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# A LOCAL INVERSE FOR NONLINEAR MAPPINGS

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*Dedicated to Professor Richard Varga on his Seventieth Birthday*

ABSTRACT. A mapping $\phi : \mathbb{R}^n \to \mathbb{R}^m$, $n \le m$, with Jacobian of full column-rank, has a local inverse that is analogous to the Moore-Penrose inverse of linear mappings.

## 1. INTRODUCTION

1.1. **Assumptions and notation.** Throughout this paper:
(a) $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, with $m \ge n$, and $\phi : U \to V$ a continuously differentiable bijection.
(b) The $m \times n$ Jacobian matrix $J(\phi)(\mathbf{u}) = \left( \dfrac{\partial \phi_i}{\partial u_j}(\mathbf{u}) \right)$ has full column-rank in $U$.
(c) Given a vector $\mathbf{v} \in \mathbb{R}^m$, consider the system of equations

$$\begin{aligned} \phi_i(u_1, \cdots, u_n) &= v_i \ , \ i = 1, \cdots, m \ , \\ \text{or} \quad \phi(\mathbf{u}) &= \mathbf{v} \ . \end{aligned} \tag{1}$$

By Assumption (b), for any $\mathbf{u} \in U$ there is at least one subsystem of (1), with $n$ equations,

$$\phi_I(\mathbf{u}) = \mathbf{v}_I \ , \tag{2}$$

that is invertible near $\phi(\mathbf{u})$. Here $I$ is an index subset of $\{1, \cdots, m\}$, and $\phi_I$ , $\mathbf{v}_I$ are the corresponding subvectors of $\phi$ and $\mathbf{v}$. Such a system, and its solution,

$$\mathbf{u} = \phi_I^{-1}(\mathbf{v}_I) \tag{3}$$

are called **basic**. The index set of basic subsystems (2) is denoted by $\mathcal{I}(\mathbf{u})$.
(d) The basic inverses $\{\phi_I^{-1} : I \in \mathcal{I}(\mathbf{u})\}$ are assumed continuously differentiable.
If equation (1) is consistent, i.e. $\mathbf{v} = \phi(\mathbf{u})$ for some $\mathbf{u} \in U$, then any basic solution $\phi_I^{-1}(\mathbf{v}_I)$ gives the solution $\mathbf{u}$. A certain convex combination of basic solutions is useful also for inconsistent equations, analogous to the Moore-Penrose inverse in the linear case. To see this we need the facts collected in §§1.2–1.4.

1.2. **The Moore-Penrose inverse.** Given a matrix $A \in \mathbb{R}^{m\times n}$, its **Moore-Penrose inverse** $A^\dagger$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ such that

$$AX\mathbf{v} = \mathbf{v} \ , \quad \forall \mathbf{v} \in R(A) \ , \tag{4a}$$

$$XA\mathbf{u} = \mathbf{u} \ , \quad \forall \mathbf{u} \in R(A^T) \ , \tag{4b}$$

$$(I - AX)\mathbf{v} \perp R(A) \ , \quad \forall \mathbf{v} \in \mathbb{R}^m \ , \tag{4c}$$

$$(I - XA)\mathbf{u} \perp R(A^T) \ , \quad \forall \mathbf{u} \in \mathbb{R}^n \ . \tag{4d}$$

Equations (4a) and (4c) are equivalent to

$$AX = P_{R(A)}$$

the orthogonal projector on $R(A)$, so that, for all $\mathbf{v} \in \mathbb{R}^m$, $A^\dagger \mathbf{v}$ is a **least squares solution** of

$$A\mathbf{x} = \mathbf{v} \ . \tag{5}$$

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Similarly, (4b) and (4d) are equivalent to $XA = P_{R(A^T)}$, so that, for all $\mathbf{v} \in R(A)$, the vector $A^\dagger \mathbf{v}$ is the least norm solution of (5).

If $A$ is of full column rank then $A^\dagger$ is a left inverse of $A$, and equation (4d) is redundant.

1.3. **Volume.** The volume of an $m \times n$ matrix $A$ of rank $r$ is defined as

$$\operatorname{vol} A := \sqrt{\sum_{(I,J)\in\mathcal{N}} \det^2 A_{IJ}} \tag{6}$$

where $A_{IJ}$ is the submatrix of $A$ with rows $I$ and columns $J$, and $\mathcal{N}$ is the index set of $r \times r$ nonsingular submatrices of $A$.

The matrices used in this paper are of full column rank, in which case the volume is simply

$$\operatorname{vol} A = \sqrt{\det A^T A} \, ,$$

and the Moore-Penrose inverse is a convex combination of inverses of all $n \times n$ nonsingular submatrices $A_{I*}$ of $A$, see [4],

$$A^\dagger = \sum_{I\in\mathcal{N}} \frac{\det^2 A_{I*}}{\operatorname{vol}^2 A} \widehat{A_{I*}^{-1}} \tag{7}$$

where $A_{I*}$ is the submatrix of $A$ with rows in $I$, and $\widehat{A_{I*}^{-1}}$ is the $n \times m$ matrix obtained by padding $A_{I*}^{-1}$ by zeros in columns $j \notin I$. This result is a special case of [5], see also [2], giving the Moore–Penrose inverse of a general matrix as a convex combination of inverses of basic submatrices.

1.4. **Change of variables in integration.** Let $\phi$, $U$, $V$ be as above, and let $f$ be an integrable function $V \to \mathbb{R}$. The integral $\int_V f$ can be computed as an integral on $U$ using the **change–of–variables formula**, see [3],

$$\int_V f = \int_U (f \circ \phi) \operatorname{vol} J(\phi) \tag{8}$$

where $\operatorname{vol} J(\phi)$ is the volume of $J(\phi)$. If $m = n$ then (8) reduces to the classical change–of–variables formula

$$\int_V f = \int_U (f \circ \phi) \, |\det J(\phi)|$$

see e.g. [6, Theorem 11.1].

**Back to the Introduction**: Consider an inverse $\phi^{-1}$ satisfying

$$\phi(\phi^{-1}(\mathbf{v})) = \mathbf{v} \, , \quad \forall \, \mathbf{v} \in V \, , \tag{9a}$$

$$\phi^{-1}(\phi(\mathbf{u})) = \mathbf{u} \, , \quad \forall \, \mathbf{u} \in U \, . \tag{9b}$$

The formula (8) is useful if the $U$–integral is simpler than the $V$–integral. In this case, it is not practical to repeat the trick, and switch back to a $V$–integral

$$\int_U (f \circ \phi) \operatorname{vol} J(\phi) = \int_V \left(f \circ \phi \circ \phi^{-1}\right) \left(\operatorname{vol} J(\phi) \operatorname{vol} J(\phi^{-1})\right) \, . \tag{10}$$

If so tempted, expect to get from (8) and (10),

$$\int_V f = \int_V \left(f \circ \phi \circ \phi^{-1}\right) \left(\operatorname{vol} J(\phi) \operatorname{vol} J(\phi^{-1})\right) \quad \text{for all integrable } f \, , \tag{11}$$

where $J(\phi)$ is evaluated at $\mathbf{u}$, and $J(\phi^{-1})$ at $\phi(\mathbf{u})$. Now $f = f \circ \phi \circ \phi^{-1}$ by definition, therefore (11) implies

$$\operatorname{vol} J(\phi) \operatorname{vol} J(\phi^{-1}) = 1 \, , \quad \text{throughout } V \, . \tag{12}$$

Note that (12) does not follow from the inverse relations (9). Indeed, the Jacobian $J(\phi^{-1})$ is a left–inverse of $J(\phi)$,

$$J(\phi^{-1}) \, J(\phi) = I_n \, , \quad \text{as shown by differentiating (9b)} \, .$$

Therefore, see [1],

$$\operatorname{vol} J(\phi^{-1}) \;\geq\; \frac{1}{\operatorname{vol} J(\phi)} \tag{13a}$$

$$\text{with equality iff} \quad J(\phi^{-1}) \;=\; (J(\phi))^{\dagger}\;, \tag{13b}$$

the Moore–Penrose inverse of $J(\phi)$. An inverse $\phi^{-1}$ satisfying

$$\operatorname{vol} J(\phi)(\mathbf{u}) \operatorname{vol} J(\phi^{-1})(\phi(\mathbf{u})) \;=\; 1 \tag{14}$$

or equivalently, satisfying (13b), is called **volume preserving** at $\mathbf{u}$.

A volume-preserving inverse is constructed below as a convex combination of basic inverses. It is analogous to the Moore-Penrose inverse, and reduces to it if $\phi$ is linear.

## 2. Construction

Let $\phi$, $U$, $V$ and $\mathcal{I}(\mathbf{u})$ be as above (see § 1.1). Let $\boldsymbol{\lambda} = (\lambda_I)$ be a vector of weights $\lambda_I$, and consider the convex combination of basic inverses of $\phi$

$$\phi_{\boldsymbol{\lambda}}^{-1}(\mathbf{v}) := \sum_{I \in \mathcal{I}(\mathbf{u})} \lambda_I\, \phi_I^{-1}(\mathbf{v}_I) \tag{15}$$

Then

$$\phi(\phi_{\boldsymbol{\lambda}}^{-1}(\mathbf{v})) \;=\; \mathbf{v}\;, \quad \text{for all } \mathbf{v} = \phi(\mathbf{u}) \tag{16a}$$

$$\phi_{\boldsymbol{\lambda}}^{-1}(\phi(\mathbf{u})) \;=\; \mathbf{u}\;, \quad \text{for all } \mathbf{u} \in U \tag{16b}$$

showing that $\phi_{\boldsymbol{\lambda}}^{-1}(\cdot)$ is an inverse of $\phi$. As shown below, it is useful to have the weights $\{\lambda_I : I \in \mathcal{I}(\mathbf{u})\}$ depend on the point $\mathbf{u}$, in the following manner (the resulting inverse (15) is denoted by $\phi^{-1}(\cdot|\mathbf{u})$, and is a local inverse).

$$\phi^{-1}(\mathbf{v}|\,\mathbf{u}) \;:=\; \sum_{I \in \mathcal{I}(\mathbf{u})} \lambda_I(\mathbf{u})\, \phi_I^{-1}(\mathbf{v}_I) \tag{17a}$$

$$\text{with } \lambda_I(\mathbf{u}) \;=\; \frac{\det^2\,(J(\phi_I)(\mathbf{u}))}{\operatorname{vol}^2\,(J(\phi)(\mathbf{u}))} \tag{17b}$$

**Theorem 1.** $\phi^{-1}(\mathbf{v}|\,\mathbf{u})$ defined by (17a)–(17b) is a volume–preserving inverse of $\phi$ at $\mathbf{u}$.

*Proof.* Let the convex weights $\lambda_I(\mathbf{u})$ be given by (17b). We prove that $\phi^{-1}$ of (17a) satisfies (13b). The derivative, w.r.t. $\mathbf{v}$, of $\phi^{-1}(\mathbf{v}|\,\mathbf{u})$ is represented by the Jacobian matrix

$$\begin{aligned} J(\phi^{-1})(\mathbf{v}|\,\mathbf{u}) \;&=\; \sum_{I \in \mathcal{I}(\mathbf{u})} \lambda_I(\mathbf{u})\, J(\phi_I^{-1})(\mathbf{v})\;, \quad \text{since } \lambda_I(\mathbf{u}) \text{ are constants in this differentiation,} \\ &=\; \sum_{I \in \mathcal{I}(\mathbf{u})} \lambda_I(\mathbf{u})\, (J(\phi_I))^{-1}(\mathbf{v})\;. \end{aligned}$$

In particular, for $\mathbf{v} = \phi(\mathbf{u})$,

$$\begin{aligned} J(\phi^{-1})(\phi(\mathbf{u})|\,\mathbf{u}) \;&=\; \sum_{I \in \mathcal{I}(\mathbf{u})} \frac{\det^2\,(J(\phi_I)(\mathbf{u}))}{\operatorname{vol}^2\,(J(\phi)(\mathbf{u}))}\,(J(\phi_I))^{-1}(\phi(\mathbf{u})_I) \\ &=\; (J(\phi))^{\dagger}(\phi(\mathbf{u}))\;, \quad \text{by (7)}\;. \end{aligned}$$

□

If $\phi$ is a linear mapping

$$\phi : \mathbf{u} \to A\mathbf{u}\;, \tag{18}$$

with $A$ of full column rank, then $\phi^{-1}$ reduces to the Moore-Penrose inverse of $A$, by (7).

**Example 1.** Let $A$, $B$ be intervals in $\mathbb{R}$, and let $f : A \to B$ be differentiable and bijective. We parametrize the graph of $f$ as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \phi(u) := \begin{pmatrix} u \\ f(u) \end{pmatrix} \tag{19}$$

(a) The basic subsystems (2) and solutions (3) are

$$\begin{array}{llll} 1: & \phi_1(u) = u, & \phi_1'(u) = 1, & u = \phi_1^{-1}(x) = x \\ 2: & \phi_2(u) = f(u), & \phi_2'(u) = f'(u), & u = \phi_2^{-1}(y) = f^{-1}(y) \end{array}$$

(b) The Jacobian of $\phi$ is $J(\phi)(u) = \begin{pmatrix} 1 \\ f'(u) \end{pmatrix}$ with volume $\operatorname{vol} J(\phi)(u) = \sqrt{1 + f'(u)^2}$.

(c) The inverse (17) is

$$\phi^{-1}(\begin{pmatrix} x \\ y \end{pmatrix} \mid u) = \frac{1}{1 + f'(u)^2} \left( x + f'(u)^2 \, f^{-1}(y) \right) \tag{20}$$

(d) The Jacobian of $\phi^{-1}$ is

$$J(\phi^{-1}(\begin{pmatrix} x \\ y \end{pmatrix} |u) = \frac{1}{1 + f'(u)^2} \left( 1 \; , \; \frac{f'(u)^2}{f'(f^{-1}(y))} \right)$$

which reduces to the Moore-Penrose inverse of $J(\phi)(u)$ for $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ f(u) \end{pmatrix}$.

**Example 2.** The mapping

$$\phi(u) := \begin{pmatrix} u \\ \sqrt{1 - u^2} \end{pmatrix} \tag{21}$$

maps points $u$ in the interval $[0, 1]$ to points $\begin{pmatrix} x \\ y \end{pmatrix}$ on the top right quarter of the unit circle. The inverse $\phi^{-1}(\begin{pmatrix} x \\ y \end{pmatrix} \mid u)$ is computed, using (20) with $f(u) = \sqrt{1 - u^2}$, as

$$\phi^{-1}(\begin{pmatrix} x \\ y \end{pmatrix} \mid u) = x - u^2 \, x + u^2 \, \sqrt{1 - y^2} \; . \tag{22}$$

**Example 3.** Given $a, h > 0$, the mapping

$$\phi(\theta) = \begin{pmatrix} a \, \cos\theta \\ a \, \sin\theta \\ h \, \theta \end{pmatrix} \tag{23}$$

takes points $\theta \in \mathbb{R}$ to points $(x, y, z)$ on a helix with radius $a$ and pitch $h$. There are three basic subsystems:

| subsystem | Jacobian | inverse |
|---|---|---|
| $\phi_1(\theta) = a\cos\theta = x$ | $-a \, \sin\theta$ | $\phi_1^{-1}(x) = \cos^{-1}\left(\frac{x}{a}\right)$ |
| $\phi_2(\theta) = a\sin\theta = y$ | $a \, \cos\theta$ | $\phi_2^{-1}(y) = \sin^{-1}\left(\frac{y}{a}\right)$ |
| $\phi_3(\theta) = h\theta = z$ | $h$ | $\phi_3^{-1}(z) = \frac{z}{h}$ |

and the inverse of $\phi$ is, by (17),

$$\phi^{-1}(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \theta) = \frac{a^2}{a^2 + h^2} \left( \sin^2\theta \, \cos^{-1}\left(\frac{x}{a}\right) + \cos^2\theta \, \sin^{-1}\left(\frac{y}{a}\right) + \frac{h \, z}{a^2} \right) \; . \tag{24}$$

## 3. Orthogonality

The inverse $\phi^{-1}(\mathbf{v}|\,\mathbf{u})$ of (17) depends on $\mathbf{u}$ (as parameter) and therefore cannot be used directly to solve

$$\phi(\mathbf{u}) \;=\; \mathbf{v}\ , \quad \text{for given } \mathbf{v}\ .$$

The inverse $\phi^{-1}$ may still be useful, because of the following orthogonality property.

**Theorem 2.** Let $\phi : U \to V$ be as above, and let $\mathbf{u} \in U$. The set of $\mathbf{v} \in \mathbb{R}^m$ which are mapped by $\phi^{-1}(\cdot|\,\mathbf{u})$ into $\mathbf{u}$,

$$S(\mathbf{u}) \;:=\; \{\mathbf{v} : \phi^{-1}(\mathbf{v}|\,\mathbf{u}) = \mathbf{u}\} \tag{25}$$

is orthogonal to the tangent manifold of $V$ at $\phi(\mathbf{u})$.
We call $S(\mathbf{u})$ the **u–trajectory** of $\phi^{-1}$.

*Proof.* The trajectory $S(\mathbf{u})$ is nonempty (it contains $\phi(\mathbf{u})$),

$$S(\mathbf{u}) \;=\; \{\phi(\mathbf{u}) + \mathbf{x} \,:\, \phi^{-1}(\phi(\mathbf{u}) + \mathbf{x}) = \mathbf{u}\}$$

Therefore, in a sufficiently small neighborhood $W$ of $\phi(\mathbf{u})$,

$$S(\mathbf{u}) \cap W \;\subset\; \{\phi(\mathbf{u}) + \mathbf{x} \,:\, \mathbf{u} + J(\phi^{-1})(\phi(\mathbf{u}))\mathbf{x} \;=\; \mathbf{u} + \circ(\|\mathbf{x}\|)\}$$

showing that the tangent of $S(\mathbf{u})$ at $\phi(\mathbf{u})$ is in the null space of $J(\phi^{-1})(\phi(\mathbf{u}))$. This null space is, by (13b), the same as the null space of $(J(\phi)(\phi(\mathbf{u})))^{\dagger}$, which is orthogonal to the range space of $J(\phi)(\phi(\mathbf{u}))$, the tangent manifold of $V$ at $\phi(\mathbf{u})$. □

Theorem 2 shows that, for $\mathbf{v}$ near $\phi(\mathbf{u})$, the vector $\mathbf{v} - \phi(\phi^{-1}(\mathbf{v}|\,\mathbf{u}))$ is approximately orthogonal to $V$. This is analogous to property (4c) of the Moore-Penrose inverse, suggesting the use of $\phi^{-1}$ for approximating least squares solutions of $\phi(\mathbf{u}) = \mathbf{v}$ locally (for $\mathbf{v}$ near $\phi(\mathbf{u})$).

**Example 1 continued**. For $\phi(u) = \begin{pmatrix} u \\ f(u) \end{pmatrix}$ and $\phi^{-1}(\begin{pmatrix} x \\ y \end{pmatrix}|u)$ given by (20), the trajectory (25) is

$$S(u) \;=\; \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \frac{1}{1 + f'(u)^2}\,\left(x + f'(u)^2\, f^{-1}(y)\right) \;=\; u \right\} \tag{26}$$

Differentiating w.r.t. $x$ gives

$$\frac{1}{1 + f'(u)^2}\,\left(1 + f'(u)^2\,\frac{y'}{f'(f^{-1}(y))}\right) \;=\; 0$$

$$\therefore \quad y' \;=\; -\frac{1}{f'(x)}\ , \quad \text{along (19)}\ ,$$

showing that the trajectories $S(u)$ are perpendicular to the curve $y = f(x)$.
**Example 2 continued**. Recall the mappings $\phi$ and $\phi^{-1}$ of Example 2. The trajectories (25) are

$$S(u) \;=\; \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,:\, x - u^2\,x + u^2\,\sqrt{1 - y^2} = u \right\} \tag{27}$$

Figure 1 shows the unit circle, and eleven trajectories $S(u)$ for $u = 0$ to $1$ in steps of $0.1$. These trajectories intersect the unit circle orthogonally. In particular, $S(0)$ and $S(1)$ coincide with the $y$-axis and $x$-axis respectively.

The implicit derivative of (27) is

$$\begin{aligned} 1 - u^2 - \frac{u^2 y y'}{\sqrt{1 - y^2}} \;&=\; 0 \\ \text{or} \quad y' \;&=\; \frac{1 - u^2}{u^2}\,\frac{\sqrt{1 - y^2}}{y} \\ &=\; \frac{y}{x}\ , \quad \text{for } x = u\,,\ y = \sqrt{1 - u^2}\ , \end{aligned}$$

showing that the trajectories (27) are perpendicular to the unit circle $x^2 + y^2 - 1 = 0$, with derivative

$$y' = -\frac{x}{y} .$$

Note that every point $(x, y)$ with $x \in [0, 1]$ and $y = 0$ lies on two trajectories, one of these is $S(1)$, the $x$-axis. On the other hand, points $(x, y)$ with $y > 1$ do not belong to any trajectory.

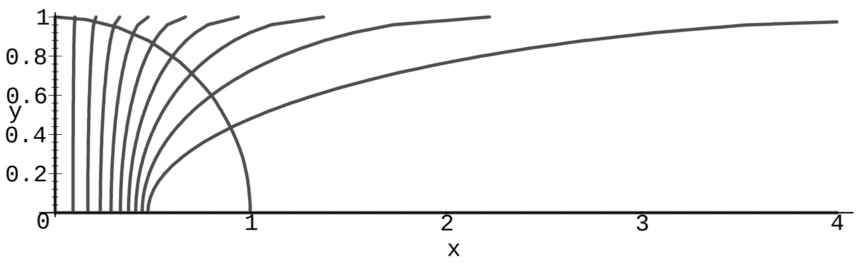


FIGURE 1. Illustration of the trajectories (27).

## 4. Projections

By Theorem 2, a trajectory $S(\mathbf{u})$ (regardless of its dimension) intersects the set $V$ in exactly one point, namely $\phi(\mathbf{u})$. The point $\phi(\mathbf{u})$ is called the **projection** of $S(\mathbf{u})$ on $V$.

In the linear case, (18), the trajectories are

$$S(\mathbf{u}) = \{A\mathbf{u} + \mathbf{x} : A^T\mathbf{x} = \mathbf{0}\} ,$$

and the projection of $S(\mathbf{u})$ on $V = R(A)$ is the orthogonal projection of $S(\mathbf{u})$ on $R(A)$,

$$P_{R(A)}S(\mathbf{u}) = A\mathbf{u} .$$

In general, projections need not exist, or be unique. For example, in Figure 1,
(a) points $(x, 0)$, with $x \in [0, 1]$, have 2 projections on the unit circle, and
(b) points $(x, y)$ with $y > 1$ do not have a projection.

If a projection of $\mathbf{v}$ on $V$ exists, it can be computed by solving $\phi^{-1}(\mathbf{v}|\,\mathbf{u}) = \mathbf{u}$ , or equivalently

$$\phi^{-1}(\mathbf{v}|\,\mathbf{u}) - \mathbf{u} = \mathbf{0} , \tag{28}$$

for $\mathbf{u}$ and then computing $\phi(\mathbf{u})$. This is illustrated in the next example.

**Example 1 continued**. For the trajectories (26), equation (28) becomes,

$$\frac{1}{1 + f'(u)^2} \left(x + f'(u)^2 f^{-1}(y)\right) = u \tag{29}$$

The Newton iteration for solving (29) is

$$u := u - \frac{x - u + f'(u)^2(f^{-1}(y) - u)}{2f'(u)f''(u)(f^{-1}(y) - u) - (1 + f'(u)^2)} \tag{30}$$

with fixed point $u = x = f^{-1}(y)$.

**Example 2 continued**. For the trajectories (27), associated with the unit circle, equation (28) becomes,

$$u^2 \sqrt{1 - y^2} - u^2\, x - u + x = 0 . \tag{31}$$

The Newton iteration for solving (31) is

$$u := u - \frac{u^2 \sqrt{1 - y^2} - u^2\, x - u + x}{2u\sqrt{1 - y^2} - 2ux - 1} . \tag{32}$$

For example, the projection of $(3, 0.8)$ on the unit circle is computed by iterating

$$u := u - \frac{2.4\,u^2 + u - 3}{4.8u + 1} \,.$$

Starting at $u = 1$, it takes 3 iterations to get the solution $u = 0.9289472539$. The projection of $(3, 0.8)$ on the unit circle is then

$$\phi(0.9289472539) \;=\; \begin{pmatrix} 0.9289472539 \\ 0.3702169063 \end{pmatrix} \,.$$

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