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GENERALIZED INVERSES OF DIFFERENTIAL-ALGEBRAIC OPERATORS*

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Abstract. In the theoretical treatment of linear differential-algebraic equations one must deal with inconsistent initial conditions, inconsistent inhomogeneities, and undetermined solution components. Often their occurrence is excluded by assumptions to allow a theory along the lines of differential equations. This paper aims at a theory that generalizes the well-known least squares solution of linear algebraic equations to linear differential-algebraic equations and that fixes a unique solution even when the initial conditions or the inhomogeneities are inconsistent or when undetermined solution components are present. For that a higher index differential-algebraic equation satisfying some mild assumptions is replaced by a so-called strangeness-free differential-algebraic equation with the same solution set. The new equation is transformed into an operator equation and finally generalized inverses are developed for the underlying differential-algebraic operator.

Key words. differential-algebraic equations, standard form, Moore–Penrose pseudoinverse, generalized inverse, least squares regularization

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1. Introduction.

We study the solution of linear differential-algebraic equations (DAEs)

$$E(t)\dot{x}(t) = A(t)x(t) + f(t) \tag{1}$$

with initial condition

$$x(a) = x_0, \tag{2}$$

where $t \in [a,b]$ and $E, A \in C([a,b],\mathbb{R}^{m,n})$, $f \in C([a,b],\mathbb{R}^m)$, $x_0 \in \mathbb{R}^n$. Here $C^r([t_0,t_1],\mathbb{R}^{m,n})$ denotes the set of r-times continuously differentiable functions from the interval $[t_0,t_1]$ to the vector space $\mathbb{R}^{m,n}$ of real $m \times n$ matrices.

Although problems of the form (1), (2) can easily be seen as generalizations of possibly under- or overdetermined systems of linear equations

$$Ax = b \tag{3}$$

with $A \in \mathbb{R}^{m,n}$, $b \in \mathbb{R}^m$, theoretical investigations of (1) mostly require the DAE to have a unique solution for consistent initial values x_0. This reduces the considerations not only to the case $m = n$ but also prohibits the occurrence of undetermined solution components. This, however, excludes DAEs that may be found, e.g., in the study of optimal control problems for descriptor systems; see [16].

In the theory of linear equations the problem of undetermined solution components or inconsistent right-hand sides is overcome by embedding (3) into the minimization problem

$$\frac{1}{2}\|x\|_2^2 = \min! \quad \text{s.t.} \quad \frac{1}{2}\|Ax - b\|_2^2 = \min!, \tag{4}$$

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which has a unique solution in any case. This unique solution, also called *least squares solution*, can be written in the form

$$x = A^+ b \tag{5}$$

with the help of the Moore–Penrose pseudoinverse A^+ of A. A more detailed interpretation is that the matrix A induces a homomorphism $A: \mathbb{R}^n \to \mathbb{R}^m$ by $x \mapsto Ax$. For fixed A the mapping which maps b on the unique solution x of (4) is found to be linear. A matrix representation of this homomorphism with respect to canonical bases is then given by A^+. It is well known that A^+ satisfies the four Penrose axioms

$$\begin{array}{ll} \text{(a)} & AA^+A = A, \\ \text{(b)} & A^+AA^+ = A^+, \\ \text{(c)} & (AA^+)^T = AA^+, \\ \text{(d)} & (A^+A)^T = A^+A; \end{array} \tag{6}$$

see, e.g., [1, 6]. In turn, for given $A \in \mathbb{R}^{m,n}$, the four axioms fix a unique matrix $A^+ \in \mathbb{R}^{n,m}$, whose existence follows, for example, by the solvability of (4).

It is the aim of this paper to generalize this concept for linear equations to a large class of linear DAEs with variable coefficients. In more detail, we first replace the given problem by an equivalent (in the sense that there is a special one-to-one correspondence of the solution sets) so-called strangeness-free problem. For this we then develop an appropriate generalized inverse of some operator representation of the new problem in the spirit of the Moore–Penrose pseudoinverse. In particular, the explicit representation of this Moore–Penrose pseudoinverse mainly consists of the solution of a linear quadratic control problem.

In [11], Hanke treated similar questions in the context of integrable functions. In a Hilbert space setting he was able to show that, in general, the operators describing (1), (2) with $m = n$ are closable. He then gave a representation of the associated closed operator for which the Moore–Penrose pseudoinverse then exists but is in general not continuous. Finally he showed that it is indeed continuous for problems with (differentiation) index at most one and not continuous when the index exceeds one. In contrast to his approach, we replace a higher index problem by an equivalent strangeness-free problem, we work in spaces of continuous functions (i.e., we have no Hilbert space structure), we allow for undetermined solution components and nonsquare systems, and we give an explicit representation of the Moore–Penrose pseudoinverse, thus showing continuity of the pseudoinverse.

Note that besides the Moore–Penrose pseudoinverse one can find other kinds of generalized inverses when dealing with differential-algebraic equations or special cases of them. The so-called Drazin inverse, see, e.g., [7] or [6, Chap. 9], is used for equations with constant coefficients to give an explicit representation of the set of solutions and consistent initial values. This theory, however, is not extendable to the case of variable coefficients. In the theory of boundary value problems for linear ordinary differential equations, so-called generalized Green's functions are used; see, e.g., [22, Chap. III, §10]. These functions define operators that yield a specific solution for a given inhomogeneity even when the solution is not unique due to the choice of the boundary conditions. Note, however, that due to the unique solvability of initial value problems these operators are Fredholm operators; i.e., the given problem is essentially finite dimensional. In the present paper we only treat initial value problems for linear DAEs. But these allow for the presence of undetermined solution components such that the kernel of the associated operator may have infinite dimension. Thus the operator is, in general, not Fredholm. The main focus of this paper, therefore, will be to handle this infinite dimensional kernel. The extension to boundary value problems seems to be possible but is beyond the scope of this paper.

The present paper is organized as follows. In §2, we give a standard form of DAEs required for the subsequent construction, thus specifying the class of DAEs we can treat in the theory to follow. The appropriate analytical context on the basis of dual systems is outlined in §3. We then treat two possible embeddings of (1), (2) into minimization problems in §4, both leading to generalizations of the Moore–Penrose pseudoinverse for matrices. Finally, we give some conclusions in §5.

2. Standard form of DAEs. In order to treat (1) as generalization of linear equations on the one hand as well as of differential equations on the other we must carefully select suitable definitions for solvability and related questions fitting to both extreme cases. Even finding an appropriate notion of solvability of (1) seems to be a hard problem. See, e.g., [2, 4, 5, 8, 10, 12] for different definitions of solvability in the context of DAEs. Many of them are orientated at properties of linear differential equations and ignore results known for the special case (3), one of which, for example, is that (3) is solvable (in the sense that there is a solution) if and only if $\operatorname{rank} A = \operatorname{rank}(A, b)$. In view of (1) the weakest possible meaning of a (strong or classical) solution without additional assumptions on the smoothness of the coefficients is given in the following definition.

DEFINITION 2.1. (a) *A function* $x \in C^1([a,b],\mathbb{R}^n)$ *is called a* solution *of* (1) *if and only if it satisfies* (1) *pointwise.*

(b) *The DAE* (1) *is called* solvable *and the inhomogeneity* f *is called* consistent *if and only if* (1) *has at least one solution.*

(c) *An initial condition* (2) *is called* consistent *if and only if* (1) *has a solution that satisfies* (2).

(d) *An initial value problem* (1), (2) *is called* (uniquely) solvable *if and only if there is a* (*unique*) *solution of* (1) *satisfying* (2).

Under certain circumstances it is possible and necessary to weaken the smoothness requirements for a solution. We shall come back to this point when it becomes important.

Unfortunately it seems to be impossible to deal with (1) in full generality. Without any further restrictions many undesired phenomena can occur. Compare the observations made in the following examples with the fact that linear differential equations, corresponding to E being pointwise nonsingular, are uniquely solvable for any continuous coefficients E, A, and f.

Example 2.2. Consider the singular differential equation

$$t\dot{x}(t) = f(t)$$

on $[-1,1]$ with initial condition $x(-1) = 0$. For $\dot{x}$ to be continuous, we must require f to be continuous $f(0) = 0$ and f differentiable at $t = 0$. The unique solution is then given by

$$x(t) = \int_{-1}^{t} \frac{f(s)}{s}\, ds.$$

Example 2.3. Consider the so-called standard problem of index two

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix},$$

which has the unique solution

$$
\begin{aligned}
x_1(t) &= -\dot{f}_2(t) - f_1(t),\\
x_2(t) &= -f_2(t),
\end{aligned}
$$

independent of the interval of interest. Obviously we must at least require f to be continuously differentiable on the entire interval to be able to write down the solution. Because of the special shape of E, which cancels the entry $\dot{x}_1$, we may be theoretically satisfied with this smoothness requirement, although the above definition would need f to be twice continuously differentiable.

Hence the set of possible inhomogeneities may be restricted even in the case of uniquely solvable problems by additional smoothness requirements or even by inner point conditions depending on the given matrix functions E and A. For a unified treatment we must therefore impose some restrictions on the functions E and A. It must, however, be clear that the remaining class of DAEs is reasonably large.

In [17, 19, 18, 21] it has been shown that under some constant rank and smoothness assumptions concerning the matrix functions E and A a given (higher index) DAE can be transformed in such a way that the set of solutions remains the same and the new equation is *strangeness-free*. The latter property can be defined in the following way.

DEFINITION 2.4. *The DAE* (1) *is called strangeness-free if there exist* $P \in C([a,b],\mathbb{R}^{m,m})$ *and* $Q \in C^1([a,b],\mathbb{R}^{n,n})$*, both pointwise orthogonal, such that we can transform* (1) *to the standard form*

$$
\tilde{E}(t)\dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t) + \tilde{f}(t), \tag{7}
$$

where

$$
\begin{aligned}
\tilde{E}(t) &= P(t)E(t)Q(t) = \begin{bmatrix} \Sigma_E(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\\
\tilde{A}(t) &= P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ A_{21}(t) & \Sigma_A(t) & 0 \\ 0 & 0 & 0 \end{bmatrix},\\
\tilde{x}(t) &= Q(t)^T x(t),\\
\tilde{f}(t) &= P(t)f(t)
\end{aligned} \tag{8}
$$

with Σ_E *and* Σ_A *pointwise nonsingular and all blocksizes are allowed to be zero.*

Observe that this definition is more general than requiring the differentiation index (see, e.g., [5]) to be at most one. This is due to the occurrence of the third block row and column in $\tilde{E}$ and $\tilde{A}$ which yields an infinite dimensional solution space for the homogeneous problem $\tilde{E}(t)\dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t)$. If these blocks are not present (implying $m = n$) the assumption of (1) being strangeness-free reduces to the assumption of (1) having differentiation index zero or one.

Note that the above transformation onto a strangeness-free DAE is numerically implementable; i.e., for a numerical treatment of (1) we may assume that the problem is strangeness-free; see [19]. At this point we should mention that the transformation procedure in [17]–[19] does not determine E, A, and f uniquely but up to multiplication by a pointwise orthogonal matrix function from the left. Thus we must take care that our approach does not depend on such transformations. We also remark that it is currently under investigation how far the necessary constant rank assumptions can be relaxed if one still requires classical solutions or how weaker solvability concepts can be obtained by dropping further assumptions; see [3, 21].

In order to treat problems of the form (1), (2) that have no unique solution along the lines of the treatment of (3), a necessary condition is that in the uniquely solvable case the mapping which maps f on the unique solution x for fixed E and A is linear. In particular, we must have the trivial solution for $f \equiv 0$. Necessary for this is that the initial condition is homogeneous, i.e., that $x_0 = 0$. This, however, can be obtained without loss of generality by shifting $x(t)$ to $x(t)-x_0$ which changes the inhomogeneity from $f(t)$ to $f(t) + A(t)x_0$.

Summarizing this section, considering the current state of research, it seems reasonable to concentrate on those linear DAEs with homogeneous initial condition

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \quad x(a) = 0, \tag{9}$$

that are strangeness-free, i.e., that can be transformed into the standard form indicated by (7) and (8).

3. Dual systems. Following the lines of the construction of the Moore–Penrose pseudoinverse for matrices as sketched in the introduction, we must deal with homomorphisms between function spaces, preferably some linear spaces of continuous functions or appropriate subspaces. In view of (4) the norm of choice would be given by

$$\|x\| = \sqrt{(x,x)}, \quad (x,y) = \int_a^b x(t)^T y(t)\,dt. \tag{10}$$

Because spaces of continuous functions cannot be closed with respect to this norm, we are not in the pure setting of Banach spaces nor of Hilbert spaces. See [1, Chap. 8] for details on generalized inverses of operators on Hilbert spaces. In this section we therefore build up a scenario for defining a Moore–Penrose pseudoinverse which is general enough to be applicable in the setting of linear spaces of continuous functions.

Looking at (6) we find two essential ingredients in imposing the four Penrose axioms. These are the binary operation of matrix multiplication and the transposition of square matrices. In the language of mappings they must be interpreted as composition of homomorphisms (we shall still call it multiplication) and the adjoint of endomorphisms. While the first item is trivial in any setting, the notion of an adjoint is heavily based on the presence of a Hilbert space structure. The most general substitute we can find here is the concept of conjugates with respect to dual systems (pairs); cf. [14, Chap. IX].

DEFINITION 3.1. *Let (X, X^*) be a pair of (real) vector spaces equipped with a bilinear form $(\cdot,\cdot): X \times X^* \to \mathbb{R}$.*

(a) *The pair (X, X^*) is called a* left dual system *if and only if $(x, x^*) = 0$ for all $x \in X$ implies $x^* = 0$.*

(b) *The pair (X, X^*) is called a* right dual system *if and only if $(x, x^*) = 0$ for all $x^* \in X^*$ implies $x = 0$.*

(c) *The pair (X, X^*) is called a* dual system *if and only if it is a left as well as a right dual system.*

It is common sense not to state the bilinear form explicitly. Requiring (X, X^*) to be some dual system therefore includes that there is a related fixed bilinear form with the above properties.

DEFINITION 3.2. *Let (X, X^*) be a left dual system and $A: X \to X$ be an endomorphism. An endomorphism $A^*: X^* \to X^*$ is called a* conjugate *of A if and only if*

$$(Ax, x^*) = (x, A^*x^*) \tag{11}$$

holds for all $x \in X$ and $x^* \in X^*$.

For a unique declaration of a Moore–Penrose pseudoinverse we of course need at least uniqueness of a conjugate. In addition we also need the inversion rule for the conjugate of a product.

LEMMA 3.3. *Let (X, X^*) be a left dual system and $A: X \to X$ be an endomorphism. There is at most one endomorphism $A^*: X^* \to X^*$ being conjugate to A. Let the endomorphisms $A^*, B^*: X^* \to X^*$ be conjugate to the endomorphisms $A, B: X \to X$. Then AB has a conjugate $(AB)^*$ which is given by*

$$(AB)^* = B^*A^*. \tag{12}$$

Proof. See, e.g., [14]. □

Observing that the third and fourth Penrose axioms in (6) require some endomorphisms to be self-conjugate, we must restrict to *self-dual systems*, i.e., to $X^* = X$. At this point we have everything prepared to define a Moore–Penrose pseudoinverse for an appropriate class of homomorphisms.

DEFINITION 3.4. *Let (X, X) and (Y, Y) be (left) dual systems and $D: X \to Y$ be a homomorphism. A homomorphism $D^+: Y \to X$ is called a* Moore–Penrose pseudoinverse *of D if and only if DD^+ and D^+D possess conjugates $(DD^+)^*$ and $(D^+D)^*$ and the relations*

$$\begin{array}{ll} \text{(a)} & DD^+D = D, \\ \text{(b)} & D^+DD^+ = D^+, \\ \text{(c)} & (DD^+)^* = DD^+, \\ \text{(d)} & (D^+D)^* = D^+D \end{array} \tag{13}$$

hold.

As for matrices, the four axioms (13) guarantee uniqueness of the Moore–Penrose pseudoinverse, whereas existence in general cannot be shown.

LEMMA 3.5. *Let (X, X) and (Y, Y) be (left) dual systems and $D: X \to Y$ be a homomorphism. Then D has at most one Moore–Penrose pseudoinverse $D^+: Y \to X$.*

Proof. Let $D^+, D^-: Y \to X$ be two Moore–Penrose pseudoinverses of D. Then we have

$$\begin{aligned} D^+ &= D^+DD^+ = D^+DD^-DD^+ \\ &= (D^+D)^*(D^-D)^*D^+ = (D^-DD^+D)^*D^+ \\ &= (D^-D)^*D^+ = D^-DD^+ = D^-(DD^+)^* \\ &= D^-(DD^-DD^+)^* = D^-(DD^+)^*(DD^-)^* \\ &= D^-DD^+DD^- = D^-DD^- = D^-. \qquad \square \end{aligned}$$

We finish this section with the remark that a Euclidean space X, i.e., a (real) vector space with an inner product, trivially forms a dual system (X, X) with itself.

4. Generalized inverses. According to (4) and (10) we consider the minimization problem

$$\frac{1}{2}\|x\|^2 = \min! \quad \text{s.t.} \quad \frac{1}{2}\|Dx - f\|^2 = \min! \tag{14}$$

with D defined by

$$Dx(t) = E(t)\dot{x}(t) - A(t)x(t) \tag{15}$$

from (9) or more explicitly

$$\frac{1}{2}\int_a^b \|x(t)\|_2^2 \, dt = \min! \quad \text{s.t.} \quad \frac{1}{2}\int_a^b \|E(t)\dot{x}(t) - A(t)x(t) - f(t)\|_2^2 \, dt = \min!. \tag{16}$$

In this form the specification of the problem is not complete. We still have to specify the appropriate spaces X and Y for D to act between. Requiring x to be continuously differentiable in general yields a continuous $f = Dx$. But even in the uniquely solvable case, f being continuous cannot guarantee the solution x to be continuously differentiable as the case $E \equiv 0$ shows.

We circumvent this problem by setting

$$\begin{aligned} X &= \{x \in C([a,b],\mathbb{R}^n) \mid \boldsymbol{E}^+\boldsymbol{E}x \in C^1([a,b],\mathbb{R}^n),\ \boldsymbol{E}^+\boldsymbol{E}x(a) = 0\}, \\ Y &= C([a,b],\mathbb{R}^m), \end{aligned} \tag{17}$$

and defining $D: X \to Y$ indirectly via the standard form (7) by

$$D = \boldsymbol{P}^T \tilde{D} \boldsymbol{Q}^T \tag{18}$$

where $\tilde{D}: \tilde{X} \to \tilde{Y}$ with

$$\tilde{D}\tilde{x}(t) = \tilde{E}(t)\dot{\tilde{x}}(t) - \tilde{A}(t)\tilde{x}(t) \tag{19}$$

and

$$\begin{aligned} \tilde{X} &= \{\tilde{x} \in C([a,b],\mathbb{R}^n) \mid \tilde{\boldsymbol{E}}^+\tilde{\boldsymbol{E}}\tilde{x} \in C^1([a,b],\mathbb{R}^n),\ \tilde{\boldsymbol{E}}^+\tilde{\boldsymbol{E}}\tilde{x}(a) = 0\}, \\ \tilde{Y} &= C([a,b],\mathbb{R}^m). \end{aligned} \tag{20}$$

To simplify notation, here and in the following we use bold letters to denote operators standing for pointwise application of the corresponding matrix function; e.g., $\boldsymbol{E}x(t) = E(t)x(t)$. Similarly, one has to interpret superscripts at such operators; e.g., $\boldsymbol{Q}^T x(t) = Q(t)^T x(t)$. In this way the matrix functions P and Q fix operators $\boldsymbol{P}: Y \to \tilde{Y}$ and $\boldsymbol{Q}: \tilde{X} \to X$. The latter property holds, because for $\tilde{x} \in \tilde{X}$ and $x = \boldsymbol{Q}\tilde{x}$ we get

$$\begin{aligned} \boldsymbol{E}^+\boldsymbol{E}x &= (\boldsymbol{P}^T\tilde{\boldsymbol{E}}\boldsymbol{Q}^T)^+(\boldsymbol{P}^T\tilde{\boldsymbol{E}}\boldsymbol{Q}^T)x \\ &= \boldsymbol{Q}\tilde{\boldsymbol{E}}^+\boldsymbol{P}\boldsymbol{P}^T\tilde{\boldsymbol{E}}\tilde{x} = \boldsymbol{Q}\tilde{\boldsymbol{E}}^+\tilde{\boldsymbol{E}}\tilde{x} \in C^1([a,b],\mathbb{R}^n), \end{aligned}$$

because $Q \in C^1([a,b],\mathbb{R}^{n,n})$ and P, Q are pointwise orthogonal, and hence $x \in X$.

Setting $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ according to the block structure of the standard form (8) and observing the special form of $\tilde{E}$, the condition $\tilde{x} \in \tilde{X}$ implies $\tilde{x}_1$ to be continuously differentiable. But only this part of $\tilde{x}$ actually appears on the right-hand side of (19). Thus (18) indeed defines an operator $D: X \to Y$ allowing the use of less smooth functions x compared with Definition 2.1. In addition, it is easy to see that D is a homomorphism. Compare this construction with the introduction of a so-called modified matrix pencil in [10] which also aimed at admitting less smooth solutions.

In accordance with the theory of differential equations, we call D a *differential-algebraic operator.*

Because

$$\|x\| = \|\boldsymbol{Q}^T x\| = \|\tilde{x}\|, \quad \|Dx - f\| = \|\boldsymbol{P}(\boldsymbol{P}^T\tilde{D}\boldsymbol{Q}^T x - f)\| = \|\tilde{D}\tilde{x} - \tilde{f}\|, \tag{21}$$

the minimization problem (14) transforms covariantly with the application of the operators $\boldsymbol{P}$ and $\boldsymbol{Q}$. Consequently, we can first solve the minimization problem for DAEs in standard form and then transform the solution back to get a solution of the original problem. Moreover, having found the Moore–Penrose pseudoinverse $\tilde{D}^+$ of $\tilde{D}$ the relation

$$D^+ = \boldsymbol{Q}\tilde{D}^+\boldsymbol{P} \tag{22}$$

immediately gives the Moore–Penrose pseudoinverse of D.

Inserting the explicit form of $\tilde{E}$ and $\tilde{A}$ into (16) for the transformed problem yields

$$\begin{aligned} &\frac{1}{2}\int_a^b (\tilde{x}_1(t)^T\tilde{x}_1(t) + \tilde{x}_2(t)^T\tilde{x}_2(t) + \tilde{x}_3(t)^T\tilde{x}_3(t))\,dt = \min! \\ &\qquad\text{s.t. } \frac{1}{2}\int_a^b (\tilde{w}_1(t)^T\tilde{w}_1(t) + \tilde{w}_2(t)^T\tilde{w}_2(t) + \tilde{w}_3(t)^T\tilde{w}_3(t))\,dt = \min! \end{aligned} \tag{23}$$

with

$$\begin{aligned} \tilde{w}_1(t) &= \Sigma_E(t)\dot{\tilde{x}}_1(t) - A_{11}(t)\tilde{x}_1(t) - A_{12}(t)\tilde{x}_2(t) - A_{13}(t)\tilde{x}_3(t) - \tilde{f}_1(t), \\ \tilde{w}_2(t) &= -A_{21}(t)\tilde{x}_1(t) - \Sigma_A(t)\tilde{x}_2(t) - \tilde{f}_2(t), \\ \tilde{w}_3(t) &= -\tilde{f}_3(t), \end{aligned} \tag{24}$$

where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ are partitioned according to the given block structure. For given $\tilde{f} \in \tilde{Y}$ minimization is to be taken over the whole of $\tilde{X}$ from (20) which can be written as

$$\tilde{X} = \{(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in C([a,b], \mathbb{R}^n) \mid \tilde{x}_1 \text{ continuously differentiable, } \tilde{x}_1(a) = 0\}. \tag{25}$$

The constraint is easily satisfied by choosing an arbitrary continuous function $\tilde{x}_3$, taking $\tilde{x}_1$ to be the solution of the linear initial value problem

$$\begin{aligned} \dot{\tilde{x}}_1(t) = {} & \Sigma_E(t)^{-1}[A_{11}(t) - A_{12}(t)\Sigma_A(t)^{-1}A_{21}(t)]\tilde{x}_1(t) \\ & + \Sigma_E(t)^{-1}[A_{13}(t)\tilde{x}_3(t) + \tilde{f}_1(t) - A_{12}(t)\Sigma_A(t)^{-1}\tilde{f}_2(t)], \ \tilde{x}_1(a) = 0, \end{aligned} \tag{26}$$

and finally setting

$$\tilde{x}_2(t) = -\Sigma_A(t)^{-1}[A_{21}(t)\tilde{x}_1(t) + \tilde{f}_2(t)]. \tag{27}$$

Thus we remain with the problem of minimizing $\frac{1}{2}\|x\|^2$ under the constraints (26) and (27).

Observe that this is the place where pointwise invertibility of the matrices Σ_E, Σ_A is needed in order to satisfy the constraints. But it is clear that weak solvability or smooth completion of solutions may be applied to generalize our results.

Taking a closer look at system (23), one immediately recognizes a linear quadratic control problem where $\tilde{x}_3$ takes the role of the control. But compared with the standard problem of linear quadratic control, the constraints appear to be more general due to the occurrence of inhomogeneities. See, e.g., [20] and references therein for related results on the homogeneous linear quadratic control problem.

4.1. Linear quadratic control problems with inhomogeneities. Because the solution of linear quadratic control problems with inhomogeneities is an interesting topic by itself, we treat this problem with a new simplified and adapted notation which should not be mixed up with the one used so far.

THEOREM 4.1. *Let*

$$
\begin{aligned}
&A \in C([a,b],\mathbb{R}^{d,d}), \quad B \in C([a,b],\mathbb{R}^{d,k}), \quad C \in C([a,b],\mathbb{R}^{l,d}),\\
&f \in C([a,b],\mathbb{R}^{d}), \quad g \in C([a,b],\mathbb{R}^{l}).
\end{aligned} \tag{28}
$$

Then the linear quadratic control problem

$$
\begin{aligned}
&\frac{1}{2}\int_a^b (x(t)^T x(t) + y(t)^T y(t) + u(t)^T u(t))\,dt = \min!\\
&\qquad\qquad s.t.\ \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t),\ x(a) = 0,\\
&\qquad\qquad\qquad y(t) = C(t)x(t) + g(t)
\end{aligned} \tag{29}
$$

possesses a unique solution $x \in C^1([a,b],\mathbb{R}^d)$, $y \in C([a,b],\mathbb{R}^l)$, $u \in C([a,b],\mathbb{R}^k)$. *This solution coincides with the corresponding part of the unique solution of the boundary value problem*

$$
\begin{aligned}
\dot{\lambda}(t) &= (I + C(t)^T C(t))x(t) - A(t)^T\lambda(t) + C(t)^T g(t),\ \lambda(b) = 0,\\
\dot{x}(t) &= A(t)x(t) + B(t)u(t) + f(t),\ x(a) = 0,\\
y(t) &= C(t)x(t) + g(t),\\
u(t) &= B(t)^T\lambda(t)
\end{aligned} \tag{30}
$$

which can be obtained by the successive solution of the initial value problems

$$
\begin{aligned}
\dot{P}(t) &= I + C(t)^T C(t) - P(t)A(t) - A(t)^T P(t) - P(t)B(t)B(t)^T P(t),\ P(b) = 0,\\
\dot{v}(t) &= C(t)^T g(t) - P(t)f(t) - A(t)^T v(t) - P(t)B(t)B(t)^T v(t),\ v(b) = 0,\\
\dot{x}(t) &= A(t)x(t) + B(t)B(t)^T(P(t)x(t) + v(t)) + f(t),\ x(a) = 0,\\
\lambda(t) &= P(t)x(t) + v(t),\\
y(t) &= C(t)x(t) + g(t),\\
u(t) &= B(t)^T\lambda(t).
\end{aligned} \tag{31}
$$

Proof. Eliminating y with the help of the algebraic constraint and using a Lagrangian multiplier λ (see, e.g., [13]), problem (29) is equivalent to (omitting arguments)

$$
J[x,\dot{x},u,\lambda] = \int_a^b \left[\frac{1}{2}(x^T x + (Cx+g)^T(Cx+g) + u^T u) + \lambda^T(\dot{x} - Ax - Bu - f)\right] dt = \min!
$$

with $x, \lambda \in C^1([a,b],\mathbb{R}^d)$, and $u \in C([a,b],\mathbb{R}^k)$. Variational calculus then yields

$$
\begin{aligned}
&J[x+\varepsilon\delta x, \dot{x}+\varepsilon\delta\dot{x}, u+\varepsilon\delta u, \lambda+\varepsilon\delta\lambda]\\
&\quad = \int_a^b \left[\frac{1}{2}\big((x+\varepsilon\delta x)^T(x+\varepsilon\delta x) + (u+\varepsilon\delta u)^T(u+\varepsilon\delta u)\big)\right.\\
&\qquad + (C(x+\varepsilon\delta x)+g)^T(C(x+\varepsilon\delta x)+g)\\
&\qquad \left. + (\lambda+\varepsilon\delta\lambda)^T\big((\dot{x}+\varepsilon\delta\dot{x}) - A(x+\varepsilon\delta x) - B(u+\varepsilon\delta u) - f\big)\right] dt
\end{aligned}
$$

$$
\begin{aligned}
&= J[x, \dot{x}, u, \lambda] \\
&\quad + \varepsilon \left[\lambda^T \delta x \Big|_a^b + \int_a^b (x^T + (Cx+g)^T C - \lambda^T A - \dot{\lambda}^T) \delta x \, dt \right. \\
&\qquad \left. + \int_a^b (u^T - \lambda^T B) \delta u \, dt + \int_a^b \delta\lambda^T (\dot{x} - Ax - Bu - f) \, dt \right] \\
&\quad + \varepsilon^2 \left[\frac{1}{2} \int_a^b (\delta x^T (I + C^T C) \delta x + \delta u^T \delta u) \, dt + \int_a^b \delta\lambda^T (\delta\dot{x} - A\delta x - B\delta u) \, dt \right]
\end{aligned}
$$

after sorting and integration by parts.

For (x, u, λ) to be a minimum, a necessary condition is that for all variations the coefficient of ε vanishes. This at once yields (30).

Now let $(x+\varepsilon\delta x, u+\varepsilon\delta u, \lambda+\varepsilon\delta\lambda)$ be a second minimum. Without loss of generality we have $\varepsilon > 0$. Then $(\delta x, \delta u, \delta\lambda)$ must solve the corresponding homogeneous problem. In particular, we must have

$$\delta\dot{x} = A\delta x + B\delta u.$$

Thus, in this case,

$$
\begin{aligned}
&J[x + \varepsilon\delta x, \dot{x} + \varepsilon\delta\dot{x}, \lambda + \varepsilon\delta\lambda, u + \varepsilon\delta u] \\
&\qquad = J[x, \dot{x}, \lambda, u] + \varepsilon^2 \int_a^b \frac{1}{2} (\delta x^T (I + C^T C) \delta x + \delta u^T \delta u) \, dt.
\end{aligned}
$$

It follows that $\delta x \equiv 0$, $\delta u \equiv 0$, and consequently $\delta\lambda \equiv 0$. Hence, there is at most one solution of the linear quadratic control problem (29) and thus also of the boundary value problem (30).

To determine the unique solution of (30) we set

$$\lambda = Px + v, \quad \dot{\lambda} = P\dot{x} + \dot{P}x + \dot{v},$$

with some $P \in C^1([a,b], \mathbb{R}^{d,d})$, $v \in C^1([a,b], \mathbb{R}^d)$. Inserting into (30), we obtain

$$P\dot{x} + \dot{P}x + \dot{v} = (I + C^T C)x - A^T(Px + v) + C^T g$$

and

$$P\dot{x} = PAx + PBB^T(Px + v) + Pf.$$

Combining these equations, we obtain

$$
\begin{aligned}
&(PA + A^T P + PBB^T P - (I + C^T C) + \dot{P})x \\
&\qquad + (PBB^T v + Pf + A^T v - C^T g + \dot{v}) = 0.
\end{aligned}
$$

Now we choose P and v to be the solutions of the initial value problems

$$
\begin{aligned}
\dot{P} &= I + C^T C - PA - A^T P - PBB^T P, \ P(b) = 0, \\
\dot{v} &= C^T g - Pf - A^T v - PBB^T v, \ v(b) = 0.
\end{aligned}
$$

This choice is possible because the second equation is linear and the first equation is a Riccati differential equation of a kind for which one can show that a symmetric solution exists for any interval of the form $[a, b]$; see, e.g., [15, Chap. 10].

It remains to show that (31) indeed solves (30). This is trivial for the third and fourth equations. For the second equation we of course have $x(a) = 0$ but also

$$\begin{aligned}&\dot{x} - Ax - Bu - f \\ &\quad = Ax + BB^TPx + BB^Tv + f - Ax - BB^TPx - BB^Tv - f = 0.\end{aligned}$$

For the first equation we have $\lambda(b) = P(b)x(b) + v(b) = 0$ and also

$$\begin{aligned}&\dot{\lambda} - (I + C^TC)x + A^T\lambda - C^Tg \\ &= P\dot{x} + \dot{P}x + \dot{v} - (I + C^TC)x + A^TPx + A^Tv - C^Tg \\ &= PAx + PBB^TPx + PBB^Tv + Pf \\ &\quad + (I + C^TC)x - PAx - A^TPx - PBB^TPx \\ &\quad + C^Tg - Pf - A^Tv - PBB^Tv - (I + C^TC)x + A^TPx + A^Tv - C^Tg = 0. \quad \square\end{aligned}$$

We remark here that the objective functional in a standard linear quadratic control problem often contains pointwise symmetric and positive definite matrix functions as additional parameters. Problem (29), however, represents no loss of generality because using the Cholesky decomposition of such matrix-valued functions, which is smooth, we can rescale the unknowns by linear transformations such that these matrix functions become pointwise identities.

4.2. The Moore–Penrose inverse of differential-algebraic operators. We now apply the results obtained for linear quadratic control problems with inhomogeneities to construct the Moore–Penrose inverse of a differential-algebraic operator.

COROLLARY 4.2. *Problem* (23) *with constraints* (26) *and* (27) *has a unique solution* $\tilde{x} \in \tilde{X}$.

Proof. The claim follows from Theorem 4.1 by the following substitutions (again without arguments)

$$\begin{aligned}&A = \Sigma_E^{-1}(A_{11} - A_{12}\Sigma_A^{-1}A_{21}),\ B = \Sigma_E^{-1}A_{13},\ C = -\Sigma_A^{-1}A_{21}, \\ &f = \Sigma_E^{-1}(\tilde{f}_1 - A_{12}\Sigma_A^{-1}\tilde{f}_2),\ g = -\Sigma_A^{-1}\tilde{f}_2.\end{aligned}$$

The unique solution is then given in the form $\tilde{x} = (x, y, u)$. $\square$

We are now ready to define an appropriate operator $\tilde{D}^+ : \tilde{Y} \to \tilde{X}$ as follows. For $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \in \tilde{Y}$, the image $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \tilde{D}^+\tilde{f}$ shall be the unique solution of (23) with (26) and (27). Note that $\tilde{D}^+\tilde{f} \in \tilde{X}$ because $\tilde{x}_1$ as part x of (31) is continuously differentiable and $\tilde{x}_1(a) = 0$. Moreover, because the Riccati differential equation in (31) does not depend on the inhomogeneities, the operator $\tilde{D}^+$ is linear, hence a homomorphism.

THEOREM 4.3. *The operator* $\tilde{D}^+$, *defined as above, is the Moore–Penrose pseudoinverse of* $\tilde{D}$; *i.e., the endomorphisms* $\tilde{D}\tilde{D}^+$ *and* $\tilde{D}^+\tilde{D}$ *have conjugates such that* (13) *holds for* $\tilde{D}$ *and* $\tilde{D}^+$.

Proof. Let $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \in \tilde{Y}$ and $\tilde{D}\tilde{D}^+\tilde{f} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$. With (19) and the notation of Theorem 4.1 and Corollary 4.2 (for simplicity) we get

$$\begin{aligned}\hat{f}_1 &= \Sigma_E\dot{\tilde{x}}_1 - A_{11}\tilde{x}_1 - A_{12}\tilde{x}_2 - A_{13}\tilde{x}_3 \\ &= \Sigma_E\dot{x} - A_{11}x - A_{12}y - A_{13}u \\ &= \Sigma_E(Ax + BB^T(Px + v) + f) - A_{11}x - A_{12}(Cx + g) - A_{13}B^T(Px + v) \\ &= A_{11}x - A_{12}\Sigma_A^{-1}A_{21}x + \Sigma_EBB^T(Px + v) + \tilde{f}_1 - A_{12}\Sigma_A^{-1}\tilde{f}_2 \\ &\qquad - A_{11}x + A_{12}\Sigma_A^{-1}A_{21}x + A_{12}\Sigma_A^{-1}\tilde{f}_2 - \Sigma_EBB^T(Px + v) = \tilde{f}_1, \\ \hat{f}_2 &= -A_{21}\tilde{x}_1 - \Sigma_A\tilde{x}_2 = -A_{21}x - \Sigma_Ay \\ &= \Sigma_A(Cx - y) = \Sigma_A(Cx - Cx - g) = \tilde{f}_2, \\ \hat{f}_3 &= 0,\end{aligned}$$

and $\tilde{D}\tilde{D}^+$ is obviously conjugate to itself. Since $\tilde{D}\tilde{D}^+$ projects onto the first two components and, because $\tilde{f}_3$ has no influence on the solution of (23), we also have $\tilde{D}^+\tilde{D}\tilde{D}^+ = \tilde{D}^+$. Because $\tilde{D}\tilde{x}$ has a vanishing third component for all $\tilde{x} \in \tilde{X}$, the projector $\tilde{D}\tilde{D}^+$ acts as an identity on $\tilde{D}$; i.e., $\tilde{D}\tilde{D}^+\tilde{D} = \tilde{D}$. The rest of the proof deals with the fourth Penrose axiom.

Let $\tilde{x} = (x, y, u) \in \tilde{X}$ and $\tilde{D}^+\tilde{D}\tilde{x} = (\hat{x}, \hat{y}, \hat{u})$. We must now apply $\tilde{D}^+$ to the inhomogeneity

$$\tilde{D}\tilde{x} = \begin{bmatrix} \Sigma_E \dot{x} - A_{11}x - A_{12}y - A_{13}u \\ -A_{21}x - \Sigma_A y \\ 0 \end{bmatrix}.$$

Therefore we must set

$$\begin{aligned} f &= \Sigma_E^{-1}(\Sigma_E \dot{x} - A_{11}x - A_{12}y - A_{13}u + A_{12}\Sigma_A^{-1}(A_{21}x + \Sigma_A y)) \\ &= \dot{x} - Ax - Bu, \\ g &= \Sigma_A^{-1}(A_{21}x + \Sigma_A y) \\ &= -Cx + y. \end{aligned}$$

Recalling that the solution P of the Riccati differential equation does not depend on the inhomogeneity, we must solve

$$\begin{aligned} \dot{v} &= C^T(-Cx + y) - A^T v - PBB^T v - P(\dot{x} - Ax - Bu), \ v(b) = 0, \\ \dot{\hat{x}} &= A\hat{x} + BB^T(P\hat{x} + v) + (\dot{x} - Ax - Bu), \ \hat{x}(a) = 0, \\ \hat{y} &= C\hat{x} - Cx + y, \\ \hat{u} &= B^T(P\hat{x} + v). \end{aligned}$$

Setting $v = w - Px$, $\dot{v} = \dot{w} - P\dot{x} - \dot{P}x$, we obtain

$$\begin{aligned} \dot{w} &= P\dot{x} + (I + C^T C)x - PAx - A^T Px - PBB^T Px \\ &\quad - C^T Cx + C^T y - A^T w + A^T Px - PBB^T w + PBB^T Px \\ &\quad - P\dot{x} + PAx + PBu \\ &= -(A^T + PBB^T)w + (x + C^T y + PBu), \ w(b) = 0. \end{aligned} \tag{32}$$

Let $W(t, s)$ be the Wronskian matrix belonging to $A + BB^T P$ in the sense that

$$\dot{W}(t, s) = (A + BB^T P)W(t, s), \ W(s, s) = I.$$

Then $W(t, s)^{-T}$ is the Wronskian matrix belonging to $-(A + PBB^T)$. With the help of $W(t, s)$ we can represent the solution of the initial value problem (32) in the form

$$w = \int_b^t W(t, s)^{-T}(x + C^T y + PBu)\, ds,$$

or

$$v = -Px + \int_b^t W(t, s)^{-T}(x + C^T y + PBu)\, ds.$$

Here, and in the following, the arguments which must be inserted start with t, and a Wronskian matrix changes it from the first to the second argument.

Setting $\hat{x} = x + z$, we obtain

$$\begin{aligned} \dot{z} &= -\dot{x} + Ax + Az + BB^T Px + BB^T Pz \\ &\quad + BB^T w - BB^T Px + \dot{x} - Ax - Bu \\ &= (A + BB^T P)z + (BB^T w - Bu), \ z(a) = 0, \end{aligned}$$

or

$$z = \int_a^t W(t,s)(BB^T w - Bu)\,ds.$$

Thus we get $(\hat{x}, \hat{y}, \hat{u})$ according to

$$\hat{x} = x + z,\ \hat{y} = y + Cz,\ \hat{u} = B^T(Pz + w).$$

In addition, now let $(\bar{x}, \bar{y}, \bar{u}) \in \tilde{X}$ be given and $\tilde{D}^+\tilde{D}(\bar{x}, \bar{y}, \bar{u}) = (\hat{\bar{x}}, \hat{\bar{y}}, \hat{\bar{u}})$. Then we have

$$\begin{aligned}
&\int_a^b (\bar{x}^T\hat{x} + \bar{y}^T\hat{y} + \bar{u}^T\hat{u})\,dt \\
&= \int_a^b \Big[\bar{x}^T x + \bar{x}^T \int_a^t W(t,s)(BB^T \int_b^s W(s,r)^{-T}(x + C^T y + PBu)\,dr - Bu)\,ds \\
&\quad + \bar{y}^T y + \bar{y}^T C \int_a^t W(t,s)\left(BB^T \int_b^s W(s,r)^{-T}(x + C^T y + PBu)\,dr - Bu\right)ds \\
&\quad + \bar{u}^T B^T P \int_a^t W(t,s)\left(BB^T \int_b^s W(s,r)^{-T}(x + C^T y + PBu)\,dr - Bu\right)ds \\
&\quad + \bar{u}^T B^T \int_b^t W(t,s)^{-T}(x + C^T y + PBu)\,ds\Big]\,dt \\
&= \int_a^b (\bar{x}^T x + \bar{y}^T y)\,dt \\
&\quad - \int_a^b \int_a^t (\bar{x}^T + \bar{y}^T C + \bar{u}^T B^T P) W(t,s) Bu\,ds\,dt \\
&\quad + \int_a^b \int_b^t \bar{u}^T B^T W(t,s)^{-T}(x + C^T y + PBu)\,ds\,dt \\
&\quad + \int_a^b \int_b^t \int_b^s (\bar{x}^T + \bar{y}^T C + \bar{u}^T B^T P) W(t,s) B \\
&\qquad\qquad \cdot B^T W(s,r)^{-T}(x + C^T y + PBu)\,dr\,ds\,dt.
\end{aligned}$$

By transposition and changing the order of the integrations, we finally find

$$\int_a^b (\bar{x}^T\hat{x} + \bar{y}^T\hat{y} + \bar{u}^T\hat{u})\,dt = \int_a^b (x^T\hat{\bar{x}} + y^T\hat{\bar{y}} + u^T\hat{\bar{u}})\,dt,$$

which is nothing else than that $\tilde{D}^+\tilde{D}$ is conjugate to itself. □

It follows immediately that (22) yields the Moore–Penrose pseudoinverse of D. That is, we have shown the existence and uniqueness of an operator D^+ satisfying (13) and thus fixed a unique classical least squares solution for a large class of DAEs (inculding higher index problems) with possibly inconsistent initial data or inhomogeneities or free solution components.

4.3. A (1,2,3)-inverse. Using D^+ for solving DAEs with undetermined solutions components, however, bears at least two disadvantages. First, the undetermined component $\tilde{x}_3$ need not satisfy the given initial value and, second, instead of an initial value problem we must solve a boundary value problem, which means that values of the coefficients in future times influence the solution at the present time.

A simple way out of this problem is to choose the undetermined part to be zero. In the following we shall investigate this approach in the context of generalized inverses.

To do this we consider the matrix functions given by

$$F(t) = (I - E(t)E(t)^+)A(t)(I - E(t)^+E(t)) \tag{33}$$

and

$$\Pi(t) = E(t)^+E(t) + F(t)^+F(t). \tag{34}$$

Transforming to standard form, we then find (omitting arguments)

$$\begin{aligned}\tilde{F} &= (I - \tilde{E}\tilde{E}^+)\tilde{A}(I - \tilde{E}^+\tilde{E}) \\ &= (I - PEQQ^TE^+P^T)(PAQ - PE\dot{Q})(I - Q^TE^+P^TPEQ) \\ &= P(I - EE^+)(A - E\dot{Q}Q^T)(I - E^+E)Q \\ &= P(I - EE^+)A(I - E^+E)Q = PFQ.\end{aligned}$$

Thus F transforms like E and therefore

$$\begin{aligned}\tilde{\Pi} &= \tilde{E}^+\tilde{E} + \tilde{F}^+\tilde{F} \\ &= Q^TE^+P^TPEQ + Q^TF^+P^TPFQ \\ &= Q^T(E^+E + F^+F)Q = Q^T\Pi Q.\end{aligned}$$

A simple calculation now yields

$$\tilde{\Pi} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This in particular shows that Π is pointwise an orthogonal projector. Note that $I - \tilde{\Pi}$ indeed projects onto the undetermined component $\tilde{x}_3$. Hence, we are led to the problem

$$\begin{aligned}&\frac{1}{2}\int_a^b \|(I - \Pi(t))x(t)\|_2^2\,dt = \min! \\ &\qquad \text{s.t. } \frac{1}{2}\int_a^b \|E(t)\dot{x}(t) - A(t)x(t) - f(t)\|_2^2\,dt = \min!\end{aligned} \tag{35}$$

replacing (16). The preceding results say that again the problem transforms covariantly with the application of $\boldsymbol{P}$ and $\boldsymbol{Q}$ so that we only need to solve (35) for DAEs in standard form. Because (35) here implies $\tilde{x}_3 = 0$ by construction, we remain with a reduced DAE that is uniquely solvable. We can therefore carry over all results obtained so far as long as they do not depend on the specific choice of $\tilde{x}_3$. Recognizing that this choice was utilized only for the fourth Penrose axiom, we find that (35) fixes a so-called (1,2,3)-inverse $\tilde{D}^-$ of $\tilde{D}$ satisfying the axioms (13 a, b, c). Keeping the spaces as before, we arrive at the following result.

THEOREM 4.4. *The operator $\tilde{D}^-$ defined by* (35) *is a* $(1, 2, 3)$*-inverse of $\tilde{D}$; i.e., the endomorphism $\tilde{D}\tilde{D}^-$ has a conjugate such that* (13 a, b, c) *hold for $\tilde{D}$ and $\tilde{D}^-$.*

Again defining the operator D^- by $D^- = \boldsymbol{Q}\tilde{D}^-\boldsymbol{P}$ then gives a (1,2,3)-inverse of the operator D. We finish this part with a number of remarks and an example for the application of the presented theory.

Remark 1. In the case $A_{13} \equiv 0$ (including A_{13} empty, i.e., no corresponding block in the standard form), we immediately have $D^- = D^+$. Observing that for $E \equiv 0$ the existence of a standard form (8) requires $\operatorname{rank} A(t)$ to be constant on $[a, b]$, we find $D^+ = D^- = -\boldsymbol{A}^+$. In particular, this shows that both D^+ and D^- are indeed generalizations of the Moore–Penrose pseudoinverse of matrices.

Remark 2. The boundedness of the linear operators $D: X \to Y$ and $D^+, D^-: Y \to X$ where X and Y are seen as the given linear spaces equipped with the norms $\|x\|_X = \|x\|_{L_2} + \|d/dt(\boldsymbol{E}^+\boldsymbol{E}x)\|_{L_2}$ and $\|y\|_Y = \|y\|_{L_2}$ allows for their extension to the closure of X and Y with respect to these norms; see, e.g., [9, Lemma 4.3.16]. In particular, Y becomes the Hilbert space $L_2([a,b],\mathbb{R}^m)$. Other choices of the norms are possible as well.

Remark 3. For the numerical calculation of a solution of a given DAE represented by the operators D^+ or D^-, one has to discretize (16) or (35). Using fixed stepsize $h=(b-a)/N$, $N \in \mathbb{N}$, one would choose discrete spaces X_h and Y_h of finite sequences $\{x_\nu\}_{\nu=0}^N$ and $\{f_\nu\}_{\nu=0}^N$. Thus by discretization we come back to a finite dimensional problem of the form (3) where we know how to compute generalized inverses. But any numerical scheme will couple $x_{\nu+1}$ at least with x_ν due to replacing the derivative by some difference approximation. Because a (1,2,3)-inverse of a lower block triangular matrix is in general not lower block triangular, it is not clear whether there is a (1,2,3)-inverse such that x_ν does not depend on values of the coefficients at points in the future.

Example 4.5. Consider the initial value problem

$$\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}.$$

The DAE of this problem has strangeness-index one (note that in contrast the differentiation index is not defined); see [17]. To obtain a strangeness-free DAE with the same solution space according to [19], we compute

$$M = \begin{bmatrix} E & 0 \\ \dot{E}-A & E \end{bmatrix}, \quad N = \begin{bmatrix} A & 0 \\ \dot{A} & 0 \end{bmatrix}, \quad g = \begin{bmatrix} f \\ \dot{f} \end{bmatrix}$$

and obtain (with shifted initial values)

$$M(t) = \left[\begin{array}{cc|cc} -t & t^2 & 0 & 0 \\ -1 & t & 0 & 0 \\ \hline 0 & 2t & -t & t^2 \\ 0 & t & -1 & t \end{array}\right], \quad N(t) = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad g(t) = \left[\begin{array}{c} f_1(t) - x_{10} \\ f_2(t) - x_{20} \\ \hline \dot{f}_1(t) \\ \dot{f}_2(t) \end{array}\right].$$

Because $\operatorname{rank} M(t) = 2$ for all $t \in \mathbb{R}$, the procedure in [19] reduces to the computation of an orthogonal projection onto the corange of $M(t)$ given, e.g., by

$$Z(t)^T = \frac{1}{\sqrt{1+t^2}} \left[\begin{array}{cc|cc} 1 & -t & 0 & 0 \\ 0 & 0 & 1 & -t \end{array}\right].$$

Now replacing E, A, and f by $Z^T M$, $Z^T N$, and $Z^T g$ yields the strangeness-free DAE

$$\begin{aligned} 0 &= \frac{1}{\sqrt{1+t^2}}(x_1(t) + t x_2(t) + f_1(t) - x_{01} - t f_2(t) + t x_{20}), \\ 0 &= \frac{1}{\sqrt{1+t^2}}(\dot{f}_1(t) - t\dot{f}_2(t)), \end{aligned}$$

together with homogeneous initial conditions. Denoting the coefficient functions again by E, A, and f, we have $E(t)=0$ and

$$A(t) = \frac{1}{\sqrt{1+t^2}} \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix}, \quad f(t) = \frac{1}{\sqrt{1+t^2}} \begin{bmatrix} f_1(t) - x_{01} - t f_2(t) + t x_{20} \\ \dot{f}_1(t) - t\dot{f}_2(t) \end{bmatrix}.$$

According to Remark 1, the least squares solution of the latter DAE is given by $x = -\boldsymbol{A}^+ f$. Shifting back we obtain as the least squares solution of the given original problem

$$x(t) = -\frac{1}{\sqrt{1+t^2}} \begin{bmatrix} -1 & 0 \\ t & 0 \end{bmatrix} \frac{1}{\sqrt{1+t^2}} \begin{bmatrix} f_1(t) - x_{01} - tf_2(t) + tx_{20} \\ \dot{f}_1(t) - t\dot{f}_2(t) \end{bmatrix} + \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

or

$$x(t) = \frac{1}{1+t^2} \begin{bmatrix} f_1(t) - x_{01} - tf_2(t) + tx_{20} \\ -t(f_1(t) - x_{01} - tf_2(t) + tx_{20}) \end{bmatrix} + \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

5. Conclusions. Considering linear DAEs as common generalization of linear ordinary differential equations and linear algebraic equations, our aim in this paper was to define a counterpart of a least squares solution in the case of inconsistent data and/or nonuniquely solvable problems. For this, we followed the approach taken for linear algebraic equations. In particular, we embedded DAEs of a certain type into a minimization problem which was then shown to be uniquely solvable. The corresponding solution operator turned out to satisfy axioms of Penrose type in a general setting of conjugates with respect to some dual systems. In this sense we defined least squares solutions of a large class of DAEs or, in other words, Moore–Penrose pseudoinverses of the corresponding differential-algebraic operators.

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