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THE CHANGE OF VARIABLES FORMULA USING MATRIX VOLUME

ADI BEN-ISRAEL

Abstract. The matrix volume is a generalization, to rectangular matrices, of the absolute value of the determinant. In particular, the matrix volume can be used in change-of-variables formulæ, instead of the determinant (if the Jacobi matrix of the underlying transformation is rectangular). This result is applicable to integration on surfaces, illustrated here by several examples.

1. Introduction

The **change-of-variables formula** in the title is

$$\int_{\mathcal{V}} f(\mathbf{v})\, d\mathbf{v} \;=\; \int_{\mathcal{U}} (f \circ \phi)(\mathbf{u})\ |\det J_\phi(\mathbf{u})|\ d\mathbf{u} \tag{1}$$

where $\mathcal{U}, \mathcal{V}$ are sets in $\mathbb{R}^n$, ϕ is a sufficiently well-behaved function: $\mathcal{U} \to \mathcal{V}$, and f is integrable on $\mathcal{V}$. Here $d\mathbf{x}$ denotes the volume element $|dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|$, and J_ϕ is the **Jacobi matrix** (or **Jacobian**)

$$J_\phi := \left(\frac{\partial \phi_i}{\partial u_j}\right) \ , \quad \text{also denoted} \quad \frac{\partial(v_1, v_2, \cdots, v_n)}{\partial(u_1, u_2, \cdots, u_n)} \ ,$$

representing the derivative of ϕ. An advantage of (1) is that integration on $\mathcal{V}$ is translated to (perhaps simpler) integration on $\mathcal{U}$.

This formula was given in 1841 by Jacobi [8], following Euler (the case $n = 2$) and Lagrange ($n = 3$). It gave prominence to **functional** (or **symbolic**) **determinants**, i.e. (non–numerical) determinants of matrices including functions or operators as elements.

If $\mathcal{U}$ and $\mathcal{V}$ are in spaces of different dimensions, say $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^m$ with $n > m$, then the Jacobian J_ϕ is a rectangular matrix, and (1) cannot be used in its present form. However, if J_ϕ is of full column rank throughout $\mathcal{U}$, we can replace $|\det J_\phi|$ in (1) by the volume $\operatorname{vol} J_\phi$ of the Jacobian to get

$$\int_{\mathcal{V}} f(\mathbf{v})\, d\mathbf{v} \;=\; \int_{\mathcal{U}} (f \circ \phi)(\mathbf{u}) \operatorname{vol} J_\phi(\mathbf{u})\, d\mathbf{u}\ . \tag{2}$$

Recall that the **volume** of an $m \times n$ matrix of rank r is

$$\operatorname{vol} A := \sqrt{\sum_{(I,J)\in\mathcal{N}} \det{}^2 A_{IJ}} \tag{3}$$

where A_{IJ} is the submatrix of A with rows I and columns J, and $\mathcal{N}$ is the index set of $r \times r$ nonsingular submatrices of A, see e.g. [1]. Alternatively, $\operatorname{vol} A$ is the product of the singular values of A. If A is of full column rank, its volume is simply

$$\operatorname{vol} A = \sqrt{\det A^T A} \tag{4}$$

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If $m = n$ then $\operatorname{vol} J_\phi = |\det J_\phi|$, and (2) reduces to the classical result (1).

The formula (2) is well known in differential geometry, see e.g. [2, Proposition 6.6.1] and [6, § 3.2.3]. Although there are elementary accounts of this formula (see e.g. [3, Vol. II, Ch. IV, § 4], [7, § 8.1] and [13, § 3.4]), it is seldom used in applications.

The purposes of this note are: (i) to establish the usefulness of (2) for various surface integrals, (ii) to simplify the computation of the Radon and Fourier Transforms, and general integrals in $\mathbb{R}^n$ (see Examples 7–9 and Appendix A below), and (iii) to introduce the **functional matrix volume**, in analogy with the functional determinant.

We illustrate (2) for an elementary calculus example. Let $\mathcal{S}$ be a subset of a surface in $\mathbb{R}^3$ represented by

$$z = g(x, y) , \tag{5}$$

and let $f(x, y, z)$ be a function integrable on $\mathcal{S}$. Let $\mathcal{A}$ be the projection of $\mathcal{S}$ on the xy–plane. Then $\mathcal{S}$ is the image of $\mathcal{A}$ under a mapping ϕ

$$\mathcal{S} = \phi(\mathcal{A}) , \quad \text{or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix} = \phi \begin{pmatrix} x \\ y \end{pmatrix} , \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{A} . \tag{6}$$

The Jacobi matrix of ϕ is the 3×2 matrix

$$J_\phi(x, y) = \frac{\partial(x, y, z)}{\partial(x, y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{pmatrix} , \tag{7}$$

where $g_x = \frac{\partial g}{\partial x}$, $g_y = \frac{\partial g}{\partial y}$. The volume of (7) is, by (4),

$$\operatorname{vol} J_\phi(x, y) \ = \ \sqrt{1 + g_x^2 + g_y^2} \, . \tag{8}$$

Substituting (8) in (2) we get the well-known formula

$$\int\limits_{\mathcal{S}} f(x, y, z) \, ds = \int\limits_{\mathcal{A}} f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} \; dx \, dy , \tag{9}$$

giving an integral over $\mathcal{S}$ as an integral over its projection in the xy–plane.

The simplicity of this approach is not lost in higher dimensions, or with different coordinate systems, as demonstrated below by eleven elementary examples from calculus and analysis.

- Example 1 concerns line integrals, in particular the **arc length** of a curve in $\mathbb{R}^n$.
- Example 2 is an application to surface integration in $\mathbb{R}^3$ using cylindrical coordinates.
- Examples 3–4 concern integration on an **axially symmetric surface** (or **surface of revolution**) in $\mathbb{R}^3$. In these integrals, the volume of J_ϕ contains the necessary information on the surface symmetry.
- Example 5 shows integration on an $(n - 1)$–dimensional surface in $\mathbb{R}^n$.
- The area of the unit sphere in $\mathbb{R}^n$, a classical exercise, is computed in Example 6.
- Example 7 uses (2) to compute the Radon transform of a function $f : \mathbb{R}^n \to \mathbb{R}$.
- Example 8 computes an integral over $\mathbb{R}^n$ as an integral on $\mathbb{R}^{n-1}$ followed by an integral on $\mathbb{R}$.
- Example 9 applies this to the computation of the Fourier transforms of a function $f : \mathbb{R}^n \to \mathbb{R}$.
- The last two examples concern simplex faces in $\mathbb{R}^n$. Example 10 is the generalized Pythagoras theorem. Example 11 computes the largest face of the n–dimensional regular simplex.

These examples show that the full rank assumption for J_ϕ is quite natural, and presents no real restriction in applications.

The solutions given here should be compared with the "classical" solutions, as taught in calculus. We see that (2) offers a unified method for a variety of curve and surface integrals, and coordinate systems, without having to construct (and understand) the differential geometry in each application. The computational tractability of (2) is illustrated in Appendix A.

A **blanket assumption**: Throughout this paper, all functions are continuously differentiable as needed, all surfaces are smooth, and all curves are rectifiable.

2. Examples and applications

If the mapping $\phi : \mathcal{U} \to \mathcal{V}$ is given by

$$y_i \;=\; \phi_i(x_1, x_2, \cdots, x_n)\ , \quad i \in \overline{1,m}$$

we denote its Jacobi matrix J_ϕ by

$$\frac{\partial(y_1, y_2, \cdots, y_m)}{\partial(x_1, x_2, \cdots, x_n)} \;=\; \left(\frac{\partial \phi_i}{\partial x_j}\right)\ , \quad i \in \overline{1,m}\ ,\ j \in \overline{1,n} \tag{10}$$

the customary notation for Jacobi matrices (in the square case).

Example 1. Let $\mathcal{C}$ be an arc on a curve in $\mathbb{R}^n$, represented in parametric form as

$$\mathcal{C} := \phi([0,1]) = \{(x_1, x_2, \cdots, x_n)\ :\ x_i := \phi_i(t)\ ,\ 0 \le t \le 1\} \tag{11}$$

The Jacobi matrix $J_\phi(t) \;=\; \dfrac{\partial(x_1, x_2, \cdots, x_n)}{\partial t}$ is the column matrix $(\phi_i'(t))$, and its volume is

$$\operatorname{vol} J_\phi \;=\; \sqrt{\sum_{i=1}^{n} (\phi_i'(t))^2}\ .$$

The **line integral** (assuming it exists) of a function f along $\mathcal{C}$, $\int_{\mathcal{C}} f$, is given in terms of the volume of J_ϕ as follows

$$\int_{\mathcal{C}} f = \int_0^1 f(\phi_1(t), \cdots, \phi_n(t)) \sqrt{\sum_{i=1}^{n} (\phi_i'(t))^2}\, dt\ . \tag{12}$$

In particular, $f \equiv 1$ gives

$$\text{arc length}\, \mathcal{C} = \int_0^1 \sqrt{\sum_{i=1}^{n} (\phi_i'(t))^2}\, dt\ . \tag{13}$$

If one of the variables, say $a \le x_1 \le b$, is used as parameter, (13) gives the familiar result

$$\text{arc length}\, \mathcal{C} \;=\; \int_a^b \sqrt{1 + \sum_{i=2}^{n} \left(\frac{dx_i}{dx_1}\right)^2}\, dx_1\ .$$

Example 2. Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$ represented by

$$z = z(r, \theta) \tag{14}$$

where $\{r, \theta, z\}$ are **cylindrical coordinates**

$$x \;=\; r\,\cos\theta \tag{15a}$$
$$y \;=\; r\,\sin\theta \tag{15b}$$
$$z \;=\; z \tag{15c}$$

The Jacobi matrix of the mapping (15a),(15b) and (14) is

$$\frac{\partial(x, y, z)}{\partial(r, \theta)} = \begin{pmatrix} \cos\theta & -r\,\sin\theta \\ \sin\theta & r\,\cos\theta \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{pmatrix} \tag{16}$$

see also (A.4), Appendix A. The volume of (16) is

$$\operatorname{vol} J_\phi = \sqrt{r^2 + r^2 \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r\sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}\ , \tag{17}$$

see also (A.7), Appendix A. An integral over a domain $\mathcal{V} \subset \mathcal{S}$ is therefore

$$\int_{\mathcal{V}} f(x,y,z)\,dV = \int_{\mathcal{U}} f(r\,\cos\theta,\, r\,\sin\theta, z(r,\theta))\, r\sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}\, dr\, d\theta\ . \tag{18}$$

Example 3. Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$, symmetric about the z–axis. This axial symmetry is expressed in cylindrical coordinates by

$$z \;=\; z(r)\ , \quad \text{or} \quad \frac{\partial z}{\partial \theta} \;=\; 0 \quad \text{in} \quad (16)\text{–}(18).$$

The volume (17) thus becomes

$$\operatorname{vol} J_\phi \;=\; r\sqrt{1 + z'(r)^2} \tag{19}$$

with the axial symmetry "built in". An integral over a domain $\mathcal{V}$ in a z–symmetric surface $\mathcal{S}$ is therefore

$$\int_{\mathcal{V}} f(x,y,z)\,dV \;=\; \int_{\mathcal{U}} f(r\,\cos\theta, r\,\sin\theta, z(r))\, r\sqrt{1+z'(r)^2}\, dr\, d\theta\ .$$

Example 4. Again let $\mathcal{S}$ be a z–symmetric surface in $\mathbb{R}^3$. We use **spherical coordinates**

$$x \;=\; \rho\,\sin\phi\,\cos\theta \tag{20a}$$

$$y \;=\; \rho\,\sin\phi\,\sin\theta \tag{20b}$$

$$z \;=\; \rho\,\cos\phi \tag{20c}$$

The axial symmetry is expressed by

$$\rho \;:=\; \rho(\phi)$$

showing that $\mathcal{S}$ is given in terms of the two variables ϕ and θ. The volume of the Jacobi matrix is easily computed

$$\operatorname{vol}\frac{\partial(x,y,z)}{\partial(\phi,\theta)} \;=\; \rho\sqrt{\rho^2 + (\rho'(\phi))^2}\,\sin\phi$$

and the change of variables formula gives

$$\begin{aligned}&\int_{\mathcal{V}} f(x,y,z)\,dV = \\ &\quad\int_{\mathcal{U}} f(\rho(\phi)\,\sin\phi\,\cos\theta, \rho(\phi)\,\sin\phi\,\sin\theta, \rho(\phi)\,\cos\phi)\,\rho(\phi)\sqrt{\rho(\phi)^2 + (\rho'(\phi))^2}\,\sin\phi\, d\phi\, d\theta\end{aligned} \tag{21}$$

Example 5. Let a surface $\mathcal{S}$ in $\mathbb{R}^n$ be given by

$$x_n \;:=\; g(x_1, x_2, \ldots, x_{n-1})\ , \tag{22}$$

let $\mathcal{V}$ be a subset on $\mathcal{S}$, and let $\mathcal{U}$ be the projection of $\mathcal{V}$ on $\mathbb{R}^{n-1}$, the space of variables $(x_1,\ldots,x_{n-1})$. The surface $\mathcal{S}$ is the graph of the mapping $\phi : \mathcal{U} \to \mathcal{V}$, given by its components $\phi \;:=\; (\phi_1, \phi_2, \ldots, \phi_n)$,

$$\begin{aligned}\phi_i(x_1,\ldots,x_{n-1}) \;&:=\; x_i\ , \quad i = 1,\ldots,n-1\\ \phi_n(x_1,\ldots,x_{n-1}) \;&:=\; g(x_1,\ldots,x_{n-1})\end{aligned}$$

The Jacobi matrix of ϕ is

$$J_\phi \;=\; \begin{pmatrix} 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 1\\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} & \cdots & \dfrac{\partial g}{\partial x_{n-2}} & \dfrac{\partial g}{\partial x_{n-1}}\end{pmatrix}$$

and its volume is

$$\operatorname{vol} J_\phi = \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\partial g}{\partial x_i}\right)^2} \tag{23}$$

For any function f integrable on $\mathcal{V}$ we therefore have

$$\int_{\mathcal{V}} f(x_1,x_2,\cdots,x_{n-1},x_n)\,dV = \int_{\mathcal{U}} f(x_1,x_2,\cdots,x_{n-1},g(x_1,x_2,\cdots,x_{n-1}))\sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\partial g}{\partial x_i}\right)^2}\,dx_1dx_2\cdots dx_{n-1} \tag{24}$$

In particular, $f \equiv 1$ gives the area of $\mathcal{V}$

$$\int_{\mathcal{V}} 1\,dV = \int_{\mathcal{U}} \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\partial g}{\partial x_i}\right)^2}\,dx_1dx_2\cdots dx_{n-1} \tag{25}$$

Example 6. Let $\mathcal{B}_n$ be the unit ball in $\mathbb{R}^n$, $\mathcal{S}_n$ the **unit sphere**, and a_n the area of $\mathcal{S}_n$. Integrals on $\mathcal{S}_n$, in particular the area a_n, can be computed using **spherical coordinates**, e.g. [12, § VII.2], or the **surface element** of $\mathcal{S}_n$, e.g. [11]. An alternative, simpler, approach is to use the results of Example 5, representing the "upper hemisphere" as $\phi(\mathcal{B}_{n-1})$, where $\phi = (\phi_1,\phi_2,\cdots,\phi_n)$ is

$$\begin{aligned}
\phi_i(x_1,x_2,\cdots,x_{n-1}) &= x_i\ ,\ i \in \overline{1,n-1}\ ,\\
\phi_n(x_1,x_2,\cdots,x_{n-1}) &= \sqrt{1-\sum_{i=1}^{n-1} x_i^2}\ .
\end{aligned}$$

The Jacobi matrix is

$$J_\phi = \begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ -\dfrac{x_1}{x_n} & -\dfrac{x_2}{x_n} & \cdots & -\dfrac{x_{n-1}}{x_n} \end{pmatrix}$$

and its volume is easily computed

$$\operatorname{vol} J_\phi = \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{x_i}{x_n}\right)^2} = \frac{1}{|x_n|} = \frac{1}{\sqrt{1-\sum_{i=1}^{n-1} x_i^2}}\ . \tag{26}$$

The area a_n is twice the area of the "upper hemisphere". Therefore, by (26),

$$a_n = 2\int_{\mathcal{B}_{n-1}} \frac{dx_1dx_2\cdots dx_{n-1}}{\sqrt{1-\sum_{i=1}^{n-1} x_i^2}}\ , \tag{27}$$

which is easily integrated to give

$$a_n = \frac{2\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\ , \tag{28}$$

using well-known properties of the **beta function**

$$B(p,q) := \int_0^1 (1-x)^{p-1}\, x^{q-1}\, dx \;,$$

$$\text{and the } \textbf{gamma function} \quad \Gamma(p) := \int_0^\infty x^{p-1}\, e^{-x}\, dx \;.$$

Example 7 (Radon transform)**.** Let $\mathcal{H}_{\boldsymbol{\xi},p}$ be a hyperplane in $\mathbb{R}^n$ represented by

$$\mathcal{H}_{\boldsymbol{\xi},p} := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n \xi_i\, x_i \;=\; p \right\} = \{\mathbf{x} : <\boldsymbol{\xi}, \mathbf{x}>= p\} \tag{29}$$

where $\xi_n \neq 0$ in the normal vector $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ of $\mathcal{H}_{\boldsymbol{\xi},p}$ (such hyperplanes are called **non–vertical**). Then $\mathcal{H}_{\boldsymbol{\xi},p}$ is given by

$$x_n := \frac{p}{\xi_n} - \sum_{i=1}^{n-1} \frac{\xi_i}{\xi_n}\, x_i \tag{30}$$

which is of the form (22). The volume (23) is here

$$\mathrm{vol}\, J_\phi \;=\; \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\xi_i}{\xi_n}\right)^2} \;=\; \frac{\|\boldsymbol{\xi}\|}{|\xi_n|} \tag{31}$$

The **Radon transform** $(\mathbf{R}f)(\boldsymbol{\xi}, p)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is its integral over the hyperplane $\mathcal{H}_{\boldsymbol{\xi},p}$, see [4],

$$(\mathbf{R}f)(\boldsymbol{\xi}, p) := \int\limits_{\{\mathbf{x}:\, <\boldsymbol{\xi},\mathbf{x}> \;=\; p\}} f(\mathbf{x})\, d\mathbf{x} \;. \tag{32}$$

Using (30)–(31), the Radon transform can be computed as an integral in $\mathbb{R}^{n-1}$

$$(\mathbf{R}f)(\boldsymbol{\xi}, p) \;=\; \frac{\|\boldsymbol{\xi}\|}{|\xi_n|} \int\limits_{\mathbb{R}^{n-1}} f\left(x_1, \cdots, x_{n-1}, \frac{p}{\xi_n} - \sum_{i=1}^{n-1} \frac{\xi_i}{\xi_n}\, x_i \right) dx_1 dx_2 \cdots dx_{n-1} \tag{33}$$

See (A.15) and (A.17), Appendix A, for Radon transform in $\mathbb{R}^2$, and (A.20) for $\mathbb{R}^3$.

In tomography applications the Radon transforms $(\mathbf{R}f)(\boldsymbol{\xi}, p)$ are computed by the scanning equipment, so (33) is not relevant. The issue is the **inverse problem**, of reconstructing f from its Radon transforms $(\mathbf{R}f)(\boldsymbol{\xi}, p)$ for all $\boldsymbol{\xi}, p$. The inverse Radon transform is also an integral, see e.g. [4],[12], and can be expressed analogously to (33), using the method of the next example.

Example 8. Consider an integral over $\mathbb{R}^n$,

$$\int\limits_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} = \int\limits_{\mathbb{R}^n} f(x_1, x_2, \cdots, x_n)\, dx_1 dx_2 \cdots dx_n \tag{34}$$

Since $\mathbb{R}^n$ is a union of (parallel) hyperplanes,

$$\mathbb{R}^n = \bigcup_{p=-\infty}^{\infty} \{\mathbf{x} : <\boldsymbol{\xi}, \mathbf{x}>= p\} \;, \quad \text{where } \boldsymbol{\xi} \neq \mathbf{0} \;, \tag{35}$$

we can compute (34) iteratively: an integral over $\mathbb{R}^{n-1}$ (Radon transform), followed by an integral on $\mathbb{R}$,

$$\int\limits_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} = \int\limits_{-\infty}^{\infty} \frac{dp}{\|\boldsymbol{\xi}\|}\, (\mathbf{R}f)(\boldsymbol{\xi}, p) \tag{36}$$

where $dp/\|\boldsymbol{\xi}\|$ is the differential of the distance along $\boldsymbol{\xi}$ (i.e. dp times the distance between the parallel hyperplanes $\mathcal{H}_{\boldsymbol{\xi},p}$ and $\mathcal{H}_{\boldsymbol{\xi},p+1}$). Combining (33) and (36) we get the integral of f on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,d\mathbf{x} \;=\; \frac{1}{|\xi_n|}\int_{-\infty}^{\infty}\left\{\int_{\mathbb{R}^{n-1}} f\left(x_1,\cdots,x_{n-1},\frac{p}{\xi_n}-\sum_{i=1}^{n-1}\frac{\xi_i}{\xi_n}\,x_i\right)\,dx_1dx_2\cdots dx_{n-1}\right\}\,dp \tag{37}$$

see (A.21), Appendix A, for integration over $\mathbb{R}^3$.

It is possible to derive (37) directly from the classical change-of-variables formula (1), by changing variables from $\{x_1,\cdots,x_{n-1},x_n\}$ to $\{x_1,\cdots,x_{n-1},p:=\sum\limits_{i=1}^{n}\xi_i x_i\}$, and using

$$\det\left(\frac{\partial(x_1,\cdots,x_{n-1},x_n)}{\partial(x_1,\cdots,x_{n-1},p)}\right)=\frac{1}{\xi_n}\;.$$

An advantage of our development is that it can be used recursively, i.e. the inner integral in (37) can again be expressed as an integral in $\mathbb{R}^{n-2}$ followed by an integral in $\mathbb{R}$, etc.

Example 9 (Fourier transform)**.** In particular, the **Fourier transform** $(\mathbf{F}f)(\boldsymbol{\xi})$ of f is the integral

$$\begin{aligned}(\mathbf{F}f)(\boldsymbol{\xi}) &:= (2\,\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i<\boldsymbol{\xi},\mathbf{x}>}\,f(\mathbf{x})\,d\mathbf{x} = \\ &\qquad (2\,\pi)^{-n/2}\int_{\mathbb{R}^n} f\,(x_1,\cdots,x_n)\,\exp\left\{-i\sum_{k=1}^{n}\xi_k\,x_k\right\}dx_1dx_2\cdots dx_n\;.\end{aligned} \tag{38}$$

For $\xi_n\neq 0$ we can compute (38) analogously to (37) as

$$(\mathbf{F}f)(\boldsymbol{\xi})=\frac{(2\pi)^{-n/2}}{|\xi_n|}\int_{-\infty}^{\infty} e^{-ip}\left\{\int_{\mathbb{R}^{n-1}} f\left(x_1,\cdots,x_{n-1},\frac{p}{\xi_n}-\sum_{i=1}^{n-1}\frac{\xi_i}{\xi_n}x_i\right)\prod_{k=1}^{n-1}dx_k\right\}\,dp\;. \tag{39}$$

The Fourier transform of a function of n variables is thus computed as an integral over $\mathbb{R}^{n-1}$ followed by an integral on $\mathbb{R}$. The inverse Fourier transform of a function $g(\boldsymbol{\xi})$ is of the same form as (38),

$$(\mathbf{F}^{-1}g)(\mathbf{x}) := (2\,\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i<\mathbf{x},\boldsymbol{\xi}>}\,g(\boldsymbol{\xi})\,d\boldsymbol{\xi}\;, \tag{40}$$

and can be computed as in (39).

Example 10. (**The generalized Pythagorean theorem**, [10]). Consider an n–dimensional **simplex**

$$\boldsymbol{\Delta}_n \;:=\; \left\{(x_1,x_2,\cdots,x_n)\;:\;\sum_{i=1}^{n}a_i\,x_i\;\leq\;a_0\;,\;x_i\geq 0\;,\;i\in\overline{1,n}\right\}\;, \tag{41}$$

with all $a_j>0\,,\;j\in\overline{0,n}$. We denote the $n+1$ faces of $\boldsymbol{\Delta}_n$ by

$$\mathcal{F}_0 \;:=\; \left\{(x_1,x_2,\cdots,x_n)\in\boldsymbol{\Delta}_n\;:\;\sum_{i=1}^{n}a_i\,x_i\;=\;a_0\right\} \tag{42a}$$

$$\mathcal{F}_j \;:=\; \{(x_1,x_2,\cdots,x_n)\in\boldsymbol{\Delta}_n\;:\;x_j\;=\;0\}\;,\;j\in\overline{1,n} \tag{42b}$$

and denote their areas by $A_0\,,\;A_j$ respectively. The **generalized Pythagorean theorem** (see [10]) states that

$$A_0^2 \;=\; \sum_{j=1}^{n}A_j^2 \tag{43}$$

We prove it here using the change of variables formula (2). For any $j \in \overline{1,n}$ we can represent the (largest) face $\mathcal{F}_0$ as $\mathcal{F}_0 = \phi^{\{j\}}(\mathcal{F}_j)$ where $\phi^{\{j\}} = (\phi_1^{\{j\}}, \cdots, \phi_n^{\{j\}})$ is

$$\begin{aligned} \phi_i^{\{j\}}(x_1, x_2, \cdots, x_n) &= x_i \ , \ i \neq j \ , \\ \phi_j^{\{j\}}(x_1, x_2, \cdots, x_n) &= \frac{a_0}{a_j} - \sum_{i \neq j} \frac{a_i}{a_j}\, x_i \ . \end{aligned}$$

The Jacobi matrix of $\phi^{\{j\}}$ is an $n \times (n-1)$ matrix with the i^{th} unit vector in row $i \neq j$, and

$$\left(-\frac{a_1}{a_j}, -\frac{a_2}{a_j}, \cdots, -\frac{a_{j-1}}{a_j}, -\frac{a_{j+1}}{a_j}, \cdots, -\frac{a_{n-1}}{a_j}, -\frac{a_n}{a_j} \right)$$

in row j. The volume of the Jacobi matrix of $\phi^{\{j\}}$ is computed as

$$\mathrm{vol}\, J_{\phi^{\{j\}}} = \sqrt{1 + \sum_{i \neq j} \left(\frac{a_i}{a_j}\right)^2} = \sqrt{\frac{\sum_{i=1}^n a_i^2}{a_j^2}} = \frac{\|\mathbf{a}\|}{|a_j|}$$

where $\mathbf{a}$ is the vector $(a_1, \cdots, a_n)$. Therefore, the area of $\mathcal{F}_0$ is

$$\begin{aligned} A_0 &= \int_{\mathcal{F}_j} \left(\frac{\|\mathbf{a}\|}{|a_j|}\right) \prod_{i \neq j} dx_i = \left(\frac{\|\mathbf{a}\|}{|a_j|}\right) A_j \ , \quad j \in \overline{1,n} \qquad (44) \\ \therefore \quad \frac{\sum_{j=1}^n A_j^2}{A_0^2} &= \frac{\sum_{j=1}^n |a_j|^2}{\|\mathbf{a}\|^2} \end{aligned}$$

and the generalized Pythagorean theorem (43) reduces to the ordinary Pythagorean theorem

$$\|\mathbf{a}\|^2 = \sum_{j=1}^n |a_j|^2 \ .$$

Example 11. The simplex (41) with $a_j = 1$ for all $j \in \overline{0,n}$ is the n–dimensional **regular simplex**

$$\mathbf{\Delta}_n := \left\{ (x_1, x_2, \cdots, x_n) : \sum_{i=1}^n x_i \leq 1 \ , \ x_i \geq 0 \ , \ i \in \overline{1,n} \right\} . \qquad (45)$$

The area A_0 of its largest face

$$\mathcal{F}_0 = \left\{ (x_1, \cdots, x_n) : \sum_{i=1}^n x_i = 1 \ , \ x_i \geq 0 \ , \ i \in \overline{1,n} \right\} . \qquad (46)$$

in given by (44) as

$$A_0 = \sqrt{n}\, V_{n-1}$$

where V_{n-1}, the volume of the $(n-1)$–dimensional regular simplex, is

$$V_{n-1} = \frac{1}{(n-1)!} \ , \quad \text{see e.g. [9, § 47]} \ .$$

Therefore

$$A_0 = \frac{\sqrt{n}}{(n-1)!} \ . \qquad (47)$$

Note that $A_0 \to 0$ as $n \to \infty$, although the side faces of the unit cube (of which the face $\mathcal{F}_0$ is a "diagonal section") have areas 1.

References

[1] A. Ben-Israel, *A volume associated with $m \times n$ matrices*, Lin. Algeb. Appl. **167**(1992), 87–111.
[2] M. Berger and B. Gostiaux, *Differential Geometry: Manifolds, Curves and Surfaces* (translated by S. Levy), Graduate Texts in Mathematics No. 115, Springer-Verlag, New York 1988
[3] R. Courant, *Differential and Integral Calculus*, (translated by E.J. McShane), Interscience Publishers, New York, 1936
[4] S.R. Deans, *The Radon Transform and Some of its Applications*, Wiley, 1983.
[5] Derive for Windows, Soft Warehouse, Inc. 3660 Waialae Avenue, Suite 304, Honolulu, Hawaii 96816-3236, USA.
[6] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.
[7] W. Fleming, *Functions of Several Variables* (2nd edition), Springer-Verlag, New York, 1977
[8] C.G.J. Jacobi, *De Determinantibus Functionalibus*, Crelle Journal für die reine und angewandte Mathematik **22**(1841), 319-359. Reprinted in *C.G.J. Jacobi's Gesammelte Werke* (K. Weierstrass, editor), Vol. 3, 393-438, Berlin 1884.
[9] M.G. Kendall, *A Course in the Geometry of n Dimensions*, Griffin's Statistical Monographs No. 8, Hafner, 1961.
[10] S.-Y.T. Lin and Y.-F. Lin, *The n-dimensional Pythagorean theorem*, Lin. and Multilin. Algeb. **26**(1990), 9–13.
[11] C. Müller, *Spherical Harmonics*, Lecture Notes in Mathematics No. 17, Springer-Verlag, 1966.
[12] F. Natterer, *The Mathematics of Computerized Tomography*, Wiley, 1986.
[13] M. Schreiber, *Differential Forms: A Heuristic Introduction*, Springer-Verlag, 1977.

Appendix A: Illustrations with Derive.

The integrations of this paper can be done symbolically. We illustrate this for the symbolic package Derive, [5], omitting details such as the commands (e.g. `Simplify`, `Approximate`) and settings (e.g. `Trigonometry:=Expand`) that are used to obtain these results.

The Jacobi matrix is computed by the function

$$\texttt{JACOBIAN}(u,\alpha) := \texttt{VECTOR(GRAD}(u_{m\_},\alpha), m\_, \texttt{DIMENSION}(u)) \tag{A.1}$$

found in the Derive utility file `VECTOR.mth`.
Example: Define the surface as a function of r and θ,

$$Z(r,\theta) := \tag{A.2}$$

as in Example 2. Then

$$\texttt{JACOBIAN}([r * \texttt{COS}(\theta), r * \texttt{SIN}(\theta), Z(r,\theta)], [r,\theta]) \tag{A.3}$$

gives (16),

$$\begin{bmatrix} \texttt{COS}(\theta) & -r * \texttt{SIN}(\theta) \\ \texttt{SIN}(\theta) & r * \texttt{COS}(\theta) \\ \dfrac{\partial Z(r,\theta)}{\partial r} & \dfrac{\partial Z(r,\theta)}{\partial \theta} \end{bmatrix} \tag{A.4}$$

The volume of an $(n+1)\times n$ matrix of full rank is computed by

$$\texttt{VOL}(a) := \texttt{SQRT(SUM(DET(DELETE\_ELEMENT}(a,k\_))^2, k\_, 1, \texttt{DIMENSION}(a)))) \tag{A.5}$$

For example,

$$\texttt{VOL(JACOBIAN}([r * \texttt{COS}(\theta), r * \texttt{SIN}(\theta), Z(r,\theta)], [r,\theta])) \tag{A.6}$$

gives (17),

$$\sqrt{r^2 * \left(\frac{\partial Z(r,\theta)}{\partial r}\right)^2 + \left(\frac{\partial Z(r,\theta)}{\partial \theta}\right)^2 + r^2} \tag{A.7}$$

In Example 3 the surface is symmetric about the z-axis, and we use $Z(r)$ instead of $Z(r,\theta)$ in (A.6)

$$\texttt{VOL(JACOBIAN}([r * \texttt{COS}(\theta), r * \texttt{SIN}(\theta), Z(r)], [r,\theta])) \tag{A.8}$$

to get (19),

$$|r| * \sqrt{Z'(r)^2 + 1} \tag{A.9}$$

For example, the volume associated with the surface $z = 1/r$

$$\texttt{VOL(JACOBIAN(}[r * \texttt{COS}(\theta), r * \texttt{SIN}(\theta), 1/r], [r, \theta])) \quad \text{gives} \quad \frac{\sqrt{r^4+1}}{|r|} \tag{A.10}$$

The surface area of $z = 1/r$ is 2π times the integral of (A.10). In particular, the surface area from $r = 1$ to $r = 100$

$$2 * \pi * \texttt{INT}\left[\frac{\texttt{SQRT}(r^4+1)}{|r|}, r, 1, 100\right] \quad \text{gives} \quad 3.14142524131 \times 10^4$$

Consider now the Radon Transform in $\mathbb{R}^2$. A line

$$a\,x + b\,y = p \tag{A.11}$$

can be solved for y

$$y = \frac{p}{b} - \frac{a}{b}\,x$$

if $b \neq 0$. Then

$$\texttt{VOL(JACOBIAN}\left(\left[x, \frac{p}{b} - \frac{a}{b}\,x\right], [x]\right) \quad \text{gives} \quad \frac{\sqrt{a^2+b^2}}{|b|} \tag{A.12}$$

in agreement with (31). The Radon transform (33) is then computed by

$$\begin{aligned} &\texttt{RADON\_AUX}(f, x, y, a, b, p) := \\ &\qquad \texttt{INT(LIM}(f, y, p/b - a/b * x), x, -\infty, \infty) * \texttt{SQRT}(a^2+b^2)/|b| \end{aligned} \tag{A.13}$$

For example, the Radon transform of $e^{-x^2-y^2}$,

$$\texttt{RADON\_AUX(EXP}(-x^2-y^2), x, y, a, b, p) \quad \text{is} \quad \sqrt{\pi}\,\texttt{EXP}\left[-\frac{p^2}{a^2+b^2}\right] \tag{A.14}$$

The function (A.13) requires $b \neq 0$. We modify it for lines with $b = 0$ (i.e. vertical lines) as follows

$$\begin{aligned} &\texttt{RADON}(f, x, y, a, b, p) := \\ &\qquad \texttt{IF}(b = 0, \texttt{INT(LIM}(f, x, p/a), y, -\infty, \infty), \texttt{RADON\_AUX}(f, x, y, a, b, p)) \end{aligned} \tag{A.15}$$

The Radon transform of $e^{-x^2-y^2}$ w.r.t. a vertical line

$$\texttt{RADON(EXP}(-x^2-y^2), x, y, a, 0, p) \quad \text{gives} \quad \sqrt{\pi}\,\texttt{EXP}\left[-\frac{p^2}{a^2}\right]$$

which is (A.14) with $b = 0$.

An alternative way is to normalize the coefficients of the lines (A.11), say $a = \cos\alpha$ and $b = \sin\alpha$. The Radon transform is then

$$\begin{aligned} &\texttt{RADON\_NORMAL\_AUX}(f, x, y, \alpha, p) := \\ &\qquad \texttt{INT(LIM}(f, y, p/\texttt{SIN}(\alpha) - \texttt{COT}(\alpha) * x), x, -\infty, \infty)/\texttt{ABS(SIN}(\alpha)) \end{aligned} \tag{A.16}$$

provided $\alpha \neq 0$. The following function treats $\alpha = 0$ separately

$$\begin{aligned} &\texttt{RADON\_NORMAL}(f, x, y, \alpha, p) := \\ &\qquad \texttt{IF}(\alpha = 0, \texttt{INT(LIM}(f, x, p), y, -\infty, \infty), \texttt{RADON\_NORMAL\_AUX}(f, x, y, \alpha, p)) \end{aligned} \tag{A.17}$$

For example, the Radon transform of

$$f(x, y) := \begin{cases} 1 & \text{if } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is computed by

$$\texttt{RADON\_NORMAL(CHI}(0, x, 1) * \texttt{CHI}(0, y, 1), x, y, \alpha, p) \tag{A.18}$$

giving

$$\texttt{SIGN(COS}(\alpha) - p) * \left[\frac{p}{\texttt{SIN}(2\alpha)} - \frac{0.5}{\texttt{SIN}(\alpha)}\right] + \frac{|p|}{\texttt{SIN}(2\alpha)} \tag{A.19}$$

The Radon transform in $\mathbb{R}^3$ is computed w.r.t. a plane

$$a\,x + b\,y + c\,z \;=\; p$$

as follows (ignoring the case $c = 0$),

$$\begin{gathered}\texttt{RADON\_3}(f, x, y, z, a, b, c, p) := \\ \texttt{INT(INT(LIM}(f, z, p/c - a/c * x - b/c * y), x, -\infty, \infty), y, -\infty, \infty) * \texttt{SQRT}(a^2 + b^2 + c^2)/|c|\end{gathered} \tag{A.20}$$

Finally, an integral over $\mathbb{R}^3$ is computed, using (37) as follows:

$$\begin{gathered}\texttt{INT\_3}(f, x, y, z, a, b, c) := \\ \texttt{INT(RADON\_3}(f, x, y, z, a, b, c, p), p, -\infty, \infty)/\texttt{SQRT}(a^2 + b^2 + c^2)\end{gathered} \tag{A.21}$$

For example, the integral over $\mathbb{R}^3$ of $f(x, y, z) = e^{x+y+z-x^2-y^2-z^2}$,

$$\texttt{INT\_3(EXP}(x + y + z - x^2 - y^2 - z^2), x, y, z, a, b, c) \quad \text{gives} \quad \pi^{3/2}\, e^{3/4}$$

RUTCOR–Rutgers Center for Operations Research, Rutgers University, 640 Bartholomew Rd, Piscataway, NJ 08854-8003, USA

E-mail address: bisrael@rutcor.rutgers.edu