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# THE MOORE OF THE MOORE–PENROSE INVERSE

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ABSTRACT. A summary and restatement, in plain English and modern notation, of the results of E.H. Moore on the generalized inverse that bears his name.

Key words: Moore-Penrose inverse, E. H. Moore, history of mathematics.
Math. subject classification: 01A75, 15A09, 15-03, 01A70

## 1. INTRODUCTION

E. H. Moore (1862–1932) introduced and studied the *general reciprocal* during the decade 1910–1920. He stated the objective as follows:

> "The effectiveness of the reciprocal of a non–singular finite matrix in the study of properties of such matrices makes it desirable to define if possible an analogous matrix to be associated with each finite matrix $\kappa^{12}$ even if $\kappa^{12}$ is not square or, if square, is not necessarily non–singular." [20, p. 197],

Moore constructed the general reciprocal, established its uniqueness and main properties, and justified its application to linear equations. This work appears in [19], [20, Part 1, pp. 197–209].

The general reciprocal was rediscovered by R. Penrose[1] [26] in 1955, and is nowadays called the *Moore–Penrose inverse*. It had to be rediscovered because Moore's work was sinking into oblivion even during his lifetime: it was much too idiosyncratic, and used unnecessarily complicated notation, making it illegible for all but very dedicated readers.

Moore's work on the general reciprocal is summarized below, and – where necessary – restated in plain English and modern notation. To illustrate the difficulty of reading the original Moore, and the need for translation, here is a theorem from [20, Part 1, p. 202]

(29.3) **Theorem**.

$$\mathfrak{U}^C\, \mathfrak{B}^{1\ \mathrm{II}}\, \mathfrak{B}^{2\ \mathrm{II}}\, \kappa^{12} .)\ \cdot$$
$$\exists \,|\, \lambda^{21\ \text{type}\ \mathfrak{M}^2_{\kappa *}\ \overline{\mathfrak{M}}^1_{\kappa}} \ni \cdot\, S^2\, \kappa^{12}\, \lambda^{21} = \delta^{11}_{\mathfrak{M}^1_\kappa} \cdot S^1\, \lambda^{21}\, \kappa^{12} = \delta^{22}_{\mathfrak{M}^2_{\kappa *}}$$

One symbol needs explanation: $\mathfrak{U}$ stands for the *number system* used throughout, and $\mathfrak{U}^C$ denotes a number system of *type* $C$, that is a *quasi-field* with a *conjugate* and an *order relation*, see [20, Part 1, p. 174] for details. All results below are for type $C$ number systems, so this assumption will not be repeated. The rest of the theorem, in plain English, is:

(29.3) **Theorem**. For every matrix $A$ there exists a unique matrix $X : R(A) \to R(A^H)$ such that

$$AX = P_{R(A)}\ ,\ XA = P_{R(A^H)}\ .$$

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[1]Sir Roger Penrose made profound contributions to Physics, Mathematics, Geometry, Philosophy of Science, Artificial Intelligence, and theories of Mind and Consciousness (see, e.g., [7], [9], [11], [28], [29], [30], [31], [32], [33]). The 1988 Wolf Prize in Physics, shared with Stephen W. Hawking, cites their "brilliant development of the theory of general relativity, in which they have shown the necessity for cosmological singularities and have elucidated the physics of black holes. In this work they have greatly enlarged our understanding of the origin and possible fate of the Universe". Penrose was awarded many other prizes and honors, and was knighted in 1994. Sir Roger discovered the Moore–Penrose inverse while a student at Cambridge, and his seminal papers [26], [27] started the field of generalized inverses (AMS subject class 15A09). However, this work pales in comparison with Penrose's other achievements, and is not even mentioned in his available biographies [25]. See also [8], [24].

Here $A^H$ denotes the conjugate transpose of $A$, $R(A)$ the range of $A$, and $P_L$ the orthogonal projector on $L$.

The plan of this paper:

- A sketch of E. H. Moore is given in § 2.
- Section 3 summarizes the results of Moore's lecture to the American Mathematical Society in 1920 [19].
- Section 4 is a translation of the main results in [20, Part 1, pp. 197–209].

## 2. Eliakim Hastings Moore

Eliakim Hastings Moore (1862–1932) was a "true forefather of modern American mathematics", [35, p. 51]. Several comprehensive accounts of Moore's life and work are available, thanks to R. C. Archibald [1], G. A. Bliss ([5], [6]), K. H. Parshall [22], K. H. Parshall and D. E. Rowe [23], R. Siegmund-Schultze [35], and others (see [23, p. 281])[2]. These accounts describe Moore's contributions to mathematics, his stewardship of the Chicago School and service to the American Mathematical Society and its *Transactions*, and how he, his students[3] and others from the Chicago School[4] put American mathematics on the world map.

Reading the Moore story, one sees a brilliant and eclectic mathematician, a strong, principled man, and a very able leader who did much to advance mathematics (at a time when mathematical research had low standing in American académe), skillfully navigating around administrative rocks[5].

Moore made important contributions to Mathematical Analysis, Algebra and Geometry, and in his *General Analysis* ([14], [15], [20]) attempted an ambitious unification, justified by his *principle of generalization by abstraction*

> "The existence of analogies between central features of various theories implies the existence of a more fundamental general theory embracing the special theories as particular instances and unifying them as to those central features," [16, p. 239],

that Barnard put in perspective

> "The striking analogies between the theories for linear equations in $n$–dimensional Euclidean space, for Fredholm integral equations in the space of continuous functions defined on a finite real interval, and for linear equations in Hilbert space of infinitely many dimensions, led Moore to lay down his well–known principle," [20, p. 1].

This effort, which consumed the last 20 years of Moore's life, failed[6], although unification was eventually achieved by others, along different lines. For awhile, several Moore students used *General Analysis* as a framework for the study of linear equations and operators, e.g. [10], and even followed Moore's notation, but these practices were limited and short–lived. Moore's *General Analysis* was rejected, or ignored, by the leading mathematicians of his time, and was soon forgotten. At the end Moore worked alone on his abstract theory, isolated even in his own department[7]. The main treatise of *General Analysis* [20] was published posthumously, by the American Philosophical Society[8], thanks to the skillful editorship of R. W. Barnard, his former student and loyal colleague at Chicago.

Besides the general reciprocal, Moore is remembered today mainly for his contributions to the Theory of Limits ([17], [21]), and to Reproducing Kernels ([18], [20, pp. 186–187]; see also [2, p. 344]).

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[2] An on-line biography appears in [13]

[3] Moore's students included L. E. Dickson (1896), O. Veblen (1903), R. L. Moore (1905, supervised jointly with Veblen), G. D. Birkhoff (1907) and T. H. Hildebrandt (1910). The on–line *Mathematical Genealogy Project* [12] lists 5112 "descendants" (including me), as of March 2002.

[4] Notably O. Bolza and his student G. A. Bliss (1900).

[5] For example, Moore's failure to secure funds from the University of Chicago for publication of the papers read at the 1893 Mathematical Congress (held in Chicago as part of the *World Columbian Exposition*), resulted in the transformation in 1894 of the *New York Mathematical Society* (a regional organization) to the *American Mathematical Society*, see [23, pp. 402–408], [34].

[6] For a "sociological explanation" see [35, pp. 83–85].

[7] See letter of I. J. Schoenberg quoted in [35, p. 56].

[8] Why not the American Mathematical Society? G. A. Bliss cites financial reasons [20, p. iv]. I suspect the reason is that by 1934 (when Barnard finished editing the *General Analysis*) Moore's work was already far removed from the mainstream.

## 3. The 1920 lecture to the AMS, [19]

This is an abstract of a lecture given by E. H. Moore at the Fourteenth Western Meeting of the American Mathematical Society, held at the University of Chicago in April 9–10, 1920. There were 19 lectures in two afternoons; only the abstracts, written by Arnold Dresden (Secretary of the Chicago Section) appear in the *Bulletin.* Dresden writes

> "In this paper Professor Moore calls attention to a useful extension of the classical notion of the reciprocal of a nonsingular square matrix." [19, p. 394].

The details: Let $A$ be any $m \times n$ complex matrix. Then there exists a unique $n \times m$ matrix $A^\dagger$, the *reciprocal* of $A$, such that:

(1) the columns of $A^\dagger$ are linear combinations of the conjugate of the rows of $A$,
(2) the rows of $A^\dagger$ are linear combinations of the conjugate of the columns of $A$,
(3) $AA^\dagger A = A$.

If $A$ is of rank $r$, then $A^\dagger$ is given explicitly as follows:

$(r \geq 2)$:

$$A^\dagger[j_1, i_1] = \frac{\displaystyle\sum_{\substack{i_2 < \cdots < i_r \\ j_2 < \cdots < j_r}} A\begin{pmatrix} i_2 \cdots i_r \\ j_2 \cdots j_r \end{pmatrix} \overline{A\begin{pmatrix} i_1 \; i_2 \cdots i_r \\ j_1 \; j_2 \cdots j_r \end{pmatrix}}}{\displaystyle\sum_{\substack{k_1 < \cdots < k_r \\ \ell_1 < \cdots < \ell_r}} A\begin{pmatrix} k_1 \cdots k_r \\ \ell_1 \cdots \ell_r \end{pmatrix} \overline{A\begin{pmatrix} k_1 \cdots k_r \\ \ell_1 \cdots \ell_r \end{pmatrix}}} \,,$$

$(r = 1)$:

$$A^\dagger[j, i] = \frac{\overline{A[i,j]}}{\sum_{k\ell} A[k,\ell]\overline{A[k,\ell]}} \,,$$

$(r = 0)$:

$$A^\dagger[j, i] = 0 \,,$$

where $A\begin{pmatrix} g_1 \cdots g_k \\ h_1 \cdots h_k \end{pmatrix}$ denotes the determinant of the $k^2$ numbers $A[g_i, h_j]$ and $\overline{x}$ denotes the conjugate of $x$.

The relation between $A$ and $A^\dagger$ is mutual: $A$ is the reciprocal of $A^\dagger$, viz.,
(4),(5): the columns (rows) of $A$ are linear combinations of the conjugates of rows (columns) of $A^\dagger$,
(6) $A^\dagger A A^\dagger = A^\dagger$.

The linear combinations of the columns of $A$ $(A^\dagger)$ are the linear combinations of the rows of $A^\dagger$ $(A)$ and constitute the $m$–dimensional vectors $\mathbf{y}$ ($n$–dimensional vectors $\mathbf{x}$) of an $r$–dimensional subspace $M$ $(N)$ of $\mathbb{C}^m$ $(\mathbb{C}^n)$. Let $\overline{M}$ $(\overline{N})$ denote the conjugate space of the conjugate vectors $\overline{\mathbf{y}}$ $(\overline{\mathbf{x}})$. Then the matrices $A, A^\dagger$ establish 1–1 linear vector correspondences between the spaces $M, \overline{M}$ and the respective subspaces $N, \overline{N}$; $\mathbf{y} = A\mathbf{x}$ is equivalent to $\mathbf{x} = A^\dagger \mathbf{y}$ and $\overline{\mathbf{x}} = \overline{\mathbf{y}}A$ is equivalent to $\overline{\mathbf{u}} = \overline{\mathbf{v}}A^\dagger$.

## 4. The general reciprocal in *General Analysis* [20]

The centerpiece of Moore's work on the general reciprocal is Section 29 of [20], his treatise on *General Analysis*, edited by R. W. Barnard and published posthumously. These results were since rediscovered, some more than once.

For a matrix $A$ and index sets $I, J$,
$A_{I*}$ (or $A[I, *]$) denotes the submatrix of rows indexed by $I$,
$A_{*J}$ or $A[*, J]$ the submatrix of columns indexed by $J$, and

$A_{IJ}$ denotes the submatrix of $A$ with rows in $I$ and columns in $J$.
If $A$ is non-singular, its inverse $A^{-1}$ satisfies

$$AX = I,\ XA = I\ .$$

Moore begins by constructing *generalized identity matrices* to replace the identity matrices above. This is done in Lemma (29.1) and Theorem (29.2). The *general reciprocal* is then constructed in Theorems (29.3) and (29.4), and its properties are studied in the sequel.

(29.1) **Lemma**.
Let $A$ be a non–zero $m \times n$ matrix, and let $A_{IJ}$ be a maximal non–singular submatrix of $A$.
(1) $A^H_{*J} A_{*J}$ is Hermitian, positive–definite[9].
(2) $\left(A^H_{*J} A_{*J}\right)^{-1}$ is Hermitian, positive–definite.
(3) $A_{I*} A^H_{I*}$ is Hermitian, positive–definite.
(4) $\left(A_{I*} A^H_{I*}\right)^{-1}$ is Hermitian, positive–definite.
(5) $P_{R(A)} := A_{*J} \left(A^H_{*J} A_{*J}\right)^{-1} A^H_{*J}$ (the generalized identity on $R(A)$).
(6) $P_{R(A^H)} := A^H_{I*} \left(A_{I*} A^H_{I*}\right)^{-1} A_{I*}$ (the generalized identity on $R(A^H)$).
(7) $P_{R(A)}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in R(A)$ .
(8) $\mathbf{x}^H P_{R(A)} = \mathbf{x}^H$ for all $\mathbf{x} \in R(A)$ .
(9) $P_{R(A^H)}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in R(A^H)$ .
(10) $\mathbf{x}^H P_{R(A^H)} = \mathbf{x}^H$ for all $\mathbf{x} \in R(A^H)$ .
(11) Let

$$\begin{aligned} X &:= A^H_{I*} \left(A_{I*} A^H_{I*}\right)^{-1} A_{IJ} \left(A^H_{*J} A_{*J}\right)^{-1} A^H_{*J} \\ &= A^H_{I*} \left(A^H_{I*} A_{I*}\right)^{-1} P_{R(A)}[I,*] \\ &= P_{R(A^*)}[*,J] \left(A^H_{*J} A_{*J}\right)^{-1} A^H_{*J}\ , \quad \text{(the \textit{general reciprocal} of } A)\ . \end{aligned}$$

(12) $X$ maps $R(A^H)$ onto $R(A)$ .
(13) $AX = P_{R(A)}$ .
(14) $XA = P_{R(A^H)}$ .

(29.2) **Theorem**.
Let $M$ be a finite dimensional subspace.
(1) There exists a unique linear operator[10] $P_M$ such that

$$P_M \mathbf{x} = \mathbf{x},\ \mathbf{x}^H P_M = \mathbf{x}^H\ , \quad \text{for all } \mathbf{x} \in M\ .$$

(2) $P_M$ is positive semidefinite, Hermitian and idempotent.
(3) $M = R(P_M)$ .
(4) For all $\mathbf{x}$: $P_M\mathbf{x} \in M$ , $(\mathbf{x} - P_M\mathbf{x}) \in M^\perp$ .
(5) $\mathbf{x} \perp M \iff P_M\mathbf{x} = \mathbf{0}$ .
(6) For any matrix $A$

$$\begin{aligned} A = P_M &\iff \begin{cases} A\mathbf{x} = \mathbf{x}\ , \text{ for all } \mathbf{x} \in M \\ R(A^H) \subset M \end{cases} \\ &\iff \begin{cases} A\mathbf{x} = \mathbf{x}\ , \text{ for all } \mathbf{x} \in M \\ A\mathbf{x} = \mathbf{0}\ , \text{ for all } \mathbf{x} \in M^\perp \end{cases} . \end{aligned}$$

(29.3) **Theorem**.
For every matrix $A$ there exists a unique matrix $X : R(A) \to R(A^H)$ such that

$$AX = P_{R(A)}\ ,\ XA = P_{R(A^H)}\ .$$

[9]Moore calls it *proper* (i.e., the determinants of all principal minors are non–zero), *positive* (i.e., the corresponding quadratic form is non–negative) and Hermitian.
[10]Called the *generalized identity matrix* for the space $M$, and denoted by $\delta_M$, [20, p. 199].

We call $X$ the *general reciprocal* and denote it by $A^\dagger$.

(29.4) **Theorem**.
For every matrix $A$ the general reciprocal $A^\dagger$ satisfies:
(1) $A^\dagger A A^\dagger = A^\dagger$, $AA^\dagger A = A$ .
(2) $\operatorname{rank} A = \operatorname{rank} A^\dagger$ .
(3) $R(A) = R(A^{\dagger H})$ , $R(A^H) = R(A^\dagger)$ .
(4) $A^{\dagger H} = (A^H)^\dagger$ , $A = (A^\dagger)^\dagger$ .

(29.45) **Corollary**.
If $A[I,J]$ is a maximal nonsingular submatrix of $A$ then:
(1) $A^\dagger = P_{R(A^H)}[*,J]A_{IJ}^{-1}P_{R(A)}[I,*]$ .
(2) $\mathbf{x}^H A^\dagger \mathbf{y} = \mathbf{x}_I^H A_{IJ}^{-1}\mathbf{y}_J$ .

(29.5) **Theorem**.
For any matrix $A$, the following statements on a matrix $X$ are equivalent:
(1) $X = A^\dagger$
(2) $R(X) \subset R(A^H)$ , $AX = P_{R(A)}$
(3) $R(X) \subset R(A^H)$ , $R(X^H) \subset R(A)$ , $AXA = A$ .

(29.55) **Corollary**.
If $A = \begin{bmatrix} B & O \\ O & C \end{bmatrix}$ then $A^\dagger = \begin{bmatrix} B^\dagger & O \\ O & C^\dagger \end{bmatrix}$ .

(29.6) **Theorem**.
Let the matrix $A$ be Hermitian. Then
(1) $A^\dagger$ is Hermitian.
(2) If $A$ is positive semi–definite then so is $A^\dagger$. □

Consider a square matrix $A$. Then for any principal submatrix $A_{II}$,

$$A_{II} = A_{II}A_{II}^\dagger A_{II}$$

More can be said if $A$ is Hermitian positive semi–definite:

(29.7) **Theorem**.
Let $A$ be Hermitian positive semi–definite. Then for any prinicipal submatrix $A_{II}$
(1) $A_{II}A_{II}^\dagger A_{I*} = A_{I*}$ .
(2) $A_{*I}A_{II}^\dagger A_{II} = A_{*I}$ .

(29.8) **Theorem**.
Let $A$ be Hermitian positive semi–definite. Then the following statements, about a vector $\mathbf{x}$, are equivalent.
(1) $\mathbf{x}^H A\mathbf{x} = 0$ ,
(2) $\mathbf{x} \perp R(A)$ ,
(3) $\mathbf{x} \perp R(A^\dagger)$ ,
(4) $\mathbf{x}^H A^\dagger \mathbf{x} = 0$ .

The general reciprocal can be used to solve linear equations

$$A\mathbf{x} = \mathbf{b} ,$$

that are assumed consistent, i.e. $\mathbf{b} \in R(A)$, or the way Moore expresses consistency: $\operatorname{rank} A = \operatorname{rank} \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$.

(29.9) **Theorem**.
Let $A$ be a matrix, $\mathbf{b}$ a vector in $R(A)$. Then the general solution of $A\mathbf{x} = \mathbf{b}$ is

$$A^\dagger \mathbf{b} + \{\mathbf{y} : \mathbf{y} \perp R(A^H)\} .$$

Note: Moore avoids the concept of null–space, and the equivalent form of the general solution, $A^\dagger \mathbf{b} + N(A)$. Also, Moore does not consider the case where $A\mathbf{x} = \mathbf{b}$ is inconsistent. A. Bjerhammar [4], R. Penrose [27] and Yuan–Yung Tseng[11] [36] would later use $A^\dagger$ to obtain least–squares solutions. This has become the major application of the Moore–Penrose inverse.

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