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On ℓ_p–approximate solutions of linear equations

Jianming Miao* and Adi Ben-Israel †

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Abstract

For $1 \le p \le \infty$, the ℓ_p–**approximate solutions** of $A\,\mathbf{x} = \mathbf{b}$ are the minimizers of $\|A\mathbf{x} - \mathbf{b}\|_p$ where $\|\cdot\|_p$ is the ℓ_p–**norm**. We consider the special case where the null space of A^T is one-dimensional. Sample results:
(a) If $1 \le p \le \infty$ and A is $m \times (m-1)$ of rank $m-1$, then there is a matrix $A^{\{p\}}$ (depending on A and p) such that, for every $\mathbf{b} \in \mathbb{R}^m$, the vector $A^{\{p\}}\,\mathbf{b}$ is an ℓ_p–approximate solution of $A\,\mathbf{x} = \mathbf{b}$, which is unique if $1 < p < \infty$.
(b) If $1 < p < \infty$ and A is $m \times n$ of rank $m-1$, then there is a matrix $A^{\{p\}}_{\{2\}}$ (depending on A and p) such that for every $\mathbf{b} \in \mathbb{R}^m$ the vector $A^{\{p\}}_{\{2\}}\,\mathbf{b}$ is the ℓ_p–approximate solution of minimal euclidean norm.
(c) Let $1 \le p \le \infty$ and let A be $m \times n$ of rank $m-1$. Then there is a vector $\mathbf{r}^{\{p\}}$ (computed from any least squares solution of $A\,\mathbf{x} = \mathbf{b}$), such that any ordinary solution of the auxiliary equation

$$A\,\mathbf{x} \;=\; \mathbf{b} + \mathbf{r}^{\{p\}}\,,$$

is an ℓ_p–approximate solution of $A\,\mathbf{x} = \mathbf{b}$.

Key words: Linear equations. ℓ_p–approximate solutions. Minimum ℓ_p–norm solutions. Least squares solutions. Generalized inverses. Moore-Penrose inverse.

1 Introduction

Given $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, consider the linear equation

$$A\,\mathbf{x} \;=\; \mathbf{b}\,, \tag{1.1}$$

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†RUTCOR-Rutgers Center for Operations Research, Rutgers University, P.O. Box 5062, New Brunswick, NJ 08903-5062, U.S.A.

and for any $\mathbf{x} \in \mathbb{R}^n$ the **residual**

$$\mathbf{r} = \mathbf{r}(\mathbf{x}) := A\mathbf{x} - \mathbf{b} . \tag{1.2}$$

If (1.1) is inconsistent, we often must settle for a "solution" vector minimizing some norm of the residual. Using the ℓ_p–**norms**, defined for $1 \leq p \leq \infty$ and $\mathbf{u} = (u_j) \in \mathbb{R}^m$ by

$$\|\mathbf{u}\|_p := \begin{cases} \left(\sum_{j=1}^{m} |u_j|^p \right)^{\frac{1}{p}} & , \ 1 \leq p < \infty \\ \max_{1 \leq j \leq m} |u_j| & , \ p = \infty \end{cases} , \tag{1.3}$$

an ℓ_p–**approximate solution**, sometimes called a **best** ℓ_p–**approximate solution**, of (1.1) and denoted by $\mathbf{x}^{\{p\}}$, is a solution of the minimization problem

$$\min \{ \|A\,\mathbf{x} - \mathbf{b}\|_p \, : \, \mathbf{x} \in \mathbb{R}^n \} . \tag{1.4}$$

In particular, the ℓ_2–approximate solutions are the **least squares solutions**, which have their advantages (nice analytical properties and efficient algorithms) as well as disadvantages (sensitivity to deviating observations, such as would result from experimental errors). Other ℓ_p–norms used in (1.4) include the ℓ_∞–norm (giving **minimax** or **Chebyshev solutions**) [8] and the ℓ_1–norm, see e.g. [5], [13]. However, for $p = 1$ and ∞ the objective function $\|A\,\mathbf{x} - \mathbf{b}\|_p$ is typically non-differentiable at the optimal solutions, making the latter hard to characterize and compute. For $p \neq 2$, the ℓ_p–approximate solutions are in general computed iteratively, see e.g. [12].

Some notation: The **set of increasing sequences** of r elements from $\{1, \cdots, m\}$ is

$$Q_{r,m} := \{ I = \{i_1, \cdots, i_r\} : 1 \leq i_1 < i_2 < \cdots < i_r \leq m \}$$

For $A \in \mathbb{R}_r^{m \times n}$, $r > 0$, we denote the following **index sets**:
$\mathcal{I}(A) := \{ I \in Q_{r,m} : \text{rank } A_{I*} = r \}$, i.e. the **maximal sets of linearly independent rows**,
$\mathcal{J}(A) := \{ J \in Q_{r,n} : \text{rank } A_{*J} = r \}$, i.e. the **maximal sets of linearly independent columns**,
$\mathcal{N}(A) := \{ (I, J) \in Q_{r,m} \times Q_{r,n} : \text{rank } A_{IJ} = r \}$, i.e. the **maximal nonsingular submatrices**.

The index sets $\mathcal{I}(A)$, $\mathcal{J}(A)$ and $\mathcal{N}(A)$ shall be abbreviated here by $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{N}$ respectively. We have

$$\mathcal{N} = \mathcal{I} \times \mathcal{J} , \quad \text{see e.g. [1]} . \tag{1.5}$$

The **basic solutions** of the linear equation $A\,\mathbf{x} = \mathbf{b}$ are the solutions of subsystems corresponding to maximal nonsingular submatrices of A. The basic solutions are for

$$A \text{ of full column-rank} : \quad \{ A_{I*}^{-1}\, \mathbf{b}_I \ : \ I \in \mathcal{I} \} , \tag{1.6}$$

$$\text{for } A \text{ of full row-rank} : \quad \{ \widehat{A_{*J}^{-1}}\, \mathbf{b} \ : \ J \in \mathcal{J} \} , \tag{1.7}$$

$$\text{and for general } A : \quad \{ \widehat{A_{IJ}^{-1}}\, \mathbf{b}_I \ : \ (I, J) \in \mathcal{N} \} , \tag{1.8}$$

where $\mathbf{b}_I$ is the I^{th} subvector of $\mathbf{b}$, and $\widehat{\ }$ denotes a vector padded by zeros. The **convex hull of basic solutions** of the given equation $A\,\mathbf{x} = \mathbf{b}$ shall be denoted by $\mathcal{C} = \mathcal{C}(A, \mathbf{b})$.

Ben-Tal and Teboulle [3] showed that for A of full column-rank and $1 \le p < \infty$, the ℓ_p–approximate solutions of (1.1) lie in $\mathcal{C} \;=\; \text{conv}\{\, A_{I*}^{-1}\,\mathbf{b}_I \;:\; I \in \mathcal{I}\,\}$, the convex hull of the basic solutions (1.6). For $p = \infty$, there is an ℓ_∞–approximate solution in $\mathcal{C}$. These results were extended in [10] to general matrices.

If there is more than one solution (or approximate solution), we often select one of minimal ℓ_p–norm, using not necessarily the same norm as in (1.4). In particular, the selection $p = 2$ is natural in statistical applications, where $\|\mathbf{x}\|_2$ is related to the variance of the estimate $\mathbf{x}$.

Given $1 \;\le\; p_1,\; p_2 \;\le\; \infty$, we define a **minimum ℓ_{p_2}–norm ℓ_{p_1}–approximate solution** of (1.1), denoted by $\mathbf{x}^{\{p_1\}}_{\{p_2\}}$ as a solution of the two-stage minimization problem :

Stage 1 $$\min_{\mathbf{x}} \;\{\, \|A\mathbf{x} - \mathbf{b}\|_{p_1} \,:\, \mathbf{x} \in \mathbb{R}^n\} \tag{1.9}$$

Stage 2 $$\min_{\mathbf{x}} \;\{\|\mathbf{x}\|_{p_2} : \text{among all solutions } \mathbf{x} \text{ of } \textbf{Stage 1}\}\;. \tag{1.10}$$

The solution set of (1.9) is a closed convex set. Therefore $\mathbf{x}^{\{p_1\}}_{\{p_2\}}$ is unique for $1 < p_2 < \infty$, since then the ℓ_p–norm is strictly convex.

The case $p_1 = p_2 = 2$ is the easiest to handle and the one most thoroughly studied. The **minimum ℓ_2–norm least squares solution** of $A\,\mathbf{x} = \mathbf{b}$ is given in closed form as

$$\mathbf{x}^{\{2\}}_{\{2\}} = A^\dagger\,\mathbf{b}\;, \quad \forall\;\mathbf{b} \in \mathbb{R}^m\;, \tag{1.11}$$

where $A^\dagger$ is the **Moore-Penrose inverse** of A. Moreover $\mathbf{x}^{\{2\}}_{\{2\}}$ is a convex combination of the basic solutions (1.8) (see e.g. [11], [4], [1])

$$\mathbf{x}^{\{2\}}_{\{2\}} = \sum_{(I,J)\in\mathcal{N}} \lambda_{IJ}\; \widehat{A_{IJ}^{-1}\mathbf{b}_I}\;, \tag{1.12}$$

where the weights λ_{IJ} are independent of $\mathbf{b}$,

$$\lambda_{IJ} = \frac{\det^2 A_{IJ}}{\sum\limits_{(K,L)\in\mathcal{N}} \det^2 A_{KL}}\;. \tag{1.13}$$

The corresponding result for the Moore-Penrose inverse of A is

$$A^\dagger = \sum_{(I,J)\in\mathcal{N}} \lambda_{IJ}\,\widehat{A_{IJ}^{-1}}\;, \tag{1.14}$$

where $\widehat{A_{IJ}^{-1}}$ is an $n \times m$ matrix which has the inverse of A_{IJ} in position (J, I), and zeros elsewhere.

In this paper we consider the case of ℓ_p–norms, $1 \le p \le \infty$.
• Residuals, and approximation errors are studied in § 2.

• In § 3 we consider the special case where A is an $(n+1) \times n$ matrix of rank n. The ℓ_p–approximate solutions $\mathbf{x}^{\{p\}}$ are given explicitly as convex combinations of basic solutions with convex weights which are independent of $\mathbf{b}$. In this case A has a generalized inverse $A^{\{p\}}$ in the sense that for every $\mathbf{b}$, $\mathbf{x}^{\{p\}} := A^{\{p\}}\,\mathbf{b}$ is an ℓ_p–approximate solution.
• § 4 deals with $A \in \mathbb{R}^{m\times n}_{m-1}$ and $1 < p < \infty$. There the minimum ℓ_2–norm ℓ_p–approximate solutions $\mathbf{x}^{\{p\}}_{\{2\}}$ are convex combinations of basic solutions, with convex weights which are independent of $\mathbf{b}$. In this case there is a matrix $A^{\{p\}}_{\{2\}}$, which is a generalized inverse of A in the sense that for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x}^{\{p\}}_{\{2\}} := A^{\{p\}}_{\{2\}}\,\mathbf{b}$ is a minimum ℓ_2–norm ℓ_p–approximate solution.
• § 5 deals again with $A \in \mathbb{R}^{m\times n}_{m-1}$. The ℓ_p–residual $\mathbf{r}^{\{p\}} = \mathbf{r}\left(\mathbf{x}^{\{p\}}\right)$ is computed in terms of the least squares residual $\mathbf{r}^{\{2\}}$, giving an auxiliary equation

$$A\,\mathbf{x} \;=\; \mathbf{b} \;+\; \mathbf{r}^{\{p\}}$$

whose solutions are the ℓ_p–approximate solutions of (1.1).

2 Residuals

If dim $N(A) > 0$, the ℓ_p–approximate solutions of $A\,\mathbf{x} = \mathbf{b}$ are not unique. However, for any A and $1 < p < \infty$ the residual is unique:

Lemma 2.1 Given $1 < p < \infty$, the residual

$$\mathbf{r}^{\{p\}} := \mathbf{r}\left(\mathbf{x}^{\{p\}}\right) = A\mathbf{x}^{\{p\}} - \mathbf{b}$$

is the same for all ℓ_p–approximate solutions $\mathbf{x}^{\{p\}}$ of $A\mathbf{x} = \mathbf{b}$.

Proof: The uniqueness of $\mathbf{r}^{\{p\}}$ is guaranteed by the strict convexity of ℓ_p–norms, $1 < p < \infty$. □

The following lemma describes the residuals in the special case of $A \in \mathbb{R}^{m\times n}_{m-1}$, which is the case when the range $R(A)$ of A is a hyperplane in $\mathbb{R}^m$, or equivalently, the null-space $N(A^T)$ of the transpose A^T is one–dimensional.

Lemma 2.2 Let $A \in \mathbb{R}^{m\times n}_{m-1}$ and $\mathbf{b} \in \mathbb{R}^m$. Then

(a) If $\mathbf{b} \notin R(A)$ then $N(A^T)$ is spanned by the least squares residual $\mathbf{r}^{\{2\}}$.

(b) The set of residuals

$$H \;:=\; \{A\mathbf{x} - \mathbf{b} \,:\, \mathbf{x} \in \mathbb{R}^n\} \tag{2.1}$$

is a hyperplane of $\mathbb{R}^m$. In fact

$$H \;=\; \{\mathbf{r} \,: <\mathbf{r}\,,\, \mathbf{r}^{\{2\}}> \,=\, <\mathbf{r}^{\{2\}}\,,\, \mathbf{r}^{\{2\}}>\}\;. \tag{2.2}$$

Proof: (a) Clearly $N(A^T)$ is spanned by $\mathbf{r}^{\{2\}}$ since dim $N(A^T) = 1$, $\mathbf{r}^{\{2\}} \neq \mathbf{0}$ and the normal equation

$$A^T\ (A\mathbf{x}\ -\ \mathbf{b})\ =\ \mathbf{0} \tag{2.3}$$

characterizes least squares solutions.
(b) From $H = -\mathbf{b} + R(A)$, dim $R(A) = m - 1$, and (a). □

Remark 2.1 Part (b) of the lemma was proved in [8, Lemma, p. 40] under an additional (unnecessary) assumption that the columns of A are linearly independent.

The following theorem gives bounds for the error of ℓ_p-approximations. For $1 \leq p \leq \infty$, the **conjugate** $q = q(p)$ is defined as customary by

$$\frac{1}{p} + \frac{1}{q}\ =\ 1\quad ,\quad 1 < p < \infty\ , \tag{2.4}$$

and the limits $\quad q(1)\ :=\ \lim_{p\to 1} q(p)\ =\ \infty\quad ,\quad q(\infty)\ :=\ \lim_{p\to\infty} q(p)\ =\ 1\ .$

Theorem 2.1 Let $A \in \mathbb{R}_r^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $1 \leq p \leq \infty$ and q be the conjugate of p. Let $\mathbf{x}^{\{2\}}$ be any least squares solution of $A\,\mathbf{x} = \mathbf{b}$, and let

$$\mathbf{r}^{\{2\}}\ :=\ A\,\mathbf{x}^{\{2\}} - \mathbf{b} \tag{2.5}$$

be the corresponding residual. Then

$$\frac{|<\mathbf{b}, \mathbf{r}^{\{2\}}>|}{\|\mathbf{r}^{\{2\}}\|_q}\ \leq\ \min_{\mathbf{x}\in\mathbb{R}^n} \|A\,\mathbf{x} - \mathbf{b}\|_p\ \leq\ \|\mathbf{r}^{\{2\}}\|_p\ . \tag{2.6}$$

Proof. The upper bound is trivial, since $\mathbf{r}^{\{2\}}$ is a residual, but not necessarily optimal. The lower bound is a consequence of the well-known duality theorem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|A\,\mathbf{x} - \mathbf{b}\|_p\ =\ \max_{\mathbf{y}\in\mathbb{R}^m} \{<\mathbf{b}, \mathbf{y}>:\ A^T\mathbf{y} = \mathbf{0}\ ,\ \|\mathbf{y}\|_q \leq 1\} \tag{2.7}$$

where $1 \leq p \leq \infty$ and q is the conjugate of p, see e.g. [6], [7]. Note that the least squares residual $\mathbf{r}^{\{2\}}$ is in $N(A^T)$ by (2.3). Therefore the left hand side of (2.6) is not greater than the right hand side of (2.7). □

Remark 2.2 The bounds (2.6) are useful since a least squares residual is readily available, i.e.

$$\mathbf{r}^{\{2\}}\ =\ (I - AA^+)\,\mathbf{b}\ =\ P_{N(A^T)}\,\mathbf{b}\ , \tag{2.8}$$

and (2.6) becomes

$$\frac{\|P_{N(A^T)}\,\mathbf{b}\|_2^2}{\|P_{N(A^T)}\,\mathbf{b}\|_q}\ \leq\ \min_{\mathbf{x}\in\mathbb{R}^n} \|A\,\mathbf{x} - \mathbf{b}\|_p\ \leq\ \|P_{N(A^T)}\,\mathbf{b}\|_p\ . \tag{2.9}$$

Remark 2.3 If $A \in \mathbb{R}^{m\times n}_{m-1}$, it follows from Lemma 2.2(a) that the lower bounds in (2.6) and (2.9) are exact, namely

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|A\,\mathbf{x}-\mathbf{b}\|_p \;=\; \frac{|<\mathbf{b},\mathbf{r}^{\{2\}}>|}{\|\mathbf{r}^{\{2\}}\|_q} \;=\; \frac{\|P_{N(A^T)}\,\mathbf{b}\|_2^2}{\|P_{N(A^T)}\,\mathbf{b}\|_q}\,. \tag{2.10}$$

3 A is an $(n+1)\times n$ matrix of rank n

Throughout this section let $A \in \mathbb{R}^{(n+1)\times n}_n$ and $\mathbf{b} \in \mathbb{R}^{n+1}$. The basic solutions of $A\mathbf{x} = \mathbf{b}$ are

$$\mathbf{x}_{I*} \;:=\; A_{I*}^{-1}\mathbf{b}_I \;, \quad I \in \mathcal{I}\,, \tag{3.1}$$

and their convex hull,

$$\mathcal{C} \;:=\; \text{conv}\{\mathbf{x}_{I*} \;:\; I \in \mathcal{I}\}\,. \tag{3.2}$$

It follows from [3] that there exists a solution of (1.4) in $\mathcal{C}$. In this section we give explicit formulæ for the convex weights which turn out to be independent of $\mathbf{b}$. Therefore there is a matrix $A^{\{p\}}$ such that for any $\mathbf{b} \in \mathbb{R}^{n+1}$, $A^{\{p\}}\mathbf{b}$ is an ℓ_p–approximate solution. We shall construct the matrix $A^{\{p\}}$ and study its properties.

An immediate consequence of the assumption that A is $(n+1)\times n$ of rank n is that for each $I \in \mathcal{I}$, the basic solution $\mathbf{x}_{I*} = A_{I*}^{-1}\,\mathbf{b}_I$ has a residual $\mathbf{r}_I = \mathbf{r}(\mathbf{x}_{I*})$ with n zeros in positions I. The remaining component of $\mathbf{r}_I$, denoted by ϵ_I , shall now be given explicitly.

Lemma 3.1 Let $A \in \mathbb{R}^{(n+1)\times n}_n$. For any $I \in \mathcal{I}$ let

$$\epsilon_I \;:=\; A_{i*}\,\mathbf{x}_{I*} - b_i \;, \tag{3.3}$$

where $\{i\}$ is the complement of I in $\{1,\ldots,n+1\}$, A_{i*} is the i^{th} row of A , and b_i is the i^{th} component of $\mathbf{b}$. Then

$$|\epsilon_I| = \frac{|\det(A,\;\mathbf{b})|}{|\det A_{I*}|}\,. \tag{3.4}$$

Proof. This follows from

$$\det\begin{pmatrix} A_{I*} & \mathbf{b}_I \\ A_{i*} & b_i \end{pmatrix} \;=\; \left(b_i - A_{i*}A_{I*}^{-1}\mathbf{b}_I\right)\det A_{I*}\,. \qquad \square$$

Another consequence of assuming $A \in \mathbb{R}^{(n+1)\times n}_n$ is that for any vector $\mathbf{x}$ which is a convex combination of the basic solutions,

$$\mathbf{x} = \sum_{I\in\mathcal{I}} \mu_I\,\mathbf{x}_{I*}\;, \quad \sum_{I\in\mathcal{I}} \mu_I = 1\;, \quad \mu_I \ge 0\,, \tag{3.5}$$

the residual $\mathbf{r}(\mathbf{x})$ has components $(\mu_I\,\epsilon_I)$. Therefore the problem of minimizing the ℓ_p–norm of residuals of the vectors in (3.5) can be written as

$$\text{(P)} \qquad \min\left\{\|(\mu_I\,|\epsilon_I|)\|_p \;:\; \sum_{I\in\mathcal{I}}\mu_I = 1\;,\; \mu_I \ge 0\right\}\,,$$

with $|\epsilon_I|$ given by (3.4), $I \in \mathcal{I}$. This problem can be solved explicitly for all $1 \leq p \leq \infty$: First the case $p = 1$.

Theorem 3.1 Let $A \in \mathbb{R}_n^{(n+1)\times n}$. Then

$$\min_{\mathbf{x}} \ \|A\mathbf{x} - \mathbf{b}\|_1 = \frac{|\det(A,\ \mathbf{b})|}{|\det A_{L*}|} \ , \tag{3.6}$$

and an optimal solution is
$$\mathbf{x}^{\{1\}} = A_{L*}^{-1}\,\mathbf{b}_L \ , \tag{3.7}$$

where
$$L \ \in \ \arg\max_I |\det A_{I*}| \ , \tag{3.8}$$

and arg max f [arg min f] denote the **set of maximizers** [**minimizers**] of a function f .
Proof. By [3], any ℓ_1–approximate solution is given by

$$\mathbf{x}^{\{1\}} \ = \ \sum_{I\in\mathcal{I}} \mu_I\,\mathbf{x}_{I*} \ ,$$

where

$$\sum_{I\in\mathcal{I}} \mu_I = 1 \ , \quad \mu_I \geq 0 \ .$$

Then
$$\begin{aligned} \|A\,\mathbf{x}^{\{1\}} - \mathbf{b}\|_1 \ &= \ \sum_{I\in\mathcal{I}} \mu_I\,|\epsilon_I| \ , \\ &= \ |\det(A,\ \mathbf{b})| \sum_{I\in\mathcal{I}} \frac{\mu_I}{|\det A_{I*}|} \qquad \text{by (3.4)} \ . \end{aligned}$$

For $p = 1$ the optimization problem (P) can therefore be written as

$$\min \left\{ \sum_{I\in\mathcal{I}} \frac{\mu_I}{|\det A_{I*}|} \ : \ \text{s.t.} \quad \sum_{I\in\mathcal{I}} \mu_I = 1 \ , \ \mu_I \geq 0 \right\} \ ,$$

and it has an optimal solution

$$\begin{cases} \mu_L^{(1)} = 1 & \text{for } \ L \text{ selected in } \arg\max_I |\det A_{I*}| \ , \\ \mu_I^{(1)} = 0 & \text{if } \ I \neq L \ . \end{cases}$$
□

Remark 3.1 The optimal basis L is independent of $\mathbf{b}$ and the solution is unique if and only if $|\det A_{L*}| \ > \ |\det A_{I*}|$ for all $I \neq L,\ I \in \mathcal{I}$.

We now solve for $1 < p < \infty$. In this case the solution is unique.

Theorem 3.2 Let $A \in \mathbb{R}_n^{(n+1)\times n}$, $1 < p < \infty$, and q be the conjugate (2.4) of p. Then

$$\min_{\mathbf{x}} \ \|A\mathbf{x} - \mathbf{b}\|_p \ = \ \frac{|\det(A, \ \mathbf{b})|}{(\sum\limits_{K\in\mathcal{I}} |\det A_{K*}|^q)^{\frac{1}{q}}} \ , \tag{3.9}$$

and the optimal solution is

$$\mathbf{x}^{\{p\}} \ = \ \sum_{I\in\mathcal{I}} \mu_I^{(p)} \, \mathbf{x}_{I*} \ , \tag{3.10}$$

where

$$\mu_I^{(p)} = \frac{|\det A_{I*}|^q}{\sum\limits_{K\in\mathcal{I}} |\det A_{K*}|^q} \ , \quad I \in \mathcal{I} \ . \tag{3.11}$$

Proof. The solution is given by

$$\mathbf{x}^{\{p\}} \ = \ \sum_{I\in\mathcal{I}} \mu_I \, \mathbf{x}_{I*} \ ,$$

where

$$\sum_{I\in\mathcal{I}} \mu_I \ = \ 1 \ , \quad \mu_I \geq 0 \ .$$

Then

$$\begin{aligned} \|A\,\mathbf{x}^{\{p\}} - \mathbf{b}\|_p^p \ &= \ \sum_{I\in\mathcal{I}} \mu_I^p \, |\epsilon_I|^p, \\ &= \ |\det(A, \ \mathbf{b})|^p \ \sum_{I\in\mathcal{I}} \frac{\mu_I^p}{|\det A_{I*}|^p} \ . \end{aligned}$$

The optimization problem (P) here becomes

$$\min \left\{ \sum_{I\in\mathcal{I}} \frac{\mu_I^p}{|\det A_{I*}|^p} : \ \sum_{I\in\mathcal{I}} \mu_I \ = \ 1 \ , \ \mu_I \geq 0 \right\} \ . \tag{3.12}$$

The Kuhn-Tucker necessary and sufficient conditions for (3.12) are

$$\begin{aligned} \frac{\mu_I^{p-1}}{|\det A_{I*}|^p} \ &= \ c \ , \quad \forall \ I \in \mathcal{I} \ , \\ \sum_{I\in\mathcal{I}} \mu_I \ &= \ 1 \ , \\ \mu_I \ &\geq \ 0 \ , \end{aligned}$$

for some constant c. The results (3.9) and (3.10)–(3.11) then follow immediately. □

Remark 3.2 The weights $\mu_I^{(p)}$ are independent of $\mathbf{b}$.

Finally the case $p = \infty$:

Theorem 3.3 Let $A \in \mathbb{R}_n^{(n+1)\times n}$. Then

$$\min_{\mathbf{x}} \ \|A\mathbf{x} - \mathbf{b}\|_\infty \ = \ \frac{|\det(A,\ \mathbf{b})|}{\sum\limits_{K\in\mathcal{I}} |\det A_{K*}|} \ , \tag{3.13}$$

and an optimal solution is

$$\mathbf{x}^{\{\infty\}} \ = \ \sum_{I\in\mathcal{I}} \mu_I^{(\infty)} \, \mathbf{x}_{I*} \ , \tag{3.14}$$

where

$$\mu_I^{(\infty)} \ = \ \frac{|\det A_{I*}|}{\sum\limits_{K\in\mathcal{I}} |\det A_{K*}|} \ , \quad I \in \mathcal{I} \ . \tag{3.15}$$

Proof. By [3], there is a solution $\mathbf{x}^{\{\infty\}}$ in $\mathcal{C}$. Let

$$\mathbf{x}^{\{\infty\}} \ = \ \sum_{I\in\mathcal{I}} \mu_I \, \mathbf{x}_{I*} \ ,$$

where

$$\sum_{I\in\mathcal{I}} \mu_I \ = \ 1 \ , \quad \mu_I \geq 0 \ .$$

Then

$$\begin{aligned} \|A\,\mathbf{x}^{\{\infty\}} - \mathbf{b}\|_\infty \ &= \ \max_{I\in\mathcal{I}} \ \mu_I \, |\epsilon_I| \ , \\ &= \ |\det(A,\ \mathbf{b})| \ \max_{I\in\mathcal{I}} \frac{\mu_I}{|\det A_{I*}|} \ . \end{aligned}$$

The optimization problem (P) here becomes

$$\min \ \left\{ \max_{I\in\mathcal{I}} \ \frac{\mu_I}{|\det A_{I*}|} \ : \quad \sum_{I\in\mathcal{I}} \mu_I = 1 \ , \ \mu_I \geq 0 \right\} \ ,$$

with

$$\mu_I^{(\infty)} \ = \ \frac{|\det A_{I*}|}{\sum\limits_{K\in\mathcal{I}} |\det A_{K*}|} \ , \quad I \in \mathcal{I} \ ,$$

as optimal solution. □

Remark 3.3 For $p = \infty$ the weights $\mu_I^{(\infty)}$ are again independent of $\mathbf{b}$. It follows, e.g. [8, p. 42, Problems 6,7], that the solution is unique if and only if the **Haar condition** is satisfied, i.e., if every set of n rows of A is linearly independent.

Definition 3.1 For any $A \in \mathbb{R}_n^{(n+1)\times n}$ define the matrix $A^{\{p\}}$

$$A^{\{p\}} \ := \ \begin{cases} \widehat{A_{L*}^{-1}} & \text{if } p = 1, \text{ and } L \text{ is selected in } \arg\max\limits_I |\det A_{I*}| \ , \\ \sum\limits_{I\in\mathcal{I}} \mu_I^{(p)} \widehat{A_{I*}^{-1}} & \text{if } 1 < p \leq \infty \ , \end{cases} \tag{3.16}$$

where $\widehat{A_{I*}^{-1}}$ is an $n \times (n+1)$ matrix with the inverse of A_{I*} in position I, and zeros elsewhere.

Example 3.1 Let $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then

$$A^{\{p\}} = \begin{cases} \begin{pmatrix} 0 \ , \ \frac{1}{2} \end{pmatrix} & \text{if } p = 1 \ , \\ \\ \dfrac{1}{1+2^q} \left(1 \ , \ 2^{q-1} \right) & \text{for all } 1 < p \leq \infty \ , \end{cases}$$

where q is the conjugate of p.

Theorems 3.1–3.3 can be summarized as follows:

Corollary 3.1 Let $A \in \mathbb{R}_n^{(n+1)\times n}$, and let $A^{\{p\}}$ be given by (3.16). Then for any $\mathbf{b} \in \mathbb{R}^{n+1}$, $A^{\{p\}}\,\mathbf{b}$ is a solution of (1.4). □

The following example shows that the conclusion does not hold for $A \in \mathbb{R}_k^{(n+1)\times k}$ with $k < n$.

Example 3.2 Let $A = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$.
The corresponding solutions of $\min \|A\mathbf{x} - \mathbf{b}_i\|_4$ are

$$\mathbf{x}_1 = -0.321465 \ , \ \mathbf{x}_2 = 0.334298 \ , \ \mathbf{x}_3 = 0.053283 \ ,$$

respectively. However

$$\mathbf{x}_3 \neq \mathbf{x}_1 + \mathbf{x}_2 \ .$$

The matrix $A^{\{p\}}$ defined in (3.16) is a generalized inverse of A. Clearly $A^{\{p\}}$ is a left-inverse,

$$A^{\{p\}}A = I \ , \tag{3.17}$$

and consequently $A^{\{p\}}$ satisfies

$$AXA = A \ , \quad XAX = X \ , \quad (XA)^T = XA \ , \tag{3.18}$$

showing that $A^{\{p\}}$ is a **{1,2,4}-inverse** of A, see [2].

Theorem 3.4 Let $A \in \mathbb{R}_n^{(n+1)\times n}$ and $Q \in \mathbb{R}_n^{n\times n}$. Then

$$(AQ)^{\{p\}} = Q^{-1}A^{\{p\}} \ . \tag{3.19}$$

Proof. Let $B = AQ$, then $B_{I*} = A_{I*}Q$, and $\mathcal{I}(B) = \mathcal{I}(A) = \mathcal{I}$. Therefore

$$\begin{aligned} B^{\{p\}} &= \sum_{I\in\mathcal{I}} \mu_I^{(p)} \widehat{B_{I*}^{-1}} \ , \\ &= Q^{-1} \sum_{I\in\mathcal{I}} \mu_I^{(p)} \widehat{A_{I*}^{-1}} \ , \end{aligned}$$

where $\mu_I^{(p)}$ as above. $\square$

Now consider the continuity of $A^{\{p\}}$: If $\{A_j\} \subset \mathbb{R}^{(n+1)\times n}$ is a sequence of matrices converging to a matrix $A \in \mathbb{R}_n^{(n+1)\times n}$, then eventually all $\{A_j\}$ have full column-rank n. Does the corresponding sequence of generalized inverses $A_j^{\{p\}}$ converge to $A^{\{p\}}$? The affirmative answer given below, for $1 < p < \infty$, shows $A^{\{p\}}$ to be insensitive to errors in A.

Theorem 3.5 Let $1 < p < \infty$ and let $A,\ A_j \in \mathbb{R}_n^{(n+1)\times n}$ be such that $A_j \to A$. Then

$$A_j^{\{p\}} \ \to \ A^{\{p\}} \quad \text{as} \ \ j \to \infty \ . \tag{3.20}$$

Proof. Since $A_j \to A$, there is an index j_0 such that for any $j \geq j_0$

$$\mathcal{I}(A) \ \subseteq \ \mathcal{I}(A_j) \ .$$

Since

$$A_j^{\{p\}} \ = \sum_{I\in\mathcal{I}(A_j)} (\mu_j)_I^{(p)}\, \widehat{(A_j)}_{I*}^{-1} \ ,$$

we need only consider terms $(\mu_j)_I^{(p)} \widehat{(A_j)}_{I*}^{-1}$ for $I \in \mathcal{I}(A_j)\backslash\mathcal{I}(A)$. For such I it follows from $A_j \to A$ that

$$\det(A_j)_{I*} \ \to \ 0 \ ,$$

and

$$(\mu_j)_I^{(p)} \widehat{(A_j)}_{I*}^{-1} = \frac{|\det(A_j)_{I*}|^{(q-1)}}{\sum\limits_{K\in\mathcal{I}(A_j)} |\det(A_j)_{K*}|^q} \ \mathrm{adj}((A_j)_{I*}) \ \to \ 0 \ ,$$

where $\mathrm{adj}((A_j)_{I*})$ is the **adjoint** of $(A_j)_{I*}$. This completes the proof. $\square$

Remark 3.4 If the set $\arg\max_I |\det A_{I*}|$ is a singleton, then $A^{\{1\}}$ is continuous by the continuity of A_{L*}^{-1} . Otherwise $A^{\{1\}}$ depends on the choice of L.

Remark 3.5 If the Haar condition is satisfied, then $A^{\{\infty\}}$ is continuous by the proof of Theorem 3.5. The next example shows that in general $A^{\{\infty\}}$ need not be continuous.

Example 3.3 Let $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A_\epsilon = \begin{pmatrix} 1 \\ \epsilon \end{pmatrix}$. Then

$$A^{\{\infty\}} = (1,\ 0) \ , \quad A_\epsilon^{\{\infty\}} = \left(\frac{1}{1+\epsilon},\ \frac{1}{1+\epsilon}\right)$$

$$\text{and} \quad A_\epsilon^{\{\infty\}} \ \not\to \ A^{\{\infty\}} \ , \quad \text{as } \epsilon \to 0 \ .$$

For $1 \le p \le \infty$, let $\mathbf{r}^{\{p\}}$ be the residual of $\mathbf{x}^{\{p\}}$:

$$\mathbf{r}^{\{p\}} = A\,\mathbf{x}^{\{p\}} - \mathbf{b} \,. \tag{3.21}$$

Then we have

Theorem 3.6 All the residuals $\mathbf{r}^{\{p\}}$ have the same sign vectors for $1 < p \le \infty$ (or may be so chosen in the event of nonuniqueness).

Proof. Let $\mathbf{x}^{\{p\}}$ be the solution of (1.4) as given by Theorems 3.2–3.3, and let $\{i\}$ be the complement of I in $\{1, \ldots, n+1\}$. Then

$$r_i^{\{p\}} = \begin{cases} \mu_I^{(p)}\,\epsilon_I & \text{if } I \in \mathcal{I}\,, \\ 0 & \text{if } I \notin \mathcal{I}\,. \end{cases}$$

The result then follows from Theorems 3.2–3.3. □

The next example shows that Theorem 3.6 does not hold for $p = 1$, notwithstanding an assertion in [8, p. 40] which would imply that all the residuals have the same sign vectors for $1 \le p \le \infty$ (or may be so chosen in the event of nonuniqueness).

Example 3.4 Let $A = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Then $\mathbf{x}^{\{1\}} = \frac{1}{2}$, $\mathbf{x}^{\{2\}} = 1$, and their residuals

$$\mathbf{r}^{\{1\}} = \begin{pmatrix} -\frac{5}{2} \\ 0 \end{pmatrix}, \quad \mathbf{r}^{\{2\}} = \begin{pmatrix} -2 \\ -1 \end{pmatrix},$$

do not agree in sign.

4 Minimum ℓ_2–norm ℓ_p–approximate solutions

For $A \in \mathbb{R}_{m-1}^{m\times n}$ and $1 < p < \infty$ we can give explicit formulæ for the solutions of

$$\min_{\mathbf{x}} \{\|\mathbf{x}\|_2 \;:\; \mathbf{x} \in \arg\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_p\} \,. \tag{4.1}$$

The results require additional hypotheses for $p = \infty$ and $p = 1$, see Theorem 4.1(b)–(c).

For any full-rank factorization $A = CR$, the problem (4.1) can be solved in stages

$$\min_{\mathbf{y}} \|C\mathbf{y} - \mathbf{b}\|_p \,, \tag{4.2}$$

$$\min_{\mathbf{x}} \{\|\mathbf{x}\|_2 \;:\; R\mathbf{x} = \mathbf{y}\,,\; \mathbf{y} \in \arg\min_{\mathbf{y}} \|C\mathbf{y} - \mathbf{b}\|_p\} \,, \tag{4.3}$$

using the basic solutions (1.8)

$$\mathbf{x}_{IJ} := \widehat{A_{IJ}^{-1}}\,\mathbf{b}_I \,, \quad (I, J) \in \mathcal{N} \,. \tag{4.4}$$

Theorem 4.1 Let $A \in \mathbb{R}^{m\times n}_{m-1}$ and $\mathbf{b} \in \mathbb{R}^m$.
(a) Let $1 < p < \infty$. Then the solution of (4.1) is given by

$$\mathbf{x}^{\{p\}}_{\{2\}} = \sum_{I\in\mathcal{I}} \mu_I \, A^\dagger_{I*} \, \mathbf{b}_I \ , \tag{4.5}$$

$$= \sum_{(I,J)\in\mathcal{N}} \lambda_{IJ} \, \mathbf{x}_{IJ} \ . \tag{4.6}$$

The weights λ_{IJ} are
$$\lambda_{IJ} := \mu_I \, \nu_J \ , \tag{4.7}$$
and

$$\mu_I = \frac{\mathrm{vol}^{\,q} A_{I*}}{\sum\limits_{K\in\mathcal{I}} \mathrm{vol}^{\,q} A_{K*}} \ , \quad \nu_J = \frac{\mathrm{vol}^{\,2} A_{*J}}{\sum\limits_{N\in\mathcal{J}} \mathrm{vol}^{\,2} A_{*N}} \ , \tag{4.8}$$

where for a matrix $X \in \mathbb{R}^{m\times n}_m$ the **volume** is $\mathrm{vol}\, X := \sqrt{\det XX^T}$, see [1].
(b) Let $p = \infty$. If every set of $m-1$ rows of A is linearly independent, then the solution $\mathbf{x}^{\{\infty\}}_{\{2\}}$ is given by (4.5)–(4.8) with $q = 1$.
(c) Let $p = 1$. If there is $L \in \mathcal{I}$ such that for some $J \in \mathcal{J}$

$$|\det A_{LJ}| > |\det A_{IJ}|, \quad \forall \ I \neq L, \ I \in \mathcal{I}, \tag{4.9}$$

then the solution $\mathbf{x}^{\{1\}}_{\{2\}}$ of

$$\min_{\mathbf{x}} \ \{\|\mathbf{x}\|_2 \ : \ \mathbf{x} \in \arg \ \min_{\mathbf{x}} \ \|A\mathbf{x} - \mathbf{b}\|_1\} \tag{4.10}$$

is

$$\mathbf{x}^{\{1\}}_{\{2\}} = A^\dagger_{L*} \, \mathbf{b}_L \ . \tag{4.11}$$

Proof. Let $A = CR$ be any full-rank factorization of A. Then for any $I \in \mathcal{I}$

$$A_{I*} = C_{I*} R \ , \quad A^\dagger_{I*} = R^\dagger C^{-1}_{I*} \ , \tag{4.12}$$

and

$$\mathrm{vol}\, A_{I*} = |\det C_{I*}| \, \mathrm{vol}\, R. \tag{4.13}$$

(a) The solution of (4.2) is given from (Theorem 3.2) by

$$\mathbf{y} = \sum_{I\in\mathcal{I}} \mu_I \, C^{-1}_{I*} \, \mathbf{b}_I \ , \tag{4.14}$$

where

$$\mu_I = \frac{|\det C_{I*}|^q}{\sum\limits_{K\in\mathcal{I}} |\det C_{K*}|^q} = \frac{\mathrm{vol}^{\,q} A_{I*}}{\sum\limits_{K\in\mathcal{I}} \mathrm{vol}^{\,q} A_{K*}} \quad \text{by (4.13).}$$

Then the solution of (4.3) is given by

$$\mathbf{x}^{\{p\}}_{\{2\}} = R^\dagger \mathbf{y}, \tag{4.15}$$

$$= \sum_{I\in\mathcal{I}} \mu_I \, A^\dagger_{I*} \, \mathbf{b}_I \ , \quad \text{from (4.14), (4.12).}$$

(4.6) follows from (4.15) and

$$R^{\dagger} \;=\; \sum_{J\in\mathcal{J}} \nu_J\, \widehat{R_{*J}^{-1}}\ , \quad \text{see [1]},$$

where

$$\nu_J \;=\; \frac{|\det R_{*J}|^2}{\sum\limits_{N\in\mathcal{J}} |\det R_{*N}|^2} \;=\; \frac{\mathrm{vol}^{\,2} A_{*J}}{\sum\limits_{N\in\mathcal{J}} \mathrm{vol}^{\,2} A_{*N}}\ .$$

(b) The hypothesis is equivalent to a Haar condition on the full-column factor C. It follows from Remark 3.3 that the solution of $\min\ \|C\mathbf{y}-\mathbf{b}\|_\infty$ is unique. The proof then follows from Theorem 3.3.
(c) By hypothesis we have

$$|\det C_{L*}| \;>\; |\det C_{I*}|, \quad \forall\, I \neq L\ ,\ I \in \mathcal{I}.$$

Then Remark 3.1 implies the uniqueness of the solution of $\min\ \|C\mathbf{y}-\mathbf{b}\|_1$. The proof follows then from Theorem 3.1, similarly to the proof of part (a). □

If the assumption in Theorem 4.1(c) is not satisfied, the convex weights may depend on $\mathbf{b}$ even if the solution of (4.10) still lies in $\mathcal{C}$.

Example 4.1 Let $A \;=\; \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} \;=\; \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ where $b_1,\ b_2 > 0$. Then the basic solutions are $\mathbf{x}_1 = b_1,\ \mathbf{x}_2 = -b_2$, and $\arg\min\ \|A\mathbf{x}-\mathbf{b}\|_1 = \mathrm{conv}\,\{b_1,\ -b_2\}$. Clearly the solution of (4.10) is

$$\mathbf{x}^{\{1\}}_{\{2\}} = \mathbf{0} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2$$

with weights $\lambda_1 \;=\; \dfrac{b_2}{b_1+b_2}\ ,\ \lambda_2 \;=\; \dfrac{b_1}{b_1+b_2}$ which depend on $\mathbf{b}$.

Definition 4.1 For any $A \in \mathbb{R}^{m\times n}_{m-1}$, $1 < p < \infty$, let the matrix $A^{\{p\}}_{\{2\}}$ be defined as

$$A^{\{p\}}_{\{2\}} \;=\; \sum_{(I,J)\in\mathcal{N}} \lambda_{IJ}\, \widehat{A_{IJ}^{-1}}\ , \tag{4.16}$$

with weights λ_{IJ} given by (4.8).

If $A \in \mathbb{R}^{(n+1)\times n}_{n}$ then Definition 4.1 reduces to Definition 3.1. From Theorem 4.1 we have

Corollary 4.1 Let $A \in \mathbb{R}^{m\times n}_{m-1}$, $1 < p < \infty$. Then for any $\mathbf{b} \in \mathbb{R}^m$, $A^{\{p\}}_{\{2\}}\mathbf{b}$ gives the solution of (4.1).

From Theorem 4.1 we can also deduce:

Theorem 4.2 Let $A \in \mathbb{R}^{m\times n}_{m-1}$ and $1 < p < \infty$. Then for any full-rank factorization $A = CR$

$$A^{\{p\}}_{\{2\}} \;=\; R^{\dagger} C^{\{p\}} \;, \tag{4.17}$$

where $C^{\{p\}}$ is defined by (3.16). Moreover $A^{\{p\}}_{\{2\}}$ is a $\{1,2,4\}$-inverse of A.

Proof. (4.17) follows directly from the proof of Theorem 4.1. It is easy to check that $A^{\{p\}}_{\{2\}}$ satisfies the three equations in (3.18). □

Theorem 4.3 Let $1 < p < \infty$ and let $A,\ A_j \in \mathbb{R}^{m\times n}_{m-1}$ be such that $A_j \to A$. Then

$$(A_j)^{\{p\}}_{\{2\}} \;\to\; A^{\{p\}}_{\{2\}} \quad \text{as} \quad j \to \infty \;. \tag{4.18}$$

Proof. This follows by Theorem 3.5 and Theorem 4.2. □

5 Auxiliary linear equations for ℓ_p–approximate solutions

In the special case that $A \in \mathbb{R}^{m\times n}_{m-1}$ we can construct an auxiliary linear equation whose (ordinary) solutions are the ℓ_p–approximate solutions of $A\,\mathbf{x} \;=\; \mathbf{b}$. For this we define the sign of a vector $\mathbf{z} = (z_j)$ as the vector

$$\operatorname{sign}\mathbf{z} := (\operatorname{sign} z_j). \tag{5.1}$$

The **Hadamard product** $\mathbf{u} \circ \mathbf{v}$ of two vectors $\mathbf{u} = (u_j)$ and $\mathbf{v} = (v_j)$ is the vector

$$\mathbf{u} \circ \mathbf{v} := (u_j\, v_j) \;. \tag{5.2}$$

Theorem 5.1 Let $A \in \mathbb{R}^{m\times n}_{m-1}$, $\mathbf{b} \in \mathbb{R}^m$, $1 < p < \infty$, and q be the conjugate of p. Then the residual $\mathbf{r}^{\{p\}} \;:=\; A\,\mathbf{x}^{\{p\}} - \mathbf{b}$ is given as

$$\mathbf{r}^{\{p\}} \;=\; \beta\,(\operatorname{sign}\mathbf{r}^{\{2\}}) \circ \left|\mathbf{r}^{\{2\}}\right|^{q-1} \;, \tag{5.3}$$

where

$$\beta \;=\; \frac{\sum\limits_{i=1}^{m} \left|r^{\{2\}}_i\right|^2}{\sum\limits_{i=1}^{m} \left|r^{\{2\}}_i\right|^q} \;. \tag{5.4}$$

Proof: If $\mathbf{b} \in R(A)$ then $\mathbf{r}^{\{p\}} \;=\; \mathbf{0}$ for all p and (5.3) holds trivially for all β. We can therefore assume that $\mathbf{b} \notin R(A)$. From

$$\nabla\, \|A\mathbf{x} - \mathbf{b}\|_p^p \;=\; 0 \quad\Longleftrightarrow\quad A^T \left((\operatorname{sign}\mathbf{r}^{\{p\}}) \circ |\mathbf{r}^{\{p\}}|^{p-1} \right) = \mathbf{0} \tag{5.5}$$

and Lemma 2.2(a) it follows that

$$(\operatorname{sign} \mathbf{r}^{\{p\}}) \circ \left|\mathbf{r}^{\{p\}}\right|^{p-1} \;=\; \alpha\, \mathbf{r}^{\{2\}} \;,$$

for some scalar α which, written componentwise, becomes

$$\begin{aligned} (\operatorname{sign} r_i^{\{p\}}) \left|r_i^{\{p\}}\right|^{p-1} &= \alpha\, r_i^{\{2\}} \;, \quad i = 1, \ldots, m \;, \\ \text{so that} \quad \operatorname{sign} r_i^{\{p\}} &= \operatorname{sign} \alpha \; \operatorname{sign} r_i^{\{2\}} \;, \quad i = 1, \ldots, m \;. \end{aligned}$$

Therefore

$$\begin{aligned} \left|r_i^{\{p\}}\right| &= |\alpha|^{q-1} \left|r_i^{\{2\}}\right|^{q-1} \;, \\ \text{and} \quad r_i^{\{p\}} &= \beta \left(\operatorname{sign} r_i^{\{2\}}\right) \left|r_i^{\{2\}}\right|^{q-1} \end{aligned}$$

for $\beta \;=\; (\operatorname{sign} \alpha)\, |\alpha|^{q-1}$. We can determine the scalar β from the identity

$$<\mathbf{r}^{\{p\}} \,, \mathbf{r}^{\{2\}}> \,=\, <\mathbf{r}^{\{2\}} \,, \mathbf{r}^{\{2\}}> \tag{5.6}$$

which holds for all p by Lemma 2.2(b). Using (5.3) and (5.6) we obtain

$$\beta \;=\; \frac{<\mathbf{r}^{\{2\}} \,, \mathbf{r}^{\{2\}}>}{<|\mathbf{r}^{\{2\}}|^{q-1} \,, |\mathbf{r}^{\{2\}}|>} \;=\; \frac{\sum_{i=1}^{m} \left|r_i^{\{2\}}\right|^2}{\sum_{i=1}^{m} \left|r_i^{\{2\}}\right|^q} \;.$$

□

For $p = 1$ and $p = \infty$, the residuals $\mathbf{r}^{\{p\}}$ are no longer unique, but among them there exist residuals which can be given in terms of $\mathbf{r}^{\{2\}}$ analogously to (5.3). First the case $p = \infty$:

Theorem 5.2 Let $A \in \mathbb{R}^{m \times n}_{m-1}$, $\mathbf{b} \in \mathbb{R}^m$. Then there is a residual $\mathbf{r}^{\{\infty\}}$ satisfying

$$\mathbf{r}^{\{\infty\}} \;=\; \beta \operatorname{sign} \mathbf{r}^{\{2\}} \;, \tag{5.7}$$

where

$$\beta \;=\; \frac{\sum_{i=1}^{m} \left|r_i^{\{2\}}\right|^2}{\sum_{i=1}^{m} \left|r_i^{\{2\}}\right|} \;. \tag{5.8}$$

Proof. This follows from Lemma 2.2(b) and [8, Problem 1, p. 42]. □

Theorem 5.3 Let $A \in \mathbb{R}^{m \times n}_{m-1}$, $\mathbf{b} \in \mathbb{R}^m$. Then there is a residual $\mathbf{r}^{\{1\}}$ satisfying

$$\mathbf{r}^{\{1\}} \;=\; \frac{\sum_{i=1}^{m} \left|r_i^{\{2\}}\right|^2}{r_\ell^{\{2\}}}\, \mathbf{e}_\ell \;, \tag{5.9}$$

where ℓ is selected in $\arg \max_i |r_i^{\{2\}}|$, and $\mathbf{e}_\ell$ denotes the ℓ^{th} unit vector.

Proof. This follows from Lemma 2.2(b) and [8, Problem 3, p. 42]. □

From Theorems 5.1–5.3 it follows that the ℓ_p–approximate solutions can be computed as ordinary solutions of auxiliary linear equations, which are given in terms of the least-squares residual. We summarize these results as follows.

Corollary 5.1 Let $A \in \mathbb{R}^{m\times n}_{m-1}$.
(a) Let $1 < p < \infty$. Then the ℓ_p–approximate solutions of $A\mathbf{x} = \mathbf{b}$ are exactly the solutions of the linear equation

$$A\,\mathbf{x} \;=\; \mathbf{b} \;+\; \beta\,(\operatorname{sign}\mathbf{r}^{\{2\}}) \circ \left|\mathbf{r}^{\{2\}}\right|^{q-1}, \tag{5.10}$$

where β is given by (5.4).
(b) Let $p = \infty$. Then any solution of (5.10), with β given by (5.8) and $q = 1$, is an ℓ_∞–approximate solution of $A\mathbf{x} = \mathbf{b}$.
(c) Let $p = 1$ and let ℓ be selected in $\arg\max_i |r_i^{\{2\}}|$. Then any solution of the linear equation

$$A\,\mathbf{x} \;=\; \mathbf{b} \;+\; \frac{\sum_{i=1}^{m} \left|r_i^{\{2\}}\right|^2}{r_\ell^{\{2\}}}\,\mathbf{e}_\ell \tag{5.11}$$

is an ℓ_1–approximate solution of $A\mathbf{x} = \mathbf{b}$. □

Remark 5.1 For $A \in \mathbb{R}^{(n+1)\times n}_{n}$ and $p = \infty$, Corollary 5.1(b) reduces to [8, Theorem, p. 41], see also [9], [14].

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