A DYNAMIC PROGRAMMING SOLUTION OF THE JEEP PROBLEM

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Abstract. The jeep problem is solved by dynamic programming.

1. Introduction

The **jeep problem** is a logistics problem, first formulated around 1945. We quote from D. Gale, [7, p. 493],

> $\cdots$ the problem concerns a jeep which is able to carry enough fuel to travel a distance d, but is required to cross a desert whose distance is greater then d (for example $2d$). It is to do this by carrying fuel from its home base and establishing fuel depots at various points along its route so that it can refuel as it moves further out. It is then required to cross the desert on the minimum possible amount of fuel.

We use the name, **jeep problem**, for the problem of

maximizing the **distance** that can be crossed using a **given** amount of **fuel** . (MAX)

as well as for the equivalent problem of

minimizing the amount of **fuel** for crossing a **given distance** , (MIN)

The problem was solved in 1947 by N.J. Fine[1], [5]. Shortly thereafter, C.G. Phipps [9] generalized the problem, and solved it by arguing that the single jeep problem is equivalent to a problem involving a convoy of jeeps which travel together, some being used to refuel others, with only one jeep required to cross, the others abandoned along the way. The convoy problem has a simple solution, [7, p. 500]. A related problem is

> to determine the range of a fleet of n aircraft with fuel capacities g_i gallons and fuel efficiencies r_i gallons per mile $(i = 1, \cdots, n)$. It is assumed that the aircraft may share fuel in flight and that any of the aircraft may be abandoned at any stage. The range is defined to be the greatest distance which can be attained in this way. Initially the fleet is supposed to have g gallons of fuel [6, p. 541].

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[1] Fine cites a simultaneous, unpublished, solution by L. Alaoglu.

This **fleet range problem** was solved by J.N. Franklin [6] in 1960 using dynamic programming. He obtained explicit solutions for two aircraft, or for any number of identical aircraft.

The **round trip jeep problem** requires the jeep to cross the desert, then cross back and return to the start point. This problem was introduced by D. Gale [7] and solved by A. Hausrath, B. Jackson, J. Mitchem and E. Schmeichel[8]. Other references and solutions include [1], [3], [4].

One may expect the jeep problem to be naturally posed, and solved, by dynamic programming, as done by Franklin [6] for the related fleet range problem. However, to quote from Gale,

> there seems to be a feeling among many people that the jeep problem can be solved by the functional equation method of dynamic programming. In fact the problem occurs as an exercise in the book of Bellman [2], but the solution is not given there and I know of no way of solving the problem by this method. [7, p. 500].

More recently, we read

> the problems are not easily posed as either linear or dynamic programming problems [8, p. 299].

Notwithstanding this pessimistic view, the jeep problem can be formulated, and solved, naturally by dynamic programming. Our solution, in § 2, uses two lemmas: Lemma 1 states that, at any time, fuel need not be stored in more than two points. This property has been observed before, [9], [7]. Lemma 3 shows that maximal steps are optimal.

In § 3 we prove that the jeep problem is a special case of the fleet range problem. J.N. Franklin pointed that the two problems are similar, but the relationship between these two problems has not been studied before.

In § 4 we use dynamic programming to obtain an explicit solution to a more general fleet range problem, allowing aircraft with different fuel efficiencies. The optimal policy is to get rid first of the worst guzzler.

The round trip problem is solved similarly in § 5.

§ 6 gives an example, where the capacities of depots is limited. In this case, Lemma 1 is no longer valid, since more than two depots are needed. However, the maximal distance that can be covered is not changed if the capacity is restricted to one unit per depot.

2. The DP solution

We assume, without loss of generality, that the jeep carries 1 unit of fuel, and with that it covers 1 unit of distance.

Let $V(x)$ be the **optimal value function** of the jeep problem,

$$V(x) \ := \ \text{maximum distance covered using } x \text{ units of fuel} \tag{1}$$

By our assumption,

$$V(x) \ = \ x \ , \quad \text{for all } 0 \le x \le 1 \tag{2}$$

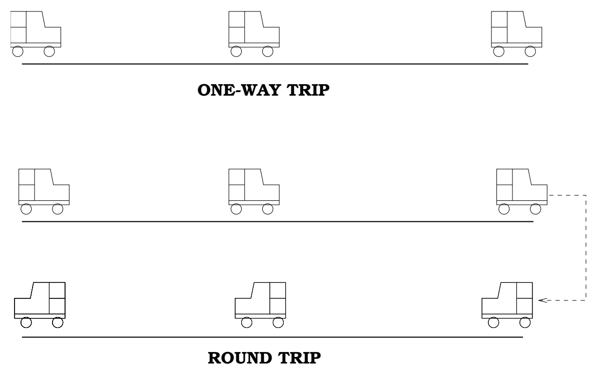

FIGURE 1. The Jeep Problem: One-way and Round trip

so the jeep problem really concerns the case $x > 1$.

Let the jeep move along a coordinate axis, towards the destination which, without loss of generality, is taken as the origin 0. At each trip (except the last) the jeep carries some fuel for storage, to be used in a later trip. The fuel is stored in n depots (n depending on x), denoted by their distances y_i from the destination $y_0 = 0$,

$$y_0 = 0 < y_1 < y_2 < \cdots < y_{n-1} < y_n \ . \tag{3}$$

Thus, y_1 is the depot nearest to the destination (at a distance y_1 from it), y_2 the second nearest, etc. The jeep saga begins with x units of fuel in the first depot, y_n, which is the total distance covered. Under an optimal policy

$$y_n \ = \ V(x) \ ,$$

i.e. the first depot is as far as possible (to cover with x units of fuel) from the destination.

Not all n depots are used to store fuel simultaneously. At any time the depots used for storage are called **active**. We show now that no more than two depots need be active at the same time. First some notation: We denote by τ_i the **number of one–way trips** (in either direction) **between** y_i **and the next depot** y_{i-1}, $i = n, n-1, \cdots, 1$. All τ_i are odd, indeed,

$$\tau_i \ = \ 2\,\rho_i + 1 \ , \tag{4}$$

where ρ_i is the **number of round trips from** y_i **to** y_{i-1}. Clearly $\tau_1 = 1$.

Lemma 1. *Let $\mathcal{P}^k(y_1, \cdots, y_n)$ be the class of policies using depots*

(3) $$y_0 = 0 < y_1 < y_2 < \cdots < y_{n-1} < y_n \ ,$$

with a maximum of k active depots at any time, $k = 1, 2, 3, \cdots, n$, and such that all fuel is used when the jeep reaches the destination.

For any $k = 3, \cdots, n$ and any policy $P \in \mathcal{P}^k(y_1, \cdots, y_n)$, there is a policy

$$\widehat{P} \in \mathcal{P}^2(y_1, \cdots, y_n),$$

with at most two active depots at any time. The two policies P and $\widehat{P}$ use the same depots, and in particular, cover the same distance y_n.

Proof. Consider a policy $P \in \mathcal{P}^k(y_1, \cdots, y_n)$, with given $k \in \{3, \cdots, n\}$ and given depots (3). When the jeep finally leaves y_1, on its way to the destination, the fuel remaining is

$$x_1 \;=\; x - \sum_{i=2}^{n} \tau_i \,(y_i - y_{i-1}) \tag{5}$$

with which it reaches the destination (i.e. $x_1 = y_1$). An explanation of (5): the remaining fuel is the initial fuel minus the fuel expended for all previous trips.

Consider an alternative policy $\widehat{P}$ with the same depots (3), the same numbers of trips τ_i, but at any time at most two, adjacent, depots are active. The policy $\widehat{P}$ moves all fuel from the initial depot y_n to the next depot y_{n-1}, depositing there

$$x_{n-1} \;=\; x - \tau_n \,(y_n - y_{n-1}) \;, \tag{6}$$

units of fuel. Next, $\widehat{P}$ moves all fuel from y_{n-1} to the next depot y_{n-2}, putting there $x_{n-2} = x_{n-1} - \tau_{n-1}\,(y_{n-1} - y_{n-2})$ units of fuel. Continuing in this way, the policy $\widehat{P}$ stores in the last depot y_1 the same amount (5) as the original policy P. □

By Lemma 1 it suffices to consider only policies with at most 2 active depots at any time. Moreover, we can assume that the numbers of trips $\{\tau_n, \tau_{n-1}, \cdots, \tau_2, \tau_1\}$ are all different. First a notation: for any real α, let $\lceil \alpha \rceil$ denote the smallest integer $\geq\; \alpha$.

Lemma 2. Let $P \in \mathcal{P}^2(y_1, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_n)$ be an optimal policy with

$$\tau_i \;=\; \tau_{i+1} \;. \tag{7}$$

Then the depot y_i can be eliminated, i.e. fuel can be moved from y_{i+1} directly to y_{i-1}.

Proof. For any depot y_i, let x_i be the amount of fuel remaining when the jeep reaches y_i, with no fuel remaining in y_{i+1}.

The optimal number of round trips between y_i and the next depot y_{i-1} is $\rho_i = \lceil x_i \rceil - 1$. Therefore the number of trips between y_i and y_{i-1} is

$$\tau_i \;=\; 2\;(\lceil x_i \rceil - 1) + 1 \;=\; 2\,\lceil x_i \rceil - 1 \;.$$

From (7) it follows that $\lceil x_i \rceil \;=\; \lceil x_{i+1} \rceil$, so the jeep need not stop at node y_i, but can proceed from y_{i+1} directly to y_{i-1}. □

Lemma 2 shows that the distances $(y_i - y_{i-1})$ between successive depots can be made sufficiently large to guarantee $\tau_{i-1} \;<\; \tau_i$, i.e. $\lceil x_{i-1} \rceil \;<\; \lceil x_i \rceil$.

This observation enables us to simplify the expression for $V(x)$, the maximal distance covered with an initial fuel stock x. The Bellman optimality principle [2] gives

$$V(x) = \max_{\substack{\epsilon > 0 \\ \rho \geq \lceil x \rceil - 1}} \{\epsilon + V(x - (2\rho + 1)\epsilon)\} \tag{8}$$

where ρ is the number of round trips to the next depot, a distance ϵ away. This can be written as

$$V(x) = \max_{\epsilon > 0} \{\epsilon + V(x - (2\lceil x \rceil - 1)\epsilon)\} \ , \tag{9}$$

since $V(x-(2\rho+1)\epsilon) \leq V(x-(2\lceil x \rceil-1)\epsilon)$ for any $\epsilon > 0$ and $\rho \geq \lceil x \rceil - 1$. The distance ϵ to the next depot need not be smaller than

$$\bar{\epsilon} := \sup\{\epsilon : \lceil x - (2\lceil x \rceil - 1)\epsilon \rceil = \lceil x \rceil\} \tag{10}$$

The interval $\{\epsilon : \lceil x - (2\lceil x \rceil - 1)\epsilon \rceil = \lceil x \rceil\}$ is closed–open, and does not include the supremum $\bar{\epsilon}$ which is easily computed:

$$\bar{\epsilon} = \frac{x - \lceil x \rceil + 1}{2\lceil x \rceil - 1} \ . \tag{11}$$

In fact, the following lemma is a restatement of Lemma 2:

Lemma 3.
(a) For any $0 \leq \epsilon < \bar{\epsilon}$

$$V(x) = \epsilon + V(x - (2\lceil x \rceil - 1)\epsilon) \ . \tag{12}$$

(b)

$$V(x) = \max_{\epsilon \geq \bar{\epsilon}} \{\epsilon + V(x - (2\lceil x \rceil - 1)\,\epsilon)\} \tag{13}$$

Proof. (a) Let $0 \leq \epsilon_1 < \bar{\epsilon}$, i.e. $\lceil x - (2\lceil x \rceil - 1)\epsilon_1 \rceil = \lceil x \rceil$. Then, from (9),

$$\begin{aligned} V(x) &= \max_{\epsilon > 0}\{\epsilon + V(x - (2\lceil x \rceil - 1)\epsilon)\} \\ &= \epsilon_1 + \max_{\epsilon > 0}\{\epsilon + V(x - (2\lceil x \rceil - 1)\epsilon_1) - (2\lceil x \rceil - 1)\epsilon)\} \\ &= \epsilon_1 + V(x - (2\lceil x \rceil - 1)\epsilon_1) \end{aligned}$$

we get the equation (12).
(b) follows from (a). ☐

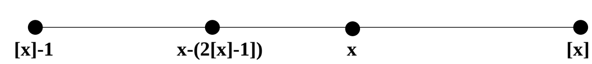

FIGURE 2. Proof of Lemma 3

Using equation (13) and (11) we get the solution of the value function.

Theorem 1. Given a positive number x unit of fuel, then the maximum distance which the jeep can travel is

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2\lceil x \rceil - 3} + \frac{x - \lceil x \rceil + 1}{2\lceil x \rceil - 1}.$$

Proof. We use induction. First we notice that for any x , $V(x) \leq x$. If $\lceil x \rceil = 1$, then it is obvious that

$$V(x) = x \,.$$

Suppose that for $\lceil x \rceil = k$ we have

$$V(x) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-3} + \frac{x - \lceil x \rceil + 1}{2k-1}$$

Now consider the case $\lceil x \rceil = k+1$. Then for $\bar{\epsilon}$ of (10),

$$\begin{aligned} V(x) &= \max_{\epsilon \geq \bar{\epsilon}} \{\epsilon + V(x - (2\lceil x \rceil - 1)\epsilon)\} &(14)\\ &= \bar{\epsilon} + V(x - (2\lceil x \rceil - 1)\bar{\epsilon}) \,. &(15)\end{aligned}$$

The equation (15) is true because that for any $\epsilon > \bar{\epsilon}$ we have

$$\begin{aligned} V(x) &\geq \bar{\epsilon} + V(x - (2\lceil x \rceil - 1)\bar{\epsilon}) \quad \text{by Lemma 3}\\ &\geq \bar{\epsilon} + (\epsilon) - \bar{\epsilon}) + V(x - (2\lceil x \rceil - 3)(\epsilon - \bar{\epsilon}))\\ &> \epsilon + V(x - (2\lceil x \rceil - 1)\epsilon)\end{aligned}$$

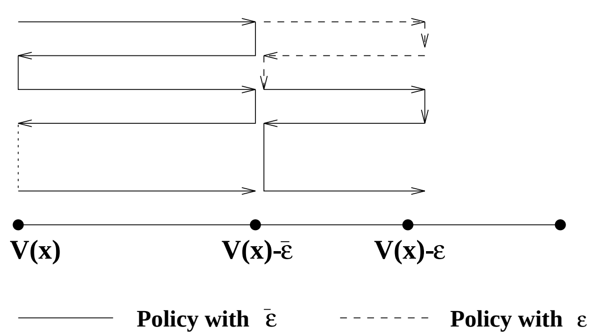

FIGURE 3. Proof of (15)

Using equation (11) we have

$$\begin{aligned} V(x) &= \frac{x - \lceil x \rceil + 1}{2\lceil x \rceil - 1} + V(\lceil x \rceil - 1)\\ &= \frac{x-k}{2k+1} + V(k)\\ &= \frac{x-k}{2k+1} + 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1}.\end{aligned}$$

This is what we need to prove. ∎

3. The fleet problem

To prove that the Jeep problem is a special case of fleet problem let us review the recursion formula for aircraft problem which was obtained by Franklin[6] in 1960.

The fleet problem states as follows. There is a fleet of n aircraft with fuel capacities g_i gallons and fuel consumptions r_i gallons per miles($i = 1, 2, \cdots, n$). The question is to determine the range of the fleet. A theoretical solution of this problem has been obtained by J.N.Franklin using dynamic programming method.

Following Franklin [6], we use C_m stands for any subset of the given n aircraft. Suppose $m > 1$ then a distance x is flown by all aircraft. Then one aircraft is leaving the fleet, the remaining subset is $C_{m-1} \subset C_m$. Note that more than one aircraft may be abandoned at one time. This is equivalent to abandon one of them first, then another one, and so on. After flying a distance x, the amount of fuel left is

$$h = g - x \sum_{i \in C_m} r_i$$

Let $M(g, C_m)$ be the maximum distance for the fleet starting with g units of fuel. Then by Bellman's principle [2, p. ??],

$$M(g, C_m) = \max_{x < \frac{g}{\sum_{i \in C_m} r_i}} \{x + \max_{C_{m-1} \subset C_m} M(g - x \sum_{i \in C_m} r_i, C_{m-1})\} \qquad (16)$$

Now let us consider the case of identical aircraft. Assume all the capacities of the aircraft are the same and all the fuel consumption are also same.

$$g_i = g_0\ ,\ r_i = R\ , \quad i = 1, 2, \cdots, n\ .$$

Let $k := \lceil g/g_0 \rceil$. Then the optimal distance is

$$M_m(g) = \frac{g_0}{R}(1 + \frac{1}{2} + \cdots + \frac{1}{k}) + \frac{g - kg_0}{(k+1)R}\ for\ g \le mg_0, k \ge 1. \qquad (17)$$

$$M_m(g) = \frac{g_0}{R}(1 + \frac{1}{2} + \cdots + \frac{1}{m})\ for\ g > mg_0, \qquad (18)$$

If we chose that $g_0 = 1$ and $R = 2$, then we get from (17) and (18) that

$$M_m(g) = \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{k}) + \frac{g - k}{y} 2(k+1)\ for\ g \le mg_0, k \ge 1. \qquad (19)$$

$$M_m(g) = \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{m})\ for\ g > mg_0, \qquad (20)$$

Remark 1. The equation (17) has a little different with the result in Franklin[6, p. 543] We use the notation $\lceil g/g_0 \rceil$ to indicate the smallest integer number great than or equal to g/g_0. Franklin use the notation $[g/g_0]$ as the greatest integer number less than or equal to g/g_0.

Now we consider the case of Jeep problem. Since the Jeep's capacity is always 1, we may consider every round trip is a new jeep with capacity 1 and fuel consumption 2, i.e. 2 units of fuel per mile, and the one-way trip is a new Jeep with capacity 1 and fuel consumption 1. This is the Jeep problem and it is also a special case of Aircraft problem with the following assumptions: all the aircraft having the same capacity(1), and all the fuel consumption r_is being the same(= 2) except one of them is one.

To solve this problem we may rewrite the Bellman's equation (16) as

$$M(g, C_m) = \max_{x<\frac{g}{2m+1}} \{x + \max_{C_{m-1}\subset C_m} M(g - x(2m-1), C_{m-1})\} \tag{21}$$

In this optimality equation, we may check the optimal procedure is that every time except the last one, we throw away the aircraft with fuel consumption 2, and the last step only the aircraft with fuel consumption 1 to used In fact we have at the first step

$$M(g - x(2m-1), C_{m-1,1}) \leq M(g - x(2m-1), C_{m-1,2}) \tag{22}$$

where $C_{m-1,i}$ stands for the subset of C_m which has $m-1$ elements and it is obtained by deleting the number of the jeep with consumption i, $i = 1$ or 2. Since we may find a feasible solution as in Jeep problem for $M(g - x(2m+1), C_{m-1,2})$ which is great than or equal to $M(g-x(2m+1), C_{m-1,1})$ and the later can be calculated using (21). Continue the discussion we may conclude that we will use the aircraft with fuel consumption 1 until the last step. Therefore the optimal solution is (Assume $m \geq g > m-1$)

$$M(g, C_m) = \frac{g-m-1}{2m-1} + \frac{1}{2m-3} + \cdots + \frac{1}{3} + 1. \tag{23}$$

Remark 2. It is noted that the solution is the same with what we have got before. If the condition $m \geq g > m-1$ is not satisfied, then if $g > m$ the solution is the same with (23) but we use only m units of fuel and the solution is

$$M(g, C_m) = \frac{1}{2m-1} + \frac{1}{2m-3} + \cdots + \frac{1}{3} + 1. \tag{24}$$

The optimal policy is each step goes forward the distance

$$\frac{1}{2m-1}, \frac{1}{2m-3}, \cdots, \frac{1}{3} \text{ and } 1.$$

If $g \leq m-1$, then use $k = \lceil g \rceil$ to replace m in above equation. The results are follows immediately.

4. Another special case

This section is to establish a solution for another special case of aircraft problem as mentioned before in the Introduction section. Suppose that all the aircraft in the fleet have the same capacity. Assume without loss of generality that their common capacity is 1. Their fuel consumption are $r_i, i = 1, 2, \cdots, m$ We may assume that

$$r_1 \geq r_2 \geq \cdots \geq r_m.$$

The optimal solutions should be

$$M(g, C_m) = \max_{x < \frac{g}{\sum_{i \in C_m} r_i}} \{x + M(g - x \sum_{i \in C_m} r_i, C_{m-1})\} \tag{25}$$

where the subsets C_i are given as following

$$C_{m-i} = \{r_{i+1}; i = 1, 2, \cdots, m-1\}, i = 0, 1, \cdots, m-1$$

The optimal policy is to throw away the aircraft with fuel consumption r_i at step i, respectively. In other words to keep the most efficient aircraft as long as possible.

Let us prove the optimality of the policy. The basic idea behind the proof is that we prove that for any $i = 1, 2, \cdots, m$

$M(g-x(2m-1), C_{m-1}) \geq M(g-x(2m-1), C'_{m-1})$, for any other $m-1$ subset of C_m, where $C'_{m-1}) = \{j, j \neq i\}$. This inequality is obvious.

Proof. For any optimal policy for $M(g-x(2m-1), C'_{m-1})$, there is a sequence of the number of aircraft,

$$r_i^0, i = 1, 2, \cdots, m-1.$$

and

$$C_{m-1}^0 = \{r_i^0, i = 1, 2, \cdots, m-1\} \subset C_m.$$

There are only two different elements in C_{m-1} and C_{m-1}^0 Suppose that

$$C_{m-1}^0 = \{r_1, r_2, \cdots, r_{k-1}, r_{k+1}, r_{k+2}, \cdots, r_m\}$$

we may find a feasible policy for $M(g - x(2m-1), C_{m-1})$ which enable us to get at least the same optimal distance as we use the optimal policy for $M(g - x(2m-1), C'_{m-1})$. In fact at each step we change the policy to the following.

i	For $M(g - x(2m-1), C_{m-1})$	For $M(g - x(2m-1), C'_{m-1})$
$\leq k-1$	r_{i+1}	r_i
$\geq k+1$	r_i	r_i

So it is clear the obtained policy for $M(g - x(2m-1), C_{m-1})$ can get at least the same distance as the optimal policy for $M(g - x(2m-1), C'_{m-1})$. This proves that at the first step we throw away the aircraft with fuel consumption r_1. Similar arguments can conclude the desired result. □

5. Round trip problem

The round trip problem is easy to solve using Franklin's recursion formula (16). In fact we may consider the round trip problem as an aircraft problem in the case that all the capacities of the aircraft are one and the consumption for all aircraft are 2.

Then by the optimal equation (17) we have

$$M_m(g) = \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{k}) + \frac{g-k}{y}2(k+1) \; for \; g \le mg_0, k \ge 1.$$

$$M_m(g) = \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{m}) \; for \; g > mg_0,$$

This is the same with Gale result(See [8, p. 300]). The optimal policy is that let the Jeep goes as far as possible using $1 - 1/(2k)$ unit of fuel at step $\lceil x \rceil - k + 1$. Leave $1/(2k)$ unit of fuel at dump k. The following example explains the policy for round-trip Jeep problem.

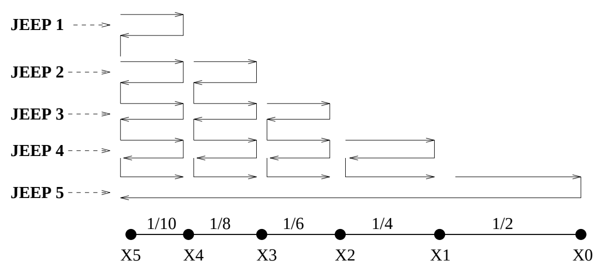

FIGURE 4. Optimal policy for round trip

step	$x5$	$x4$	$x3$	$x2$	$x1$	$x0$
0	5	0	0	0	0	0
1	0	$4+1/10$	0	0	0	0
2	0	$1/10$	$3+1/8$	0	0	0
3	0	$1/10$	$1/8$	$2+1/6$	0	0
4	0	$1/10$	$1/8$	$1/6$	$1+1/4$	0
5	0	$1/10$	$1/8$	$1/6$	$1/4$	$1/2$
6	0	0	0	0	0	0

TABLE 1. Optimal policy for round trip Jeep problem The numbers are the units of fuel left at the depots after the corresponding step

6. An example

Now let us see a example for the case that the capacity at each depot point is only one. The example shows that the capacities of the depots are not important since we may find an optimal policy which require the capacity on every depot is only i unit.

Step 1: From $X5$ to $X4$ leave $1 - 2/9$ unit of fuel, then back to $X5$.

Step 2: From $X5$ to $X4$, pick up $1 - 1/9$ unit of fuel, make up one unit of fuel in the tank of the jeep, to $X3$ and leave $1 - 2/7$ unit of fuel at $X3$, then back to $X4$, pick up $1/9$ unit of fuel then back to $X5$.

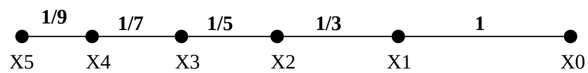

FIGURE 5. Example

step 3: From $X5$ to $X3$, pick up $1-1/9-1/7$ unit of fuel, make up one unit of fuel in the tank of the jeep, to $X2$ and leave $1/5$ unit of fuel at $X2$, then back to $X3$, pick up $1/7$ unit of fuel then back to $X4$, then pick up $1/9$ and back to $X5$.

Step 4. From $X5$ to $X1$, leave $1/3$ unit of fuel, and back to $X5$.

Step 5. From $X5$ to $X4$, pick up $1/9$ unit of fuel, to $X3$, pick up $1/7$ unit of fuel, to $X2$, pick up $1/5$ unit of fuel, to $X1$, pick up $1/3$ unit of fuel, now there is one unit of fuel in the tank, the jeep can go to $X0$ at last.

The following table shows the amount of fuel at each depot for each step: where $X5=0$, $X4=\frac{1}{9}$, $X3=\frac{1}{9}+\frac{1}{7}$, $X2=\frac{1}{9}+\frac{1}{7}+\frac{1}{5}$, $X1=\frac{1}{9}+\frac{1}{7}+\frac{1}{5}+\frac{1}{3}$, $X0=\frac{1}{9}+\frac{1}{7}+\frac{1}{5}+\frac{1}{3}+1$.

steps	X5	X4	X3	X2	X1	X0
0	5	0	0	0	0	0
1	4	7/9	0	0	0	0
2	3	5/9	5/7	0	0	0
3	2	3/9	3/7	3/5	0	0
4	1	1/9	1/7	1/5	1/3	0
5	0	0	0	0	0	0

TABLE 2. The optimal policy for one unit capacity one-way trip Jeep problem

Same idea can be applied to round trip problem. The following table shows the procedure.

steps	X5	X4	X3	X2	X1	X0
0	5	0	0	0	0	0
1	4	8/10	0	0	0	0
2	3	6/10	6/8	0	0	0
3	2	4/10	4/8	4/6	0	0
4	1	2/10	2/8	2/6	2/4	0
5	0	1/10	1/8	1/6	1/4	1/2
6	0	0	0	0	0	0

TABLE 3. The optimal policy for one unit capacity round trip Jeep problem

7. Conclusion

The Jeep problem and Aircraft Problem have been among the most interesting problems in a long time. They have many variations. Although the general aircraft problem can not solved explicitly using dynamic programming method, some important special cases can be solved.

This paper provides the solution for Jeep problem directly using dynamic programming principle. This has been considered not easy. We also proved that the both one way and round trip Jeep problem are special cases of Aircraft problem.

The optimal policy lets the jeep travel as far as possible until the minimum number of round trips needed for moving all fuel from the present depot changed. Except the first step(if the amount of fuel at the beginning is not an integer), we use only one unit of fuel each step and let the jeep travel as far as possible using the one unit of fuel. This observation also help us to prove the following jeep problem: Instead of a empty tank when the jeep stops, we want the jeep leaves some amount of fuel at the last depot. For example, suppose we have x units of fuel at beginning and need to deliver y units of fuel at the last depot. Then the maximum distance the jeep can travel is

$$V(x) - V(y). \tag{26}$$

Same discussion can be applied to round trip problem. If we let $V(x)$ and $V(y)$ be the maximum distances using x and y units of fuel respectively, and we need to deliver y units of fuel at last depot, then we obtain that the maximum round trip distance is the same with (26). This is Theorem B in A. Hausrath, et. al.[8, p. 30].

We also noted that the capacities of the depots are not necessary very large. In fact we may get the maximum distance even if we assume the capacities are all one unit.

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