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Minors of the Moore-Penrose Inverse *

Jianming Miao and Adi Ben-Israel †

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Abstract

Let $Q_{k,n} = \{\alpha = (\alpha_1, \cdots, \alpha_k) : 1 \le \alpha_1 < \cdots < \alpha_k \le n\}$ denote the strictly increasing sequences of k elements from $1, \ldots, n$. For $\alpha, \beta \in Q_{k,n}$ we denote by $A[\alpha, \beta]$ the submatrix of A with rows indexed by α, columns by β. The submatrix obtained by deleting the α-rows and β-columns is denoted by $A[\alpha', \beta']$.

For nonsingular $A \in \mathbb{R}^{n\times n}$, the **Jacobi identity** relates the **minors** of the inverse A^{-1} to those of A:

$$\det A^{-1}[\beta, \alpha] = (-1)^{\sum_{i=1}^{k} \alpha_i + \sum_{i=1}^{k} \beta_i} \frac{\det A[\alpha', \beta']}{\det A}$$

for any $\alpha,\ \beta \in Q_{k,n}$.

We generalize Jacobi's identity to matrices $A \in \mathbb{R}_r^{m\times n}$, expressing the minors of the **Moore-Penrose inverse** $A^\dagger$ in terms of the minors of the maximal nonsingular submatrices A_{IJ} of A. In our notation,

$$\det A^\dagger[\beta, \alpha] = \frac{1}{\mathrm{vol}^2 A} \sum_{(I,J)\in\mathcal{N}(\alpha,\beta)} \det A_{IJ} \frac{\partial}{\partial |A_{\alpha\beta}|} |A_{IJ}|,$$

for any $\alpha \in Q_{k,m}$, $\beta \in Q_{k,n}$, $1 \le k \le r$, where $\mathrm{vol}^2 A$ denotes the sum of squares of determinants of $r \times r$ submatrices of A. This represents the $k \times k$ minors of $A^\dagger$ as a convex combination of the minors of A. The weights of this combination are, surprisingly, the same for all k.

We apply our results to questions concerning the nonnegativity of principal minors of the Moore-Penrose inverse.

Key words: Minors. Determinants. Moore-Penrose inverse. P-matrices. M-matrices.

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†RUTCOR-Rutgers Center for Operations Research, Rutgers University, P.O. Box 5062
New Brunswick, NJ 08903-5062, U.S.A.

1 Introduction

If the matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular, then the adjoint formula for its inverse

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A, \tag{1.1}$$

has a well-known generalization, the **Jacobi identity**, which relates the **minors** of A^{-1} to those of A.

Denote the set of strictly increasing sequences of k elements from $1, \ldots, n$, by

$$Q_{k,n} = \{\alpha = (\alpha_1, \cdots, \alpha_k) \ : \ 1 \le \alpha_1 < \cdots < \alpha_k \le n\} \tag{1.2}$$

For $\alpha,\ \beta \in Q_{k,n}$, denote by:

$A[\alpha, \beta]$ the submatrix of A having row indices α and column indices β,

$A[\alpha', \beta']$ the submatrix obtained from A by deleting rows indexed by α and columns indexed by β.

Then the Jacobi identity (see [7]) is: For any $\alpha,\ \beta \in Q_{k,n}$,

$$\det A^{-1}[\beta, \alpha] = (-1)^{s(\alpha)+s(\beta)} \frac{\det A[\alpha',\ \beta']}{\det A}, \tag{1.3}$$

where $s(\alpha)$ is the sum of the integers in α. By convention,

$$\det A[\emptyset,\ \emptyset] = 1. \tag{1.4}$$

For $A \in \mathbb{R}_r^{m\times n}$, Moore [11] gave a determinantal formula for the entries of the **Moore-Penrose inverse** $A^\dagger$, a formula recently rediscovered by Berg [4]. The result was further generalized to matrices defined over an integral domain [1]. We consider here the minors of $A^\dagger$, for $A \in \mathbb{R}_r^{m\times n}$. Theorem 1 (in § 2) expresses them in terms of the minors of the maximal nonsingular submatrices A_{IJ} of A. A numerical example is given in §3. Theorem 2 (in § 4) is a somewhat surprising result: Every minor of $A^\dagger$ is the same convex combination of the corresponding minors of inverses of the A_{IJ}'s. This generalizes Berg's representation [4] of $A^\dagger$ as a convex combination of the A_{IJ}'s. Section 5 deals with the nonnegativity of principal minors of the Moore-Penrose inverse, extending some previous results of Mohan, Neumann and Ramamurthy [10], [12].

We use the following notation. For any index sets $I,\ J$, let $A_{I*},\ A_{*J},\ A_{IJ}$ denote the submatrices of A lying in rows indexed by I, in columns indexed by J, and in their intersection, respectively. The principal submatrix A_{JJ} is denoted by A_J. For $A \in \mathbb{R}_r^{m\times n}$, let

$$\begin{aligned}
\mathcal{I}(A) &= \{I \in Q_{r,m} \ : \ \operatorname{rank} A_{I*} = r\}, \\
\mathcal{J}(A) &= \{J \in Q_{r,n} \ : \ \operatorname{rank} A_{*J} = r\}, \\
\mathcal{N}(A) &= \{(I,\ J) \in Q_{r,m} \times Q_{r,n} \ : \ \operatorname{rank} A_{IJ} = r\},
\end{aligned}$$

be the index sets of maximal sets of linearly independent rows and columns, and of maximal nonsingular submatrices, respectively. For $\alpha \in Q_{k,m}$, $\beta \in Q_{k,n}$ let

$$\begin{aligned}
\mathcal{I}(\alpha) &= \{I \in \mathcal{I}(A) \ : \ \alpha \subseteq I\}, \\
\mathcal{J}(\beta) &= \{J \in \mathcal{J}(A) \ : \ \beta \subseteq J\}, \\
\mathcal{N}(\alpha, \beta) &= \{(I,\ J) \in \mathcal{N}(A) \ : \ \alpha \subseteq I,\ \beta \subseteq J\}.
\end{aligned}$$

Then by [2]

$$\mathcal{N}(A) = \mathcal{I}(A) \times \mathcal{J}(A),$$

and therefore,

$$\mathcal{N}(\alpha, \beta) = \mathcal{I}(\alpha) \times \mathcal{J}(\beta). \tag{1.5}$$

For $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$, we denote by

$$A[\beta \leftarrow \mathbf{I}_\alpha] \tag{1.6}$$

the matrix obtained from A by replacing the ${\beta_i}^{th}$ column with the unit vector $\mathbf{e}_{\alpha_i}$, $(i = 1, \ldots, k)$, and by

$$A[\beta \leftarrow \mathbf{0}] \tag{1.7}$$

the matrix obtained from A by replacing the ${\beta_i}^{th}$ column with the zero vector $\mathbf{0}$, $(i = 1, \ldots, k)$.

Finally, the coefficient $(-1)^{s(\alpha)+s(\beta)} \det A[\alpha',\ \beta']$, of $\det A[\alpha, \beta]$ in the Laplace expansion of $\det A$ is denoted by

$$\frac{\partial}{\partial |A_{\alpha\beta}|} |A| \ . \tag{1.8}$$

Using the above notation we rewrite (1.8) as

$$\frac{\partial}{\partial |A_{\alpha\beta}|} |A| = (-1)^{s(\alpha)+s(\beta)} \det A[\alpha',\ \beta'] = \det A[\beta \leftarrow \mathbf{I}_\alpha] \ , \tag{1.9}$$

and the Jacobi identity as

$$\begin{aligned}
\det A^{-1}[\beta, \alpha] &= \frac{\det A[\beta \leftarrow \mathbf{I}_\alpha]}{\det A} \ , & (1.10)\\
&= \frac{1}{\det A^T A} \det A^T \cdot A[\beta \leftarrow \mathbf{I}_\alpha] \ . & (1.11)
\end{aligned}$$

As in [2], we define the **volume** of the $m \times n$ matrix A by,

$$\operatorname{vol} A = \sqrt{\sum_{(I,\ J) \in \mathcal{N}(A)} \det{}^2 A_{IJ}} \ , \tag{1.12}$$

and in particular,

$$\operatorname{vol} A = \sqrt{\det(A^T A)}, \ \ \text{if } A \text{ has full column rank.} \tag{1.13}$$

The following lemma is used in the sequel:

Lemma 1 (Blattner, [6]) Let $A \in \mathbb{R}_r^{m\times n}$, and let $U \in \mathbb{R}^{m\times(m-r)}$ and $V \in \mathbb{R}^{n\times(n-r)}$ be matrices whose columns form orthonormal bases of $N(A^T)$ and $N(A)$, respectively. Then

$$B = \begin{pmatrix} A & U \\ V^T & \mathrm{O} \end{pmatrix} \tag{1.14}$$

is nonsingular, and its inverse is

$$B^{-1} = \begin{pmatrix} A^\dagger & V \\ U^T & \mathrm{O} \end{pmatrix} \tag{1.15}$$

If A has full column [row] rank, then V [U] is vacant. Moreover, by [2],

$$\det B^T B = \mathrm{vol}^2 A \ . \tag{1.16}$$

2 Minors of the Moore–Penrose inverse

Theorem 1 Let $A \in \mathbb{R}_r^{m\times n}$, and $1 \le k \le r$. Then for any $\alpha \in Q_{k,m}$, $\beta \in Q_{k,n}$,

$$\det A^\dagger[\beta,\alpha] = \begin{cases} 0, \text{ if } \mathcal{N}(\alpha,\beta) = \emptyset \ , \\ \dfrac{1}{\mathrm{vol}^2 A} \displaystyle\sum_{(I,J)\in\mathcal{N}(\alpha,\beta)} \det A_{IJ} \frac{\partial}{\partial|A_{\alpha\beta}|}|A_{IJ}| \\ \qquad\qquad \text{otherwise} \ . \end{cases} \tag{2.1}$$

Proof. Let B, U, V be as in Lemma 1. Then

$$\begin{aligned} \det A^\dagger[\beta,\alpha] &= \det B^{-1}[\beta,\ \alpha], \qquad \text{by Lemma 1,} \\ &= \frac{1}{\det B^T B} \det B^T \cdot B[\beta \leftarrow \mathbf{I}_\alpha] \ , \end{aligned} \tag{2.2}$$

by (1.11). Now $\det B^T \cdot B[\beta \leftarrow \mathbf{I}_\alpha] \ =$

$$\begin{aligned} &= \det \begin{pmatrix} A^T & V \\ U^T & \mathrm{O} \end{pmatrix} \begin{pmatrix} A[\beta \leftarrow \mathbf{I}_\alpha] & U \\ V^T[\beta \leftarrow \mathbf{0}] & \mathrm{O} \end{pmatrix} \\ &= \det(A^T,\ V) \begin{pmatrix} A[\beta \leftarrow \mathbf{I}_\alpha] \\ V^T[\beta \leftarrow \mathbf{0}] \end{pmatrix} \\ &= \sum_{I\in\mathcal{I}(A)} \det((A^T)_{*I},\ V) \det \begin{pmatrix} A[\beta \leftarrow \mathbf{I}_\alpha]_{I*} \\ V^T[\beta \leftarrow \mathbf{0}] \end{pmatrix} \\ &= \sum_{I\in\mathcal{I}(\alpha)} \det((A_{I*})^T,\ V) \det \begin{pmatrix} A_{I*}[\beta \leftarrow \mathbf{I}_\alpha] \\ V^T[\beta \leftarrow \mathbf{0}] \end{pmatrix} \end{aligned} \tag{2.3}$$

The penultimate equality is by the Cauchy-Binet formula, noting that the determinant of any $n \times n$ submatrix of $(A^T,\ V) \in \mathbb{R}^{n\times(m+n-r)}$ is zero if it consists of more than r columns of A^T. The last equality holds since the matrix $\begin{pmatrix} A[\beta \leftarrow \mathbf{I}_\alpha]_{I*} \\ V^T[\beta \leftarrow \mathbf{0}] \end{pmatrix}$ has at least one column of zeros, if $I \notin \mathcal{I}(\alpha)$.

We assume now (and prove later) that for any fixed $I \in \mathcal{I}(\alpha)$,

$$\begin{aligned} &\det((A_{I*})^T,\ V) \det \begin{pmatrix} A_{I*}[\beta \leftarrow \mathbf{I}_\alpha] \\ V^T[\beta \leftarrow \mathbf{0}] \end{pmatrix} = \\ &\qquad = \sum_{J\in\mathcal{J}(\beta)} \det A_{IJ} \det A_{IJ}[\beta \leftarrow \mathbf{I}_\alpha] \ . \end{aligned} \tag{2.4}$$

Then using (1.16) and (2.3), (2.2) becomes

$$\det A^\dagger[\beta,\alpha] \ =$$

$$= \frac{1}{\mathrm{vol}^2 A} \sum_{I\in\mathcal{I}(\alpha)} \ \sum_{J\in\mathcal{J}(\beta)} \det A_{IJ} \det A_{IJ}[\beta \leftarrow \mathbf{I}_\alpha] \tag{2.5}$$

$$= \frac{1}{\mathrm{vol}^2 A} \sum_{(I,\ J)\in\mathcal{N}(\alpha,\beta)} \det A_{IJ} \frac{\partial}{\partial|A_{\alpha\beta}|}|A_{IJ}| \ , \tag{2.6}$$

by (1.9). Finally we prove (2.4). For any fixed $I \in \mathcal{I}(\alpha)$, the columns of V form also an orthonormal basis of $N(A_{I*})$. Let

$$L = \begin{pmatrix} A_{I*} \\ V^T \end{pmatrix} \tag{2.7}$$

Then

$$\begin{aligned} &\det(A_{I*})^\dagger[\beta,\alpha] \ = \\ &= \det L^{-1}[\beta,\alpha] \ , \qquad \text{by Lemma 1,} \\ &= \frac{1}{\det LL^T} \det L^T \cdot L[\beta \leftarrow \mathbf{I}_\alpha] \ , \qquad \text{by (1.11),} \\ &= \frac{1}{\mathrm{vol}^2 A_{I*}} \det((A_{I*})^T,\ V) \det \begin{pmatrix} A_{I*}[\beta \leftarrow \mathbf{I}_\alpha] \\ V^T[\beta \leftarrow \mathbf{0}] \end{pmatrix} \end{aligned} \tag{2.8}$$

Writing $(A_{I*})^T = C$, so that,

$$\begin{aligned} \det(A_{I*})^\dagger[\beta,\alpha] &= \det(C^\dagger)^T[\beta,\alpha] \ , \\ &= \det C^\dagger[\alpha,\beta] \ , \end{aligned} \tag{2.9}$$

we take W to be a matrix whose columns form an orthonormal basis of $N(C^T)$, and denote,

$$M = (C,\ W) \ . \tag{2.10}$$

Then (2.9) becomes, by Lemma 1 and (1.11),

$$\begin{aligned} &\det(A_{I*})^\dagger[\beta,\alpha] \ = \\ &= \frac{1}{\det M^T M} \det M^T \cdot M[\alpha \leftarrow \mathbf{I}_\beta] \ , \\ &= \frac{1}{\mathrm{vol}^2 A_{I*}} \det A_{I*} \cdot (A_{I*})^T[\alpha \leftarrow \mathbf{I}_\beta] \ , \\ &= \frac{1}{\mathrm{vol}^2 A_{I*}} \sum_{J\in\mathcal{J}(\beta)} \det A_{IJ} \det(A_{IJ})^T[\alpha \leftarrow \mathbf{I}_\beta] \ , \\ &= \frac{1}{\mathrm{vol}^2 A_{I*}} \sum_{J\in\mathcal{J}(\beta)} \det A_{IJ} \det A_{IJ}[\beta \leftarrow \mathbf{I}_\alpha] \end{aligned} \tag{2.11}$$

The penultimate equality is by the Cauchy-Binet formula, noting that, if $J \notin \mathcal{J}(\beta)$, then the submatrix of $(A_{I*})^T[\alpha \leftarrow \mathbf{I}_\beta]$ whose rows are indexed by J has at least one column of zeros. Finally, (2.4) follows by comparing (2.8) and (2.11). □

Note that $\mathcal{N}(\alpha,\beta) = \emptyset$ is equivalent to linear dependence of either the columns of $A_{*\beta}$ or the rows of $A_{\alpha*}$.

As a special case, if $\alpha = I \in \mathcal{I}(A)$, $\beta = J \in \mathcal{J}(A)$, then $\mathcal{N}(\alpha,\beta)$ contains only one element, i.e., (I,J). Now Theorem 1 gives the identity, [2],

$$\det(A^\dagger)_{JI} = \frac{1}{\mathrm{vol}^2 A} \det A_{IJ} \ , \quad \forall (I,J) \in \mathcal{N}(A) \ . \tag{2.12}$$

3 Example

Consider the 4×4 matrix A of rank 3,

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} ,$$

and its Moore-Penrose inverse $A^\dagger$,

$$A^\dagger = \frac{1}{15} \begin{pmatrix} -5 & 0 & 20 & 25 \\ 0 & -6 & 0 & 0 \\ 5 & 0 & -5 & -10 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

A list of the 3×3 nonsingular submatrices of A and their determinants is as follows :

I	J	A_{IJ}	$\det A_{IJ}$
$1,2,3$	$1,2,3$	$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$	2
$1,2,3$	$1,3,4$	$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$	1
$1,2,4$	$1,2,3$	$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	2
$1,2,4$	$1,3,4$	$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	1
$2,3,4$	$1,2,3$	$\begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}$	-2
$2,3,4$	$1,3,4$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix}$	-1

The volume of A is given by

$$\mathrm{vol}^2 A = 2^2 + 1 + 2^2 + 1 + 2^2 + 1 = 15$$

Take now $\alpha = \{2,3\}$ and $\beta = \{1,4\}$. Then $\mathcal{N}(\alpha,\beta) = \mathcal{I}(\alpha) \times \mathcal{J}(\beta) = \{I_1,\ I_2\} \times \{J\}$, where $I_1 = \{1,2,3\},\ I_2 = \{2,3,4\}$, and $J = \{1,3,4\}$. We calculate

$$\frac{\partial}{\partial |A_{\alpha\beta}|} |A_{I_1 J}| = (-1)^{(2+3)+(1+3)}\, 3 = -3,$$

and

$$\frac{\partial}{\partial |A_{\alpha\beta}|} |A_{I_2 J}| = (-1)^{(1+2)+(1+3)} (-1) = 1.$$

Now from (2.1)

$$\det A^\dagger[\beta,\alpha] = \frac{1}{15}(1 \times (-3) + (-1) \times 1) = -\frac{4}{15} .$$

4 Convex decomposition of a matrix and its minors

Berg [4] proved that the Moore-Penrose inverse of $A \in \mathbb{R}_r^{m \times n}$ is a convex combination of ordinary inverses of $r \times r$ submatrices

$$A^\dagger = \sum_{(I,J) \in \mathcal{N}(A)} \lambda_{IJ}\ \widehat{A_{IJ}^{-1}} \tag{4.1}$$

where each $\widehat{A_{IJ}^{-1}}$ is an $n \times m$ matrix with the inverse of A_{IJ} in position (J, I) and zeros elsewhere, and

$$\lambda_{IJ} = \frac{\det^2 A_{IJ}}{\mathrm{vol}^2 A} \tag{4.2}$$

By summing (4.1) over $I \in \mathcal{I}(A)$, one obtains $A^\dagger$ as a convex combination of the Moore–Penrose inverses of maximal full column-rank submatrices A_{*J}, see [2],

$$A^\dagger = \sum_{J \in \mathcal{J}(A)} \lambda_{*J}\ \widehat{A_{*J}^\dagger} , \tag{4.3}$$

where the convex weights are

$$\lambda_{*J} = \frac{\mathrm{vol}^2 A_{*J}}{\mathrm{vol}^2 A} , \tag{4.4}$$

and $\widehat{A_{*J}^\dagger}$ is an $n \times m$ matrix with $A_{*J}^\dagger$ in rows indexed by J and zeros elsewhere.

Similarly, summing (4.1) over $J \in \mathcal{J}(A)$ gives

$$A^\dagger = \sum_{I \in \mathcal{I}(A)} \lambda_{I*}\ \widehat{A_{I*}^\dagger} \tag{4.5}$$

with convex weights

$$\lambda_{I*} = \frac{\mathrm{vol}^2 A_{I*}}{\mathrm{vol}^2 A}. \tag{4.6}$$

and $\widehat{A_{I*}^\dagger}$ the $n \times m$ matrix with $A_{I*}^\dagger$ in columns indexed by I and zeros elsewhere.

Theorem 1 allows a stronger claim than (4.1), i. e. , every minor of $A^\dagger$ in position (β, α) is the same convex combination of the minors of $\widehat{A_{IJ}^{-1}}$'s in the corresponding position:

Theorem 2 Let $A \in \mathbb{R}_r^{m\times n}$, and $1 \le k \le r$. Then for any $\alpha \in Q_{k,m}$, $\beta \in Q_{k,n}$,

$$\det A^\dagger[\beta,\alpha] = \sum_{(I,J)\in\mathcal{N}(A)} \lambda_{IJ} \ \det \widehat{A_{IJ}^{-1}}[\beta,\alpha] \ . \qquad (4.7)$$

Proof. From Theorem 1, it follows that

$$\begin{aligned}\det A^\dagger[\beta,\alpha] &= \sum_{(I,\ J)\in\mathcal{N}(\alpha,\beta)} \frac{\det^2 A_{IJ}}{\operatorname{vol}^2 A}\cdot\frac{\det A_{IJ}[\beta\leftarrow \mathbf{I}_\alpha]}{\det A_{IJ}} \ , \\ &= \sum_{(I,J)\in\mathcal{N}(\alpha,\beta)} \lambda_{IJ} \ \det \widehat{A_{IJ}^{-1}}[\beta,\alpha] \ ,\end{aligned}$$

by (1.10). We prove (4.7) by showing that the sum over $\mathcal{N}(\alpha,\beta)$ is the same as the sum over the larger set $\mathcal{N}(A)$. Indeed, if $(I,\ J)\in\mathcal{N}(A)$, and either $I\notin\mathcal{I}(\alpha)$ or $J\notin\mathcal{J}(\beta)$, then there is at least one column, or row, of zeros in $\widehat{A_{IJ}^{-1}}[\beta,\alpha]$, thus $\det \widehat{A_{IJ}^{-1}}[\beta,\alpha]=0$. □

Using the same argument we can show that summing (4.7) over $I\in\mathcal{I}(\alpha)$ gives the same sum as summing over $I\in\mathcal{I}(A)$. Similarly, summing over $J\in\mathcal{J}(\beta)$ and over $J\in\mathcal{J}(A)$ give the same result. We summarize these observations in:

Corollary 1 Let $A \in \mathbb{R}_r^{m\times n}$, and $1 \le k \le r$. Then, for any $\alpha \in Q_{k,m}$, $\beta \in Q_{k,n}$,

$$\det A^\dagger[\beta,\alpha] = 0 \ , \qquad \text{if} \qquad \mathcal{J}(\beta)=\emptyset \qquad \text{or} \qquad \mathcal{I}(\alpha)=\emptyset \ , \qquad (4.8)$$

and otherwise,

$$\det A^\dagger[\beta,\alpha] \ =$$

$$= \sum_{J\in\mathcal{J}(A)} \lambda_{*J} \ \det \widehat{A_{*J}^{\dagger}}[\beta,\alpha] = \sum_{J\in\mathcal{J}(\beta)} \lambda_{*J} \ \det A_{*J}^{\dagger}[\beta,\alpha] \ , \qquad (4.9)$$

$$= \sum_{I\in\mathcal{I}(A)} \lambda_{I*} \ \det \widehat{A_{I*}^{\dagger}}[\beta,\alpha] = \sum_{I\in\mathcal{I}(\alpha)} \lambda_{I*} \ \det A_{I*}^{\dagger}[\beta,\alpha] \ . \qquad (4.10)$$

□

By applying Berg's formula to $A^\dagger$, it follows from (2.12) that the same weights appear in the convex decomposition of A into ordinary inverses of the submatrices $(A^\dagger)_{JI}$,

$$A = \sum_{(I,J)\in\mathcal{N}(A)} \lambda_{IJ} \, \widehat{(A^\dagger)_{JI}^{-1}} \ , \qquad (4.11)$$

where $\widehat{(A^\dagger)_{JI}^{-1}}$ is the $m\times n$ matrix with the inverse of the (J,I)th submatrix of $A^\dagger$ in position (I,J) and zeros elsewhere.

Finally applying (4.7) to $A^\dagger$, we establish a remarkable property of the convex decomposition (4.11) of A: Every minor of A is the same convex combination of the minors of $\widehat{(A^\dagger)_{JI}^{-1}}$'s.

Theorem 3 Let $A \in \mathbb{R}_r^{m\times n}$, $r>0$. Then there is a convex decomposition of A

$$A = \sum_{(I,J)\in\mathcal{N}(A)} \lambda_{IJ}\, B_{IJ} \qquad (4.12)$$

such that for all $k=1,\ldots,r$, and for every $\alpha\in Q_{k,m}$, $\beta\in Q_{k,n}$,

$$\det A[\alpha,\beta] = \sum_{(I,J)\in\mathcal{N}(A)} \lambda_{IJ} \det B_{IJ}[\alpha,\beta] \qquad (4.13)$$

where B_{IJ} is an $m\times n$ matrix with a $r\times r$ nonsingular matrix in position (I,J), zeros elsewhere. □

5 Nonnegativity of principal minors of the Moore-Penrose inverse

Let $\mathbf{P}$ [$\mathbf{P}_0$] denote the real $n\times n$ matrices with **positive** [**nonnegative**] **principal minors**. We study conditions under which the Moore-Penrose inverse of a matrix is a P_0-matrix.

If A is nonsingular, then it is immediate from (1.3) that $A\in\mathbf{P}$ if and only if $A^\dagger = A^{-1}\in\mathbf{P}$. If $A\in\mathbb{R}_r^{n\times n}$, then by (2.12) it is necessary $A^\dagger\in\mathbf{P}_0$ that $\det A_J\ge 0$, $\forall J\in Q_{r,n}$. It is known that $A\in\mathbf{P}_0$ does not imply $A^\dagger\in\mathbf{P}_0$. Mohan, Neumann and Ramamurthy [10] proved that the Moore-Penrose inverse of a singular irreducible M-matrix is a P_0-matrix (an M-**matrix** is a P_0-matrix with nonpositive off-diagonal elements). Ramamurthy and Mohan [12] extended the above result to $n\times n$ M-matrices of rank $n-1$ (the rank of any singular irreducible $n\times n$ M-matrix is $n-1$, see [5]). However for an $n\times n$ M-matrix A of rank less than $n-1$, $A^\dagger$ is not necessarily in $\mathbf{P}_0$, see [8].

We apply here our representation of minors, to give a direct proof for the result of [12], and generalize to the class of $(n-r)^{th}$ compound M-matrices of rank r, a class including M-matrices of rank $n-1$. We show that if $A\in\mathbf{P}_0$, and A is a $(n-r)^{th}$ compound M-matrix of rank r, then $A^\dagger\in\mathbf{P}_0$.

For any $n\times n$ matrix A, the k^{th} **compound matrix** $C_k(A)$ is an $\binom{n}{k}\times\binom{n}{k}$ matrix whose elements are determinants of all $k\times k$ submatrices of A in lexicographic order. We call a matrix k^{th} **compound** M**-matrix** if its k^{th} compound matrix is an M-matrix. The k^{th} **supplementary compound** of A is defined by, see [9, p.42],

$$C^k(A) = \Big((-1)^{s(\alpha)+s(\beta)} \det A[\alpha',\ \beta']\Big) \ , \quad \alpha,\ \beta\in Q_{k,n} \ . \qquad (5.1)$$

Note that the $(\alpha,\beta)^{th}$ element of $C^k(A)$ is

$$(-1)^{s(\alpha)+s(\beta)} \det A[\alpha',\beta'] = \det A[\beta\leftarrow\mathbf{I}_\alpha] \ . \qquad (5.2)$$

In particular, for $k=1$,

$$C^1(A^T) = \operatorname{adj}(A) \ . \qquad (5.3)$$

Some facts about M-matrices are collected below:

Lemma 2 ([5]) If A is an M-matrix, then
(a) any principal submatrix of A is also an M-matrix,
(b) adj $(A) \geq 0$,
(c) there exist a nonnegative matrix B and a number $s \geq \rho(B)$ such that $A = sI - B$, where $\rho(B)$ is the spectral radius of B,
(d) $A^{-1} \geq 0$ if A is nonsingular. □

Theorem 4 (Ramamurthy and Mohan, [12]) If $A \in \mathbb{R}^{n\times n}_{n-1}$ is an M-matrix, then $A^\dagger \in \mathbf{P}_0$.

Proof. For any permutation matrix P, if $\widetilde{A} = PAP^T$, then $(\widetilde{A})^\dagger = PA^\dagger P^T$. Moreover, $\widetilde{A}$ is also an M-matrix. It therefore suffices to show the nonnegativity of leading principal minors,

$$\det A^\dagger[\alpha,\alpha] \geq 0 \ ,$$

for any $\alpha = \{1, 2, \ldots, k\},\ 1 \leq k \leq n-1$. By Theorem 1,

$$\det A^\dagger[\alpha,\alpha] = \frac{1}{\mathrm{vol}^2 A} \sum_{(I,\ J)\in\mathcal{N}(\alpha,\alpha)} \det A_{IJ} \det A_{IJ}[\alpha',\alpha'] \ ,$$

so enough to show that

$$\det A_{IJ} \det A_{IJ}[\alpha',\alpha'] \geq 0 \quad \text{for any} \quad (I, J) \in \mathcal{N}(\alpha,\alpha) \ . \tag{5.4}$$

Since rank $A = n - 1$, there are $i,\ j$ such that

$$I = \mathbf{N}\backslash\{i\} \quad , \quad \text{and} \quad J = \mathbf{N}\backslash\{j\}$$

where $\mathbf{N} = \{1, 2, \ldots, n\}$. From Lemma 2(b)

$$(-1)^{i+j} \det A_{IJ} \geq 0 \ . \tag{5.5}$$

Similarly, $A_{IJ}[\alpha',\alpha']$ is the submatrix of the principal submatrix $A[\alpha',\alpha']$ lying in rows indexed by $I\backslash\alpha$ and in columns indexed by $J\backslash\alpha$. Then by Lemma 2(a),(b),

$$(-1)^{(i-k)+(j-k)} \det A_{IJ}[\alpha',\alpha'] \geq 0 \ ,$$

which, together with (5.5), implies (5.4). □

Theorem 5 Let $A \in \mathbb{R}^{n\times n}_r$. If $C^{n-r}(A_J) \geq 0,\ \forall J \subseteq \mathbf{N}$, $|J| \geq n - r$, then $A^\dagger \in \mathbf{P}_0$.

Proof. For any $\alpha \in Q_{k,n}$, $1 \leq k \leq r$,

$$\det A^\dagger[\alpha,\alpha] = \frac{1}{\mathrm{vol}^2 A} \sum_{(I,\ J)\in\mathcal{N}(\alpha,\alpha)} \det A_{IJ} \ \det A_{IJ}[\alpha \leftarrow \mathbf{I}_\alpha] \ . \tag{5.6}$$

From $C^{n-r}(A) \geq 0$, we have

$$(-1)^{s(I')+s(J')} \det A_{IJ} \geq 0 \ , \tag{5.7}$$

where $I' = \mathbf{N}\backslash I$, and $J' = \mathbf{N}\backslash J$. For $(I, J) \in \mathcal{N}(\alpha,\alpha)$, let

$$A[\alpha, J' \leftarrow \mathbf{I}_{\alpha,I'}] \ , \tag{5.8}$$

denote the matrix obtained from A by replacing the $\alpha_i{}^{th}$ column with the unit vector $\mathbf{e}_{\alpha_i},\ i = 1, \ldots, |\alpha|$, and replacing the $(j'_t)^{th}$ column with the unit vector $\mathbf{e}_{i'_t},\ t = 1, \ldots, n - r$. Then

$$\begin{aligned} \det A_{IJ}[\alpha \leftarrow \mathbf{I}_\alpha] \ &= \\ &= (-1)^{s(I')+s(J')} \det A[\alpha, J' \leftarrow \mathbf{I}_{\alpha,I'}] \ , \\ &= (-1)^{s(I')+s(J')} \det A_{\mathbf{N}\backslash\alpha}[J' \leftarrow \mathbf{I}_{I'}] \ . \end{aligned} \tag{5.9}$$

Now from $C^{n-r}(A_{\mathbf{N}\backslash\alpha}) \geq 0$ it follows, using (5.2), that

$$\det A_{\mathbf{N}\backslash\alpha}[J' \leftarrow \mathbf{I}_{I'}] \geq 0 \ , \tag{5.10}$$

which together with (5.7) and (5.9), implies

$$\det A_{IJ} \det A_{IJ}[\alpha \leftarrow \mathbf{I}_\alpha] \geq 0 \ , \quad \forall\ (I,\ J) \in \mathcal{N}(\alpha,\alpha) \ . \qquad \square$$

Theorem 6 Let $A \in \mathbb{R}^{n\times n}_r$ be a P_0-matrix. If A is a $(n-r)^{th}$ compound M-matrix, $n - r \leq r < n$, then $A^\dagger \in \mathbf{P}_0$.

Proof. Since $C_{n-r}(A^T) = C_{n-r}(A)^T$, $C_{n-r}(A^T)$ is a singular M-matrix. By Lemma 2(c) there is a nonnegative matrix B such that

$$C_{n-r}(A^T) = \rho(B)I - B \tag{5.11}$$

Suppose B is positive. From the Laplace expansion theorem

$$C_{n-r}(A^T)C^{n-r}(A) = (\det A)\, I \tag{5.12}$$

it follows that every nonzero column of $C^{n-r}(A)$ is an eigenvector of B corresponding to $\rho(B)$. By the Perron Theorem, there is only one eigenvector with positive elements corresponding to $\rho(B)$, therefore in each nonzero column of $C^{n-r}(A)$ all elements are nonzero and have the same sign. The same is true for the rows of $C^{n-r}(A)$ by applying the argument to A^T. Thus all elements of $C^{n-r}(A)$ have the same sign. Since A is a P_0-matrix, $C^{n-r}(A)$ is positive.

Suppose B is only nonnegative, then B can be expressed as $B = \lim_{m\to\infty} B_m$, where B_m is positive. By a limiting process, we have

$$C^{n-r}(A) \geq 0 \tag{5.13}$$

Now for any $J \subseteq \mathbf{N}$, $|J| \geq n - r$, $C_{n-r}(A_J)$ is a principal submatrix of $C_{n-r}(A)$, so it is also an M-matrix. If A_J is nonsingular, so is $C_{n-r}(A_J)$, and from (5.12) and Lemma 2(d),

$$C^{n-r}(A_J) = (\det A_J)\left(C_{n-r}(A_J^T)\right)^{-1} \geq 0 \ . \tag{5.14}$$

If A_J is singular, then by the limiting argument,

$$C^{n-r}(A_J) \geq 0 \ . \tag{5.15}$$

The theorem follows from Theorem 5. □

Theorem 4 is a special case of Theorem 6, for which $r = n-1$.

Example 1 The following is an example of a 4×4 P_0-matrix of rank 2 whose 2^{nd} compound matrix is an M-matrix. Let

$$A = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.16}$$

Then A is a P_0-matrix of rank 2, and

$$C_2(A) = \begin{pmatrix} \begin{matrix} 3 & -3 \\ -3 & 3 \end{matrix} & O \\ O & O \end{pmatrix} \in \mathbb{R}^{6\times 6} \tag{5.17}$$

is an M-matrix. The Moore-Penrose inverse of A

$$A^{\dagger} = \frac{1}{6}\begin{pmatrix} 4 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.18}$$

is also a P_0-matrix.

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