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SOME APPLICATIONS OF THE PSEUDOINVERSE OF A MATRIX[1]

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1. INTRODUCTION

IN A PREVIOUS NOTE [1] attention was called to the notion of the pseudoinverse of a rectangular or singular matrix introduced by E. H. Moore [2, 3] and later rediscovered independently by Bjerhammar [4, 5] and Penrose [6]. It is the purpose of the present note to point out two specific applications of the pseudoinverse.[3] Among other possible uses not discussed here is its application to bivariate interpolation.[4]

As a preliminary it will be useful to redefine the pseudoinverse. An $m \times n$ matrix A of rank $r > 0$ can be expressed as a product[5]

$$A = BC, \tag{1}$$

where B is $m \times r$ and C is $r \times n$, and both are of rank r. Then the pseudoinverse of A is given by[6]

$$A^\dagger = C^T(CC^T)^{-1}(B^TB)^{-1}B^T, \tag{2}$$

where the superscript T denotes the transpose. To complete the definition, we define the pseudoinverse of a zero matrix as equal to its transpose.

It will be noted that, for the particular cases $n = r$ and $m = r$, (1) reduces to $A = BI$ and $A = IC$, respectively, and (2) therefore reduces to (1) and (2) of [1]. Equation (2) can therefore be written in the form

$$A^\dagger = C^\dagger B^\dagger.$$

The various properties of the pseudoinverse as given in [1] are now easily derived. In particular it will be convenient to recall three of these: (i) the pseudoinverse is unique, (ii) for a nonsingular matrix it reduces to the ordinary inverse, and (iii) $(A^\dagger)^\dagger = A$.

2. POLYNOMIALS ORTHOGONAL OVER DISCRETE DOMAINS

In a recent note [8] Dent and Newhouse have described a recursive procedure for obtaining orthogonal polynomials over a discrete domain,[7] making use of a

[1] Received by the editors January 6, 1959 and in revised form September 30, 1959.

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[3] A more elementary presentation of these applications (with a numerical example) was given in a paper [7] presented at the 118th Annual Meeting of the American Statistical Association, Chicago, Ill., December 29, 1958.

[4] This possibility was suggested to the writer by William Hodgkinson, Jr., of the American Telephone and Telegraph Co.

[5] To prove this choose B as a matrix whose columns form a basis for the column-space of A. It follows that B is $m \times r$, and that there exists an $r \times n$ matrix C such that $A = BC$. Both B and C are of rank r since the rank of a product does not exceed the rank of any factor.

[6] This definition was suggested to the author by A. S. Householder. It represents an improvement on that given in [1], which was essentially Moore's.

[7] The same general problem has also been treated by Forsythe [9] and Barker [10].

method of matrix inversion by means of submatrices given by Fraser, Duncan, and Collar [11]. We shall point out certain advantages that result from a modification of their procedure involving the introduction of the pseudoinverse.

Given n distinct abscissas x_i, $i = 1, 2, \cdots, n$, let y_i denote a given ordinate corresponding to x_i, and consider the problem of fitting a polynomial of degree $k < n$. It can be shown [9] that there is a unique polynomial $f(x)$ of the required degree which provides the best fit to the given ordinates in the sense of least squares. Let $q_j(x)$, $j = 0, 1, \cdots, n$, be a sequence of known polynomials such that $q_j(x)$ is of proper degree j. In general, we shall be interested in the case $q_j(x) = x^j$. It is evident that $f(x)$ has a unique representation in the form

$$f(x) = \sum_{j=0}^{k} c_j q_j(x),$$

where the coefficients c_j are to be determined. Let y denote the vector whose elements are the given ordinates, and let Q_k, $k = 0, 1, \cdots, n$, denote the $n \times (k + 1)$ matrix $(q_{j-1}(x_i))$. Parenthetically we remark that the matrix S_{k+1} of Dent and Newhouse is $Q_k^T Q_k$. The least-square fitting problem may now be restated as the problem of finding the vector d_k which is the "best" solution of the matrix equation

$$Q_k d_k = y$$

in the sense that the length of $y - Q_k d_k$ is a minimum. Bjerhammar [4, 5] and Penrose [12] have shown that this solution is

$$d_k = Q_k^\dagger y.$$

Since $k < n$, the columns of Q_k are linearly independent. Thus, taking $B = Q_k$ and $C = I_{k+1}$, the identity matrix of order $k + 1$, in (2) gives

$$Q_k^\dagger = (Q_k^T Q_k)^{-1} Q_k^T. \tag{3}$$

We note that $Q_k^\dagger Q_k = I_{k+1}$, while $M_k = Q_k Q_k^\dagger$ is the smoothing matrix [13] which gives the fitted ordinates in terms of the given ones, since

$$M_k y = Q_k d_k.$$

We note also that the sum of the squared differences between the given ordinates and the fitted ones, which Forsythe denotes by δ_k, is given by[8]

$$\delta_k = y^T(I - M_k)y.$$

Now, let q_k denote the last column of Q_k and consider the vector $p_k = q_k - M_{k-1} q_k$. Since $Q_{k-1}^T p_k = 0$ by (3), we see that p_k is orthogonal to every column of Q_{k-1}. Moreover, its elements are ordinates (corresponding to the abscissas x_i) of a polynomial $p_k(x)$ of proper degree k. For $k < n$, $p_k(x)$ is uniquely determined by these n ordinates, while q_n is necessarily a linear combination of the columns of Q_{n-1}, so that $p_n = 0$, and therefore

$$p_n(x) = h \prod_{i=1}^{n} (x - x_i),$$

[8] See [9] for explanation of the use of these quantities in judging the degree of polynomial best suited to the given data.

where h is arbitrary. The polynomials $p_k(x)$, $k = 0, 1, \cdots, n$, are an orthogonal set over the discrete domain $x_1, x_2, \cdots, x_n$. If the $q_k(x)$ are monic polynomials (in particular, if $q_k(x) = x^k$, the $p_k(x)$ are the same polynomials found by Dent and Newhouse. Otherwise, they differ at most by a constant factor.

We have

$$p_k = q_k - Q_{k-1}a_k$$

where

$$a_k = Q_{k-1}^{\dagger}q_k$$

is the same as the A_k of Dent and Newhouse. Thus, if a_{ki}, $i = 0, 1, \cdots, k-1$, are the elements of a_k,

$$p_k(x) = q_k(x) - \sum_{i=0}^{k-1} a_{ki}q_i(x) \tag{4}$$

as given by them.

To recapitulate, if we can find a convenient method of obtaining the pseudoinverses $Q_k^{\dagger}$ for $k = 0, 1, \cdots, n$, then the problem is practically solved, for:

$d_k = Q_k^{\dagger}y$ gives the coefficients by which the fitted polynomial of degree k is expressed in terms of the known polynomials $q_i(x)$.

$M_k y = Q_k d_k$ gives the ordinates of the fitted polynomial corresponding to the given ordinates.

$y^T(I - M_k)y = y^T(y - Q_k d_k)$ gives the sum of the squared residuals when a polynomial of degree k is fitted by least squares.

$a_k = Q_{k-1}^{\dagger}q_k$ gives the coefficients by which $p_k(x)$, the orthogonal polynomial of degree k, is expressed in terms of the known polynomials $q_i(x)$ in accordance with (4).

$p_k = q_k - Q_{k-1}a_k$ gives the ordinates of $p_k(x)$ corresponding to the given abscissas.

3. MULTILINEAR REGRESSION COEFFICIENTS

Let a variate y depend on n variates $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$, and let it be required to determine the coefficients a_j in the regression equation

$$y = \textstyle\sum_{j=1}^{n} a_j x^{(j)}.$$

It is assumed that corresponding numerical values y_i, $x_i^{(j)}$ are given for $i = 1, 2, \cdots, m$. If y denotes the column-vector whose ith component is y_i, a the column-vector whose jth component is a_j and X the matrix $(x_i^{(j)})$, the regression coefficients are given by

$$a = X^{\dagger}y. \tag{5}$$

If the columns of X are linearly independent,[9] as will usually be the case, the least squares regression equation is unique. Otherwise, there will be many solutions which yield the minimum value for the sum of the squared residuals.

[9] Linear dependence would indicate that at least one of the variates $x^{(i)}$ is completely determined by the remaining ones, and therefore redundant.

Of these possible solutions, (5) then gives the one for which the sum of the squares of the coefficients a_j is smallest.

From (5) it follows that the vector

$$y' = Xa = XX^T y$$

gives the values of the variate y predicted by the regression equation, while

$$y^T(y - y') = y^T(I - XX^T)y$$

is the sum of the squares of the residuals.

4. RECURSIVE ALGORITHM FOR THE PSEUDOINVERSE

Let a_k denote the kth column of a given matrix A, and let A_k denote the submatrix consisting of the first k columns. As previously pointed out by Dent and Newhouse, there are substantial advantages in using a recursive procedure for obtaining $A_k^\dagger$ from $A_{k-1}^\dagger$. In fitting a polynomial this makes it unnecessary to decide in advance the degree of polynomial to be fitted, or to start over from scratch if an unfortunate choice is made. Instead, one fits polynomials of successively higher degree and can stop when it appears that the most suitable degree has been reached. Though the advantage is less clear-cut in the regression application, one can attempt to arrange the variables in decreasing order of their probable importance in the regression equation, and can note how much the coefficients change as the less significant variables are introduced, and, if desired, the reduction at each step in the standard error of estimate.

In order to derive the desired recursive procedure, let us consider A_k in the partitioned form

$$(A_{k-1} \qquad a_k), \tag{6}$$

and similarly partition $A_k^\dagger$ in the form

$$A_k^\dagger = \begin{pmatrix} B_k \\ b_k \end{pmatrix}.$$

Multiplication then gives

$$A_k A_k^\dagger = A_{k-1}B_k + a_k b_k . \tag{7}$$

As shown in [1], $A_k A_k^\dagger$ is symmetric, and also, when used as a left multiplier, it leaves unchanged any matrix with columns in the column-space of A_k. It follows that, as a right multiplier, it leaves unchanged any matrix with rows in the transposed column-space of A_k. Now [1, p. 40], $A_{k-1}^\dagger$ has rows in the transposed column-space of A_{k-1} (which is contained in that of A_k). Therefore

$$A_{k-1}^\dagger A_k A_k^\dagger = A_{k-1}^\dagger.$$

By similar reasoning, $A_{k-1}^\dagger A_{k-1}$ as a left multiplier leaves unchanged any matrix with columns in the transposed row-space of A_{k-1}. Now, since $A_k^\dagger$ has columns in the transposed row-space of A_k, a moment's reflection will convince the reader that B_k (as a submatrix of $A_k^\dagger$) has columns in the transposed row-space of A_{k-1} (submatrix of A_k). Therefore,

$$A_{k-1}^\dagger A_{k-1} B_k = B_k .$$

It follows that multiplying (7) on the left by $A_{k-1}^\dagger$ gives

$$A_{k-1}^\dagger = B_k + A_{k-1}^\dagger a_k b_k .$$

Thus we may write

$$A_k^\dagger = \begin{pmatrix} A_{k-1}^\dagger - d_k\, b_k \\ b_k \end{pmatrix}, \tag{8}$$

where

$$d_k = A_{k-1}^\dagger a_k \tag{9}$$

and b_k remains to be determined.

From (6) and (8) we obtain

$$A_k A_k^\dagger = A_{k-1} A_{k-1}^\dagger - A_{k-1}\, d_k b_k + a_k b_k = A_{k-1} A_{k-1}^\dagger + c_k b_k , \tag{10}$$

where

$$c_k = a_k - A_{k-1}\, d_k . \tag{11}$$

Multiplying (11) on the left by $A_{k-1}^\dagger$ and making use of (9) and the relations given in [1], we obtain

$$A_{k-1}^\dagger c_k = 0. \tag{12}$$

Now the rows of $A_{k-1}^\dagger$ are in the transposed column-space of A_{k-1}, and, in fact, they span that space (since the equation $A_{k-1} A_{k-1}^\dagger A_{k-1} = A_{k-1}$ shows that $A_{k-1}^\dagger$ is not of lower rank than A_{k-1}). Therefore (12) shows that c_k is orthogonal to the column-space of A_{k-1}.

It is necessary now to consider two cases, according to whether $c_k = 0$ or not. From (11) we see that $c_k = 0$ implies that a_k is in the column-space of A_{k-1}: in other words, A_k and A_{k-1} have the same rank. Let us first deal with the case $c_k \neq 0$, and let us consider the matrix

$$P_k = A_{k-1} A_{k-1}^\dagger + c_k c_k^\dagger. \tag{13}$$

Now, (11) shows that c_k is in the column-space of A_k, and it follows that $c_k^\dagger$ is in the transposed column-space of A_k. It follows from (13) that the rows of P_k are in the transposed column-space of A_k. From (2), taking $B = c_k$ and C as the identity matrix of order one, it is easily verified that $c_k^\dagger$ is a scalar multiple of c_k^T, and that

$$c_k^\dagger c_k = 1.$$

Further, multiplying (11) on the left by $c_k^\dagger$ and making use of the fact that c_k is orthogonal to the column-space of A_{k-1} gives

$$c_k^\dagger a_k = 1,$$

and it follows from (13) that

$$P_k a_k = A_{k-1} d_k + c_k = a_k$$

by (11), while we observe also that

$$P_k A_{k-1} = A_{k-1} .$$

Thus, (6) shows that

$$P_k A_k = A_k .$$

We see then that P_k has both the properties which uniquely determine the left identity matrix [1, p. 39] of A_k , and therefore

$$P_k = A_k A_k^{\dagger}. \tag{14}$$

From (13), (14) and (10) we see that

$$c_k c_k^{\dagger} = c_k b_k , \tag{15}$$

since both are equal to $A_k A_k^{\dagger} - A_{k-1} A_{k-1}^{\dagger}$. Multiplying (15) on the left by $c_k^{\dagger}$ gives

$$b_k = c_k^{\dagger}. \tag{16}$$

Turning now to the case $c_k = 0$, (11) shows that we then have

$$a_k = A_{k-1} d_k . \tag{17}$$

Let G_k denote the submatrix of $A_k^{\dagger} A_k$ obtained by deleting the last row and the last column. Then it follows from (8) and (6) that

$$G_k = A_{k-1}^{\dagger} A_{k-1} - d_k b_k A_{k-1} .$$

The first term of the right member is symmetric [1, p. 39], as is also G_k , being a principal minor of a symmetric matrix. It follows that $d_k b_k A_{k-1}$ is symmetric. Since $b_k A_{k-1}$ is a one-rowed matrix, this implies[10] that

$$b_k A_{k-1} = h\, d_k^{T}, \tag{18}$$

where h is some scalar.

From (8), (6), (17) and (18) we have

$$A_k^{\dagger} A_k = \begin{pmatrix} A_{k-1}^{\dagger} A_{k-1} - h d_k d_k^{T} \\ h d_k^{T} \end{pmatrix} (I \quad d_k). \tag{19}$$

Now, (9) shows that d_k is in the column-space of $A_{k-1}^{\dagger}$, which is the transposed row-space of A_{k-1} . It follows that

$$A_{k-1}^{\dagger} A_{k-1} d_k = d_k .$$

Thus (19) becomes

$$A_k^{\dagger} A_k = \begin{pmatrix} A_{k-1}^{\dagger} A_{k-1} - h d_k d_k^{T} & d_k - h d_k d_k^{T} d_k \\ h d_k^{T} & h d_k^{T} d_k \end{pmatrix}.$$

[10] If $d_k = 0$, then (since $c_k = 0$), (11) implies $a_k = 0$. Equating the last row of $A_k^{\dagger} A_k$ to the transpose of its last column gives $b_k A_{k-1} = 0$, so that the conclusion still holds.

In view of the symmetry of this matrix and the fact that $d_k^T d_k$ is a scalar, we have

$$(h\, d_k^T)^T = h\, d_k = d_k - h(d_k^T d_k)\, d_k ,$$

and solving for h gives[11]

$$h = (1 + d_k^T d_k)^{-1}. \tag{20}$$

Since the rows of $A_k^\dagger$ are in the transposed column-space of A_k, b_k is in that space, which, in this case is identical with the transposed column-space of A_{k-1}. Thus $b_k A_{k-1} A_{k-1}^\dagger = b_k$, and therefore multiplying (18) on the right by $A_{k-1}^\dagger$ gives

$$b_k = h\, d_k^T A_{k-1}^\dagger. \tag{21}$$

On substituting (20) this gives

$$b_k = (1 + d_k^T d_k)^{-1}\, d_k^T A_{k-1}^\dagger. \tag{22}$$

The desired recursive procedure for obtaining $A_k^\dagger$ from $A_{k-1}^\dagger$ then consists in applying formulas (9), (11), (16) or (22), and (8), in that order. In order to initiate the process, we note that $A_1^\dagger$ is a zero vector if a_1 is a zero vector; otherwise it can be computed from (2).

5. "STREAMLINED" ALGORITHM FOR STATISTICAL APPLICATIONS

For the purpose of statistical applications, some "streamlining" of the algorithm can be effected by noting that in these situations it is unnecessary to obtain the pseudoinverse explicitly. Rather, what is wanted is the "best" (in the sense of least squares) solution $x = A^\dagger \alpha$ of an inconsistent system $Ax = \alpha$. The algorithm can be modified to give $A_k^\dagger \alpha$ for $k = 1, 2, \cdots$ successively. To this end it is convenient to define a matrix A' obtained by enlarging A through the addition of two columns on the right: (i) the vector α and (ii) a total column, which is the sum of all the preceding column vectors. Then (8) gives

$$A_k^\dagger A' = \begin{pmatrix} A_{k-1}^\dagger A' - d_k(b_k A') \\ b_k A' \end{pmatrix}. \tag{23}$$

The penultimate column of this matrix is $A_k^\dagger \alpha$, while the final column should be the sum of the preceding column vectors if the arithmetic has been correctly performed. Moreover, (9) shows that d_k is the kth column of $A_{k-1}^\dagger A'$.

In order to obtain $b_k A'$ for use in (23) we must first compute the right member of (11). If this vector vanishes, (22) shows that

$$b_k A' = (1 + d_k^T d_k)^{-1}\, d_k^T A_{k-1}^\dagger A'. \tag{24}$$

If (11) does not vanish, it gives c_k, and, in view of (16) and (2), we have

$$b_k A' = (c_k^T c_k)^{-1} c_k^T A'. \tag{25}$$

If we first compute the vector $c_k^T A'$, we note that its kth element is $c_k^T a_k$. Multiplying (11) on the left by c_k^T and noting, as previously shown, that c_k is

[11] Provided $d_k \neq 0$. If $d_k = 0$, (21) shows that $b_k = 0$, and (22) still holds.

orthogonal to the column-space of A_{k-1}, we obtain $c_k{}^T c_k = c_k{}^T a_k$. It follows from (25) that $b_k A'$ is obtained from the computed vector $c_k{}^T A'$ upon "normalizing" it by dividing by its kth element. With these explanations, (11), (25) or (24), and (23) constitute the recursive procedure desired.

REFERENCES

1. Greville, T. N. E. *The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations*, this Review, 1 (1959), pp. 38–43.
2. Moore, E. H. Bull. Amer. Math. Soc., 26 (1920), pp. 394–395.
3. ———, *General Analysis*, Part 1, Mem. Amer. Philos. Soc., 1 (1935), especially pp. 197–209.
4. Bjerhammar, A. *Application of calculus of matrices to method of least squares; with special reference to geodetic calculations*, Trans. Roy. Inst. Tech. Stockholm. No. 49 (1951), pp. 1–86.
5. ———, *Rectangular reciprocal matrices with special reference to geodetic calculations*, Bull. Géodésique 1951, pp. 188–220.
6. Penrose, R. *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc., 51 (1955), pp. 406–413.
7. Greville, T. N. E. *The pseudoinverse of a rectangular matrix and its statistical applications*, Amer. Statist. Assoc., Proc. Soc. Statist. Sect., 1958, pp. 116–121. The numerical example is given only in the mimeographed version, available from the author.
8. Dent, B. A. and Newhouse, A. *Polynomials orthogonal over discrete domains*, this Review, 1 (1959), pp. 55–59.
9. Forsythe, G. E. *Generation and use of orthogonal polynomials for data fitting with a digital computer*, J. Soc. Indust. Appl. Math., 5 (1957), pp. 74–88.
10. Barker, J. E. *Use of orthogonal polynomials in fitting curves and estimating their first and second derivatives*, NPG Report No. 1553, NAVORD Report No. 5138, First Revision, U. S. Naval Proving Ground, Dahlgren, Va., June, 1958.
11. Frazer, R. A., Duncan, W. J., and Collar, A. R. *Elementary Matrices*, Cambridge, 1938.
12. Penrose, R. *On best approximate solutions of linear matrix equations*, Proc. Cambridge Philos. Soc., 52 (1956), pp. 17–19.
13. Greville, T. N. E. *On smoothing a finite table: a matrix approach*, J. Soc. Indust. Appl. Math., 5 (1957), pp. 137-154.