \tag{68}$$

The P^*–divergence in the RHS of (66) is then

$$\begin{aligned} I_{P^*}(\mathbf{q},\mathbf{p}) &= \sum p_i \left\{ \left(\frac{q_i}{p_i}\right) \log\left(\frac{q_i}{p_i}\right) + 1 - \frac{q_i}{p_i} \right\} \\ &= \sum q_i \log\left(\frac{q_i}{p_i}\right) + \sum p_i - \sum q_i \\ &= \sum q_i \log\left(\frac{q_i}{p_i}\right) \ , \quad \text{since } \mathbf{q},\mathbf{p} \in \mathbb{P}^n \end{aligned} \tag{69}$$

the **Kullback–Leibler distance**, [13], [14]. Substituting (23) and (69) in (66) we get the interesting identity

$$\log \sum p_i\, e^{x_i} = \sup_{\mathbf{q}\in\mathbb{P}^n} \left\{ \sum q_i\, x_i - \sum q_i \log\left(\frac{q_i}{p_i}\right) \right\} \tag{70}$$

for all $\mathbf{x} = (x_i) \in \mathbb{R}^n$ and $\mathbf{p} = (p_i) \in \mathbb{P}^n$. □

Example 7. The conjugate of the penalty function

$$P(z) := z + \frac{1}{2} z^2 \ , \quad z \geq -1 \tag{26}$$

is

$$\begin{aligned} P^*(t) &= \sup_t \left\{ tz - z - \frac{1}{2} z^2 \right\} \ , \quad \text{and is easily computed to be} \\ &= \frac{1}{2}\,(t-1)^2 \end{aligned} \tag{71}$$

The P^*–divergence in the RHS of (66) is then

$$\begin{aligned} I_{P^*}(\mathbf{q},\mathbf{p}) &= \sum p_i \left\{ \frac{1}{2}\left(\frac{q_i}{p_i} - 1\right)^2 \right\} \\ &= \frac{1}{2} \sum \frac{(q_i - p_i)^2}{p_i} \end{aligned} \tag{72}$$

which is half the $\boldsymbol{\chi}^2$**–distance**, [4, Example 2.4]. We verify (27) by substituting (72) in (66), and following the steps in the proof of Theorem 6, obtaining

$$\begin{aligned} B_P([\mathbf{x},\mathbf{p}]) &= \sup_{\mathbf{q}\in\mathbb{P}^n} \left\{ \sum q_i\, x_i - \frac{1}{2} \sum \frac{(q_i - p_i)^2}{p_i} \right\} \\ &= \inf_\lambda \left\{ \lambda + \sum p_i \left[(x_i - \lambda) + \frac{1}{2}(x_i - \lambda)^2 \right] \right\} \\ &= E\mathbf{X} + \frac{1}{2}\,\mathrm{Var}\mathbf{X} \end{aligned} \tag{73}$$

since the minimizing λ in (73) is $\lambda = \sum p_i x_i$. □

7. A duality interpretation

As in [3], we interpret the RCE $B_P(\mathbf{X})$ as the optimal value of the problem

$$\inf \{ z : \text{“}z \ \geq \ \mathbf{X}\text{”} \} \tag{P}$$

where optimality is in the sense of **recourse** or **two stage optimization** ([11], [12]), using the **penalty function** P to account for the stochastic constraint

$$\text{“}z \ \geq \ \mathbf{X}\text{”}.$$

We interpret the RHS of (66) as the following dual of (P),

$$\sup \{ z : \text{“}z \ \leq \ \mathbf{X}\text{”} \} \tag{D}$$

where the stochastic constraint

$$\text{“}z \ \leq \ \mathbf{X}\text{”} \tag{74}$$

is “enforced” by using a **stochastic penalty function**, [2].

Given the RV

$$\mathbf{X} \;:=\; [\mathbf{x},\mathbf{p}]$$

and z, define the subset of probability vectors

$$R\{z\} \;:=\; \left\{\mathbf{q} \in \mathbb{P}^n :\; z \leq \sum_{i=1}^{n} q_i\, x_i\right\}, \tag{75}$$

representing the set of RV's

$$\{\mathbf{Z} = [\mathbf{x},\mathbf{q}] :\; \mathbf{q} \in R\{z\}\}$$

with the same support as $\mathbf{X}$, which satisfy (74) "in the mean".

We then interpret the problem (D) as

$$\sup\;\{z - Q(z)\} \tag{D1}$$

where $Q(z)$ is the **penalty**

$$Q(z) \;:=\; \text{“dist”}\,(\mathbf{p}, R\{z\}) \;=\; \inf_{\mathbf{q}\in R\{z\}} \;\text{“dist”}\,(\mathbf{q},\mathbf{p}) \tag{76}$$

and "dist" is the "distance" induced by the ϕ–divergence,

$$Q(z) \;=\; \inf_{\mathbf{q}\in R\{z\}}\; I_\phi(\mathbf{q},\mathbf{p}) \tag{77}$$

Therefore, the problem (D1) is

$$\begin{aligned}
\sup_z \left\{z - \inf_{\mathbf{q}\in R\{z\}} \; I_\phi(\mathbf{q},\mathbf{p})\right\} \;&=\; \sup_z\; \sup_{\mathbf{q}\in R\{z\}}\; \{z - I_\phi(\mathbf{q},\mathbf{p})\}\\
&=\; \sup_{\mathbf{q}\in\mathbb{P}^n} \left\{\sum_{i=1}^{n} q_i\, x_i - I_\phi(\mathbf{q},\mathbf{p})\right\}
\end{aligned}$$

which is the RHS of (66).

8. Extremal principles for I_ϕ

The extremal principle (66) characterizes B_P in terms of the Csiszár ϕ–divergence I_ϕ. A converse principle, giving I_ϕ in terms of B_P, is as follows.

Theorem 7. Let $\phi : \mathbb{R}_{++} \to \mathbb{R}$ be convex, and let $P \;:=\; \phi^*$. Then for all $\mathbf{p},\mathbf{q} \in \mathbb{P}^n$,

$$I_\phi(\mathbf{q},\mathbf{p}) \;=\; \sup_{\substack{\mathbf{x}\,\in\,\mathbb{R}^n\\ \mathbf{q}\cdot\mathbf{x}=0}} \;\{-B_P([-\mathbf{x},\mathbf{p}])\} \tag{78a}$$

$$=\; -\inf_{\substack{\mathbf{x}\,\in\,\mathbb{R}^n\\ \mathbf{q}\cdot\mathbf{x}=0}} \; B_P([\mathbf{x},\mathbf{p}]) \tag{78b}$$

Proof. Follows, as in Remark 3, from (12) and the proof of [4, Theorem 4.2]. □

Another extremal principle for I_ϕ in terms of B_P is:

Theorem 8. Let $\phi : \mathbb{R}_{+} \to \mathbb{R}$ be convex, and let $P \;:=\; \phi^*$. Then for all $\mathbf{p},\mathbf{q} \in \mathbb{P}^n$,

$$I_\phi(\mathbf{q},\mathbf{p}) \;=\; \sup_{\mathbf{x}\in\mathbb{R}^n} \left\{\sum q_i\, x_i - B_P([\mathbf{x},\mathbf{p}])\right\}\;. \tag{79}$$

Proof. Given $\phi, \mathbf{p}, \mathbf{q}$ as above, it was shown in [4, Theorem 4.3] that

$$I_\phi(\mathbf{q},\mathbf{p}) \;=\; \sup_{\mathbf{x}\in\mathbb{R}^n} \left\{S_U([\mathbf{x},\mathbf{p}]) - \sum q_i\, x_i\right\} \tag{80}$$

and (79) follows, as in Remark 3, from (12). □

9. S_U and B_P as equilibrium prices

The prices S_U and B_P have the following characterizations.

Theorem 9. Let the value–risk function U and the penalty function P be dual. Then

$$S_U([\mathbf{x},\mathbf{p}]) = \inf_{\mathbf{q}\in\mathbb{P}^n} \sup_{\mathbf{u}\in\mathbb{R}^n} \left\{\sum q_i\,(x_i+u_i) - B_P([\mathbf{u},\mathbf{p}])\right\}, \tag{81a}$$

$$B_P([\mathbf{x},\mathbf{p}]) = \sup_{\mathbf{q}\in\mathbb{P}^n} \inf_{\mathbf{u}\in\mathbb{R}^n} \left\{\sum q_i\,(x_i+u_i) - S_U([\mathbf{u},\mathbf{p}])\right\}. \tag{81b}$$

Proof.

$$\begin{aligned} S_U([\mathbf{x},\mathbf{p}]) &= \inf_{\mathbf{q}\in\mathbb{P}^n} \left\{\sum_{i=1}^n q_i\,x_i + I_\phi(\mathbf{q},\mathbf{p})\right\}, \quad \text{by (64)} \\ &= \inf_{\mathbf{q}\in\mathbb{P}^n} \left\{\sum_{i=1}^n q_i\,x_i + \sup_{\mathbf{u}\in\mathbb{R}^n}\left\{\sum q_i\,u_i - B_P([\mathbf{u},\mathbf{p}])\right\}\right\}, \quad \text{by (79)} \\ &= \inf_{\mathbf{q}\in\mathbb{P}^n} \sup_{\mathbf{u}\in\mathbb{R}^n} \left\{\sum q_i\,(x_i+u_i) - B_P([\mathbf{u},\mathbf{p}])\right\} \end{aligned}$$

proving (81a). Similarly,

$$\begin{aligned} B_P([\mathbf{x},\mathbf{p}]) &= \sup_{\mathbf{q}\in\mathbb{P}^n} \left\{\sum_{i=1}^n q_i\,x_i - I_\phi(\mathbf{q},\mathbf{p})\right\}, \quad \text{by (66)} \\ &= \sup_{\mathbf{q}\in\mathbb{P}^n} \left\{\sum_{i=1}^n q_i\,x_i - \sup_{\mathbf{u}\in\mathbb{R}^n}\left\{S_U([\mathbf{u},\mathbf{p}]) - \sum q_i\,u_i\right\}\right\}, \quad \text{by (80)} \\ &= \sup_{\mathbf{q}\in\mathbb{P}^n} \inf_{\mathbf{u}\in\mathbb{R}^n} \left\{\sum q_i\,(x_i+u_i) - S_U([\mathbf{u},\mathbf{p}])\right\} \end{aligned}$$

proving (81b). □

Remark 4. The results (81a)–(81b) are dual characterizations of S_U and B_P as **equilibrium prices**.

The vector $\mathbf{u} = (u_i)$ appearing in (81a)–(81b) can be interpreted as **perturbations** of the (nominal) values x_i. Then $\mathbf{q} = (q_i)$ is the vector of (normalized) **prices** at which the combined amounts $\mathbf{x}+\mathbf{u} = (x_i+u_i)$ are traded.

For example, (81a) describes the situation where a perturbation $\mathbf{u} = (u_i)$ is bought at the price $B_P([\mathbf{u},\mathbf{p}])$, and is added to the vector $\mathbf{x} = (x_i)$ to be sold for $\sum q_i(x_i+u_i)$. The profit of this transaction is

$$\pi(\mathbf{q},\mathbf{u}) := \sum q_i\,(x_i+u_i) - B_P([\mathbf{u},\mathbf{p}]). \tag{82}$$

Then (81a) guarantees that for any price vector $\mathbf{q} = (q_i)$ there is no perturbation $\mathbf{u} = (u_i)$ such that

$$\pi(\mathbf{q},\mathbf{u}) < S_U([\mathbf{x},\mathbf{p}]),$$

i.e. with profit less than the minimum selling price.

Similarly, in (81b) the $\mathbf{u} = (u_i)$ are perturbations of the nominal amounts $\mathbf{x} = (x_i)$, changing the price to $\sum q_i(x_i+u_i)$. If the perturbation $\mathbf{u}$ is sold for $S_U([\mathbf{u},\mathbf{p}])$, the net cost is

$$\gamma(\mathbf{q},\mathbf{u}) := \sum q_i\,(x_i+u_i) - S_U([\mathbf{u},\mathbf{p}]) \tag{83}$$

For any price $\mathbf{q} = (q_i)$ these costs are bounded below by

$$\inf_{\mathbf{u}\in\mathbb{R}^n} \gamma(\mathbf{q},\mathbf{u})$$

The maximum such bound is the maximum buying price, by (81b).

Corollary 3. Let U and P be dual. Then

$$\inf_{\mathbf{q}\in\mathbb{P}^n}\left\{\sum q_i\,(x_i+u_i)-B_P([\mathbf{u},\mathbf{p}])\right\}\le\ S_U([\mathbf{x},\mathbf{p}])\ \le\sup_{\mathbf{u}\in\mathbb{R}^n}\left\{\sum q_i\,(x_i+u_i)-B_P([\mathbf{u},\mathbf{p}])\right\}\tag{84a}$$

$$\inf_{\mathbf{u}\in\mathbb{R}^n}\left\{\sum q_i\,(x_i+u_i)-S_U([\mathbf{u},\mathbf{p}])\right\}\le\ B_P([\mathbf{x},\mathbf{p}])\ \le\sup_{\mathbf{q}\in\mathbb{P}^n}\left\{\sum q_i\,(x_i+u_i)-S_U([\mathbf{u},\mathbf{p}])\right\}\tag{84b}$$

for all $\mathbf{q}\in\mathbb{P}^n$, $\mathbf{u}\in\mathbb{R}^n$.

Proof. (84a) follows from (81a) and the fact that $B_P([\mathbf{u},\mathbf{p}])$ is convex in $\mathbf{u}$, see Theorem 2(b). Similarly, (84b) follows from (81b) and the concavity of $S_U([\mathbf{u},\mathbf{p}])$ in $\mathbf{u}$. □

10. Insurance

We assume in this section that U and P are dual, and recall the insurance model of [3, § 9]. The elements of this model are:

n	**states of nature**	$\mathbf{p}=$	$(p_1,\dots,p_n)$ their **probabilities** (assumed positive)
$\hat q_i=$	**premium** for a unit coverage in state i, $\hat q_i>0$	$\hat B=$	**insurance budget**
$q_i=$	$\dfrac{\hat q_i}{\sum_{j=1}^n \hat q_j}$ = **normalized premium**	$B=$	$\dfrac{\hat B}{\sum_{j=1}^n \hat q_j}$ = **normalized budget**
$x_i=$	desired **income** in state i	$\mathbf{x}=$	$(x_1,\dots,x_n)$ the **decision variable**

The feasible decisions x_i are subject to the budget constraint

$$\sum_{i=1}^n q_i\,x_i\ =\ B.\tag{85}$$

The buyer of insurance coverage buys (from the insurance company) the RV $[\mathbf{x},\mathbf{p}]$, at the cost $\sum_{i=1}^n q_i\,x_i$. Assuming the buyer knows the probabilities $\mathbf{p}$ and the (normalized) premiums $\mathbf{q}$, his problem is to determine the optimal coverage $\mathbf{x}$.

The insurance company has a different problem: assuming it knows the probabilities $\mathbf{p}$ and the desired $\mathbf{x}$, it may be tempted to set the premiums so as to maximize its profit. The insurance problem then becomes a game between the company (maximizing its profit for sale of the RV $[\mathbf{x},\mathbf{p}]$) and the buyer (selecting an optimal coverage $\mathbf{x}$, for the given $\mathbf{p}$ and $\mathbf{q}$.)

10.1. **The insurance buyer's problem.** An insurance buyer who maximizes the RCE of his coverage, will determine the optimal coverage $\mathbf{x}$ by solving

$$\begin{aligned}\max\quad & S_U([\mathbf{x},\mathbf{p}])-\sum_{i=1}^n q_i\,x_i\\ \text{s.t.}\quad & \sum_{i=1}^n q_i\,x_i\ =\ B\end{aligned}\tag{86}$$

see e.g [4, § 4.4]. Changing the variables from x_i to

$$\hat x_i\ :=\ x_i-B\tag{87}$$

the problem (86) becomes

$$\begin{aligned}\max\quad & S_U([\hat{\mathbf{x}},\mathbf{p}])\\ \text{s.t.}\quad & \sum_{i=1}^n q_i\,\hat x_i\ =\ 0\end{aligned}\tag{88}$$

whose value, by (78a) and (12), is $I_\phi(\mathbf{q},\mathbf{p})$. Therefore, the optimal value of the insurance plan is the distance $I_\phi(\mathbf{q},\mathbf{p})$ between the (normalized) premiums $\mathbf{q}$ and the probabilities $\mathbf{p}$.

If $\mathbf{p}=\mathbf{q}$ (such premiums are called **actuarially fair**), the optimal coverage is

$$x_i\ =\ B\ ,\quad i=1,\cdots,n\ ,\tag{89}$$

making the buyer indifferent to the various states of nature, see [3, Theorem 10.1].

10.2. **The company's problem.** If the insurance company knows the probabilities $\mathbf{p}$ and the desired coverage $\mathbf{x}$, the company may select the premiums $\mathbf{q}$ which maximize its objective

$$\sum_{i=1}^{n} q_i\, x_i - I_\phi(\mathbf{q},\mathbf{p})$$

the difference between its revenue $\sum_{i=1}^{n} q_i\, x_i$ and the value $I_\phi(\mathbf{q},\mathbf{p})$ it delivers. The optimal value of the insurance company is

$$\sup_{\mathbf{q}\in\mathbb{P}^n} \left\{ \sum_{i=1}^{n} q_i\, x_i - I_\phi(\mathbf{q},\mathbf{p}) \right\} \tag{90}$$

which by (66) is the maximum buying price $B_P([\mathbf{x},\mathbf{p}])$ of the RV $[\mathbf{x},\mathbf{p}]$.

We show now how to compute the optimal premiums $\mathbf{q}$, given the probabilities $\mathbf{p}$ and the coverage $\mathbf{x}$. The Lagrangian of the maximization problem in (90) is

$$L(\mathbf{q},\lambda) \;=\; \sum q_i x_i - \sum p_i \phi\left(\frac{q_i}{p_i}\right) - \lambda\left(\sum q_i - 1\right) \tag{91}$$

where λ is the Lagrange multiplier of the constraint $\sum q_i = 1$. The inequality constraints $q_i \geq 0$ and their Lagrange multipliers are ignored for now.

The necessary optimality conditions

$$\frac{\partial L}{\partial q_i} \;=\; x_i - \phi'\left(\frac{q_i}{p_i}\right) - \lambda \;=\; 0\ , \quad i = 1,\cdots,n\ , \tag{92}$$

are also sufficient because the maximand in (90) is concave in $\mathbf{q}$. To continue, we need:

Lemma 2. Let $P \in \mathcal{P}$ and let $\phi \;:=\; P^*$. Then:

$$\phi(1) \;=\; 0\ , \tag{93a}$$

$$\text{and} \quad \phi(0) \;=\; -\inf P(x)\ . \tag{93b}$$

If the supremum in (65) is attained in $x = x(y)$, then the derivative

$$\phi'(y) \;=\; x(y) \tag{94a}$$

is increasing in $(0,+\infty)$. In particular:

$$\phi'(1) \;=\; 0\ , \tag{94b}$$

$$\lim_{y\to 0^+} \phi'(y) \;=\; -\infty\ . \tag{94c}$$

Proof. (93b) follows from the definition (65) with $y = 0$, and (93a) follows from Assumptions 2(p1),(p3). If the supremum in (65) is attained in $x = x(y)$,

$$\phi(y) \;=\; yx(y) - P(x(y))$$

then

$$\phi'(y) \;=\; x(y) + yx'(y) - P'(x(y))x'(y)$$

which simplifies to (94a), by the Envelope Theorem (see e.g.[17], [19]). The rest follows from Assumptions 2. □

We solve (92) to get

$$q_i \;=\; p_i \left(\phi'\right)^{-1}(x_i - \lambda) \tag{95}$$

where the inverse $(\phi')^{-1}$ exists by Lemma 2. Summing over i we get an equation for λ,

$$1 \;=\; \sum p_i \left(\phi'\right)^{-1}(x_i - \lambda) \tag{96}$$

Since the derivative ϕ' is increasing in $(0,+\infty)$ it follows from Lemma 2 that its inverse $(\phi')^{-1}$ is positive, and increasing in $(-\infty,+\infty)$. Therefore, the premiums q_i in (95) are positive, showing that it was justified to ignore the nonnegativity constraints $q_i \geq 0$.

Also, from the fact that $(\phi')^{-1}$ is increasing, it follows that the ratios $\dfrac{q_i}{p_i}$ of premiums to probabilities increase with the coverage x_i,

$$x_i \;\leq\; x_j \quad \Longrightarrow \quad \frac{q_i}{p_i} \;\leq\; \frac{q_j}{p_j}\ .$$

If the buyer chooses equal income in all states of nature

$$x_1 = x_2 = \cdots = x_n$$

then it follows from (95) that the optimal premiums are actuarially fair

$$q_i = p_i \ , \quad i = 1, \cdots, n \ .$$

Example 8. Consider the quadratic penalty P

$$P(z) \ := \ z + \tfrac{1}{2} z^2 \ , \quad z \geq -1 \tag{26}$$

and the corresponding $\phi = P^*$

$$\phi(t) \ = \ \tfrac{1}{2}\,(t-1)^2 \tag{71}$$

Because of the restriction $z \geq -1$ in (26) the result (94c) is now replaced by

$$\phi'(0) \ = \ -1 \tag{97}$$

and the nonnegativity of the premiums q_i in (95) is no longer guaranteed. To get it we assume the values of x_i satisfy

$$x_i \ \geq \ E\mathbf{X} - 1 \ , \quad i = 1, \cdots, n \tag{98}$$

which is the case, in particular, if all x_i lie in a unit interval.

Since $\phi'(t) \ = \ t - 1$ it follows that $(\phi')^{-1}(x) \ = \ x + 1$, and we get from (96),

$$\lambda \ = \ \sum p_i\, x_i \ .$$

Substituting $\lambda \ = \ E\mathbf{X}$ in (95) we get the premiums

$$q_i \ = \ p_i \ + \ p_i\,(x_i - E\mathbf{X}) \ , \quad i = 1, \cdots, n \tag{99}$$

where $p_i + p_i(x_i - E\mathbf{X}) < 0$ is excluded by the assumption (98). These (normalized) premiums satisfy

$$\begin{aligned} q_i \ &> \ p_i \quad \text{if} \quad x_i \ > \ E\mathbf{X} \ , \\ q_i \ &< \ p_i \quad \text{if} \quad x_i \ < \ E\mathbf{X} \ , \end{aligned}$$

as expected. The divergence $I_\phi(\mathbf{q}, \mathbf{p})$ is, by (72),

$$I_\phi(\mathbf{q}, \mathbf{p}) \ = \ \frac{1}{2} \sum p_i\,(x_i - E\mathbf{X})^2 \ = \ \frac{1}{2}\,\mathrm{Var}\mathbf{X} \ ,$$

giving the optimal value of the coverage as half the income variance.

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