Lecture 7: The Drazin Inverse
**Extension to matrices with index $k$**

For a matrix $A \in \mathbb{C}^{n \times n}$ with index 1, the **group inverse** $A^\#$ is the unique solution of the system

\[
AXA = A, \quad (1)
\]
\[
XAX = X, \quad (2)
\]
\[
AX = XA, \quad (5)
\]

also called the $\{1, 2, 5\}$–inverse of $A$. 
Extension to matrices with index \( k \)

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\[
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\]
\[
XAX = X, \\
\]
\[
AX =XA, \\
\]

also called the \( \{1, 2, 5\} \)–inverse of \( A \).

For matrices with index \( k \), the appropriate system is

\[
A^kXA = A^k, \\
\]
\[
XAX = X, \\
\]
\[
AX =XA, \\
\]

and its unique solution is the Drazin inverse, or \( \{1^k, 2, 5\} \)–inverse, of \( A \), denoted \( A^D \).
Motivation

If $A \in \mathbb{C}^{n \times n}$, $\text{Ind} A = k$, then $\mathbb{C}^n = R(A^k) \oplus N(A^k)$, and the restriction $A|_{R(A^k)} \in \mathcal{L}(R(A^k), R(A^k))$ is invertible.

Let $X \in \mathbb{C}^{n \times n}$ be defined by

$$X u = \begin{cases} A|_{R(A^k)}^{-1} u, & \text{if } u \in R(A^k); \\ 0, & \text{if } u \in N(A^k). \end{cases}$$

Then $X$ is an $\{1^k, 2, 5\}$–inverse of $A$. 
The Drazin inverse

Lemma. If $Y$ is a $\{1^\ell, 5\}$–inverse of square matrix $A$, then

$$X = A^\ell Y^{\ell+1}$$

is a $\{1^\ell, 2, 5\}$–inverse.

Theorem (Drazin). Let $A \in \mathbb{C}^{n \times n}$ have index $k$. Then $A$ has a unique $\{1^k, 2, 5\}$–inverse, which is expressible as a polynomial in $A$, and is also the unique $\{1^\ell, 2, 5\}$–inverse for every $\ell \geq k$.

Proof. If $A$ has index $k$, its minimum polynomial is

$$m(\lambda) = \lambda^k p(\lambda), \text{ where } p(0) \neq 0.$$  

$$\therefore \quad m(\lambda) = c\lambda^k (1 - \lambda q(\lambda)), \text{ for some polynomial } q(\cdot)$$  

$$\therefore \quad A^{k+1}q(A) = A^k.$$  

showing that $q(A)$ is a $\{1^k, 5\}$–inverse of $A$. Use lemma to get a $\{1^k, 2, 5\}$–inverse. Proof of uniqueness omitted. □
**Jordan form**

**Theorem.** Let $A \in \mathbb{C}^{n \times n}$ have the Jordan form

$$A = X J X^{-1} = X \begin{bmatrix} J_1 & O \\ O & J_0 \end{bmatrix} X^{-1}, \quad (1)$$

where $J_0$ and $J_1$ are the parts of $J$ corresponding to zero and non-zero eigenvalues. Then

$$A^D = X \begin{bmatrix} J_1^{-1} & O \\ O & O \end{bmatrix} X^{-1}. \quad (2)$$

**Proof.** Let $A$ be singular of index $k$ (i.e., the biggest block in the submatrix $J_0$ is $k \times k$). Then the matrix given by (2) is a \{1^k, 2, 5\}–inverse of $A$. 
Properties of the Drazin inverse

(a) If $X$ is nonsingular, then

$$A = XBX^{-1} \implies A^D = XB^D X^{-1}.$$

(b) $(A^*)^D = (A^D)^*$

(c) $(A^\ell)^D = (A^D)^\ell$ for $\ell = 1, 2, \ldots$

(d) $(A^D)^D = A$ if and only if $A$ has index 1.

(e) $AA^D = A^D A = P_{R(A^D), N(A^D)} = P_{R(A^\ell), N(A^\ell)}$ for all $\ell \geq \text{Ind } A$.

(f) If Ind $A = k$ then

$$A^D = \lim_{\alpha \to 0} (A^{k+1} + \alpha^2 I)^{-1} A^k, \quad \alpha \text{ real},$$

and the approximation error is

$$\|A^D - (A^{k+1} + \alpha^2 I)^{-1} A^k\| \leq \frac{\alpha^2 \|A^D\|^{k+2}}{1 - \alpha^2 \|A^D\|^{k+1}},$$

where $\| \cdot \|$ is the spectral norm.
The Wedderburn decomposition

Theorem (Wedderburn). A square matrix $A$ with index $k$ has a unique decomposition

$$A = C + N,$$

such that $C$ has index $0$ or $1$, $N$ is nilpotent of index $k$, and $NC = CN = O$.

Proof. Let $A \in \mathbb{C}^{n \times n}$ be given in Jordan form

$$A = X J X^{-1} = X \begin{bmatrix} J_1 & O \\ O & J_0 \end{bmatrix} X^{-1}$$

where $J_0$ and $J_1$ are the parts of $J$ corresponding to zero and non-zero eigenvalues. Then

$$C = X \begin{bmatrix} J_1 & O \\ O & O \end{bmatrix} X^{-1}, \quad N = X \begin{bmatrix} O & O \\ O & J_0 \end{bmatrix} X^{-1}$$
The Wedderburn decomposition (cont’d)

Ex.

\[
A = X^{-1} \begin{bmatrix}
J_1(5) & O & O & O & O \\
O & J_2(3) & O & O & O \\
O & O & J_3(-7) & O & O \\
O & O & O & J_4(0) & O \\
O & O & O & O & J_5(0)
\end{bmatrix} X,
\]

\[
N = X^{-1} \begin{bmatrix}
O & O & O & O & O \\
O & O & O & O & O \\
O & O & O & O & O \\
O & O & O & J_4(0) & O \\
O & O & O & O & J_5(0)
\end{bmatrix} X, \quad C = A - N.
\]

**Theorem.** If \( A = C + N \) is a Wedderburn decomposition then
\[
C = A^2 A^D = (A^D)^\# , \quad N = A(I - A^D A), \quad A^D = C^D \in C\{1, 2\}.
\]
Spectral property of the Drazin inverse

The definition of $S$–inverse is weakened in (b) below.

(a) Let $A \in \mathbb{C}^{n \times n}$. Then $X$ is an $S$–inverse of $A$ if they share the property that, for every $\lambda \in \mathbb{C}$ and every vector $x$,

\[
\begin{align*}
\begin{cases}
    x \text{ is a } \lambda \text{–vector of } A \\
    \text{of grade } p
\end{cases}
\end{align*}
\iff
\begin{align*}
\begin{cases}
    x \text{ is a } \lambda^{\dagger} \text{–vector of } X \\
    \text{of grade } p
\end{cases}
\end{align*}
\]

(b) $X$ is an $S'$–inverse of $A$ if, for all $\lambda \neq 0$, and every vector $x$,

\[
\begin{align*}
\begin{cases}
    x \text{ is a } \lambda \text{–vector of } A \\
    \text{of grade } p
\end{cases}
\end{align*}
\iff
\begin{align*}
\begin{cases}
    x \text{ is a } \lambda^{\dagger} \text{–vector of } X \\
    \text{of grade } p
\end{cases}
\end{align*}
\]

and $x$ is a 0–vector of $X$ if and only if it is a 0–vector of $A$ (without regard to grade).

**Theorem.** For every square matrix $A$, $A$ and $A^D$ are $S'$–inverses of each other.
Matrix functions

A **matrix function** is a mapping $f : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ that corresponds, in some sense, to a **scalar function** $f : \mathbb{C} \to \mathbb{C}$. For $f(A)$ to be defined, the scalar function $f$ is required to be **analytic** in some open set containing $\lambda(A)$. 


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Ex. Inhomogeneous linear differential equation.

Scalar equation: \( \dot{x} + ax = b(t) \)
Solution: \( x(t) = e^{-at}y + e^{-at} \int_{t}^{\infty} e^{as}b(s)ds \), \( y \) arbitrary.

Vector equation: \( \dot{x}(t) + Ax(t) = b(t) \)
Solution: \( x(t) = e^{-At}y + e^{-At} \int_{t}^{\infty} e^{As}b(s)ds \), \( y \) arbitrary.
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Vector equation: $\dot{x}(t) + Ax(t) = b(t)$
Solution: $x(t) = e^{-At}y + e^{-At} \int_{t}^{t} e^{As}b(s)ds$, $y$ arbitrary.

The matrix function $e^{-At}$ plays here the same role as the scalar function $e^{-at}$. What is the nature of the correspondence

$e^{-at} \longleftrightarrow e^{-At}$
The Fantappiè requirements

To be useful, the correspondence

\[ f(z) \leftrightarrow f(A) \]

must satisfy certain formal conditions. The following four conditions are due to Fantappiè:
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I. If \( f(z) = k \) then \( f(A) = kI \)

II. If \( f(z) = z \) then \( f(A) = A \)

III. If \( f(z) = g(z) + h(z) \) then \( f(A) = g(A) + h(A) \)
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III. If \( f(z) = g(z) + h(z) \) then \( f(A) = g(A) + h(A) \)

IV. If \( f(z) = g(z)h(z) \) then \( f(A) = g(A)h(A) \).
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IV. If \( f(z) = g(z)h(z) \) then \( f(A) = g(A)h(A) \).

A fifth condition serves to assure consistency of compositions of matrix functions:

V. If \( f(z) = h(g(z)) \) then \( f(A) = h(g(A)) \).
Spectral theorem for diagonalizable matrices

**Theorem.** Let \( A \in \mathbb{C}^{n \times n} \) with \( s \) distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_s \). Then \( A \) is **diagonalizable** if and only if there exist **projectors** \( E_1, E_2, \ldots, E_s \) such that

\[
E_i E_j = \delta_{ij} E_i ,
\]

\[
I_n = \sum_{i=1}^{s} E_i ,
\]

\[
A = \sum_{i=1}^{s} \lambda_i E_i .
\]

Let \( L_i \) be the \( i \)th **eigenspace**, \( X = (X_1 X_2 \cdots X_s) \in \mathbb{C}^{n \times n} \), cols \( X_i \) = basis for \( L_i \), \( Y_i \) the corresponding submatrix of \( X^{-1} \), then

\[
E_i = X_i Y_i , \quad \forall \ i \in \overline{1,s}
\]

the **principal idempotents** (or **Frobenius invariants**) of \( A \).
Functions of diagonalable matrices

Definition. If $A$ is diagonalable, then

$$f(A) := \sum_{i=1}^{s} f(\lambda_i) E_i$$
Functions of diagonalable matrices

Definition. If $A$ is diagonalable, then

$$f(A) := \sum_{i=1}^{s} f(\lambda_i) E_i$$

This definition satisfies the Fantappiè requirements. In particular, if $h \leftrightarrow h$ and $g \leftrightarrow g$ then

$$f := h \circ g \leftrightarrow f := h \circ g$$

defined as

$$(h \circ g)(A) := h(g(A)) = \sum_{i=1}^{s} h(g(\lambda_i)) E_i$$
A polynomial in a square matrix

Let $A \in \mathbb{C}^{n \times n}$ have $s$ distinct eigenvalues, and Jordan form ($J_i$ all blocks corresponding to the $i_{th}$ eigenvalue),

$$A = X J X^{-1} = (X_1 X_2 \cdots X_s)$$

$$\begin{bmatrix}
            J_1 & O & \cdots & O \\
            O & J_2 & \cdots & O \\
            O & \ddots & \ddots & \\
            O & O & \cdots & J_s
        \end{bmatrix}
        \begin{bmatrix}
            Y_1 \\
            Y_2 \\
            \vdots \\
            Y_s
        \end{bmatrix}$$

Then for any polynomial $p(\lambda)$,

$$p(A) = (X_1 X_2 \cdots X_s)$$

$$\begin{bmatrix}
p(J_1) & O & \cdots & O \\
O & p(J_2) & \cdots & O \\
O & \ddots & \ddots & \\
O & O & \cdots & p(J_s)
\end{bmatrix}
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_s
\end{bmatrix}$$
A power of a $k \times k$ Jordan block

$$(J_k(\lambda))^m = \begin{bmatrix}
\lambda^m & m\lambda^{m-1} & (\binom{m}{2})\lambda^{m-2} & \cdots & (\binom{m}{k-1})\lambda^{m-k+1} \\
0 & \lambda^m & m\lambda^{m-1} & \ddots & \\
& & \ddots & \ddots & (\binom{m}{2})\lambda^{m-2} \\
& & \ddots & \lambda^m & m\lambda^{m-1} \\
0 & \cdots & \cdots & 0 & \lambda^m
\end{bmatrix}$$

$$= \sum_{j=0}^{k-1} \binom{m}{j} \lambda^{m-j} (J_k(\lambda) - \lambda I_k)^j$$

$$= \sum_{j=0}^{k-1} \frac{p(j)(\lambda)}{j!} (J_k(\lambda) - \lambda I_k)^j,$$

where $p(\lambda) = \lambda^m$, $\binom{m}{\ell}$ is interpreted as zero if $m < \ell$. 
A polynomial in a Jordan block

Let $p(\lambda)$ be a polynomial. Then

$$p(J_k(\lambda)) :=
\begin{bmatrix}
p(\lambda) & \frac{1}{1!}p'(\lambda) & \frac{1}{2!}p''(\lambda) & \cdots & \frac{1}{(k-1)!}p^{(k-1)}(\lambda) \\
0 & p(\lambda) & \frac{1}{1!}p'(\lambda) & \cdots & \\
\vdots & \vdots & \ddots & \ddots & \frac{1}{2!}p''(\lambda) \\
\vdots & \vdots & \ddots & p(\lambda) & \frac{1}{1!}p'(\lambda) \\
0 & \cdots & \cdots & 0 & p(\lambda)
\end{bmatrix}$$

or

$$p(J_k(\lambda)) = \sum_{j=0}^{k-1} \frac{p(j)(\lambda)}{j!} (J_k(\lambda) - \lambda I_k)^j.$$
Spectral theorem for square matrices

**Theorem.** Let the matrix $A \in \mathbb{C}^{n \times n}$ have $s$ distinct eigenvalues. Then there exist $s$ unique projectors $\{E_\lambda : \lambda \in \lambda(A)\}$ such that

$$E_\lambda E_\mu = \delta_{\lambda \mu} E_\lambda ,$$  \hspace{1cm} (1)

$$I_n = \sum_{\lambda \in \lambda(A)} E_\lambda ,$$ \hspace{1cm} (2)

$$A = \sum_{\lambda \in \lambda(A)} \lambda E_\lambda + \sum_{\lambda \in \lambda(A)} (A - \lambda I) E_\lambda ,$$ \hspace{1cm} (3)

$$AE_\lambda = E_\lambda A , \text{ for all } \lambda \in \lambda(A) ,$$ \hspace{1cm} (4)

$$E_\lambda (A - \lambda I)^k = O , \text{ for all } \lambda \in \lambda(A) , \text{ } k \geq \nu(\lambda) .$$ \hspace{1cm} (5)

where $\nu(\lambda)$ denote the index of $(A - \lambda I)$, called the **index** of $\lambda$. 
Matrix functions on a Jordan block

Let \( \lambda \in \lambda(A) \), and let \( \nu(\lambda) \) denote the index of \((A - \lambda I)\), called the index of \( \lambda \).

Let \( J \) be a \( \nu \times \nu \) Jordan block corresponding to \( \lambda \). The matrix function \( f \) (\( \longmapsto \) a scalar function \( f \)) is defined on \( J \) as follows

\[
f(J) = E_\lambda \sum_{k=0}^{\nu-1} \frac{f^{(k)}(\lambda)}{k!} (J - \lambda I_\nu)^k
\]

\[
\begin{bmatrix}
f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \ldots & \frac{f^{(\nu-1)}(\lambda)}{(\nu-1)!} \\
\vdots & f(\lambda) & f'(\lambda) & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \frac{f''(\lambda)}{2!} \\
\vdots & \vdots & \vdots & f(\lambda) & f'(\lambda) \\
0 & \ldots & \ldots & \ldots & f(\lambda)
\end{bmatrix}
\]
Matrix functions

For any $A \in \mathbb{C}^{n \times n}$ with spectrum $\lambda(A)$, let $\mathcal{F}(A)$ denote the class of all functions $f : \mathbb{C} \to \mathbb{C}$ which are analytic in some open set containing $\lambda(A)$. For any scalar function $f \in \mathcal{F}(A)$, the corresponding matrix function $f(A)$ is defined by

$$f(A) = \sum_{\lambda \in \lambda(A)} E_{\lambda} \sum_{k=0}^{\nu(\lambda) - 1} \frac{f^{(k)}(\lambda)}{k!} (A - \lambda I_n)^k.$$  

(1)
Matrix functions

For any $A \in \mathbb{C}^{n \times n}$ with spectrum $\lambda(A)$, let $\mathcal{F}(A)$ denote the class of all functions $f : \mathbb{C} \to \mathbb{C}$ which are analytic in some open set containing $\lambda(A)$. For any scalar function $f \in \mathcal{F}(A)$, the **corresponding matrix function** $f(A)$ is defined by

$$f(A) = \sum_{\lambda \in \lambda(A)} E_{\lambda} \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} (A - \lambda I_n)^k.$$  

(1)

$f(A)$ can be evaluated from the Jordan form of $A$

$$A = X \text{ diag } (J_1, J_2, \ldots) X^{-1},$$

as

$$f(A) = X \text{ diag } (f(J_1), f(J_2), \ldots) X^{-1}.$$
The Drazin inverse is the reciprocal function

For $A \in \mathbb{C}^{n \times n}$ with spectrum $\lambda(A)$, and a scalar function $f \in \mathcal{F}(A)$, the corresponding matrix function $f(A)$ is

$$f(A) = \sum_{\lambda \in \lambda(A)} E_{\lambda} \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} (A - \lambda I_n)^k,$$

(1)
The Drazin inverse is the reciprocal function

For \( A \in \mathbb{C}^{n \times n} \) with spectrum \( \lambda(A) \), and a scalar function \( f \in \mathcal{F}(A) \), the corresponding matrix function \( f(A) \) is

\[
f(A) = \sum_{\lambda \in \lambda(A)} E_\lambda \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} (A - \lambda I_n)^k,
\]

(1)

The analogous result for the Drazin inverse is:

**Theorem.** Let \( A \in \mathbb{C}^{n \times n} \)

\[
A^D = \sum_{0 \neq \lambda \in \lambda(A)} E_\lambda \sum_{k=0}^{\nu(\lambda)-1} \frac{(-1)^k}{\lambda^{k+1}} (A - \lambda I_n)^k.
\]

(2)

This shows that the Drazin inverse is the matrix function corresponding to the reciprocal \( f(z) = 1/z \), defined on nonzero eigenvalues.
The system \( x' + Ax = f \)

The **general solution** of \( x' + Ax = f \) is

\[
x = e^{-At} \left( \int e^{At} f(t) dt \right)
\]

where

\[
e^{At} = I + At + A^2 \frac{t^2}{2!} + \cdots + A^k \frac{t^k}{k!} + \cdots
\]

If \( A \) is **nonsingular** then

\[
\int e^{At} \, dt = A^{-1} e^{At} + C
\]

where \( C \) is arbitrary (the constant of integration.)

If \( A \) is **singular**, of **index** \( k \), then

\[
\int e^{At} \, dt = A^D e^{At} + (I - AA^D) \left[ tI + A \frac{t^2}{2!} + A^2 \frac{t^3}{3!} + \cdots + A^{k-1} \frac{t^k}{k!} \right] + C
\]
The system $x' + Ax = f$ (cont’d)

If $f$ is a constant vector then

$$x' + Ax = f$$

has a particular solution that is a polynomial in $t$. The general solution is

$$x = e^{-At} \left( \int e^{At} dt \right) f + C$$

and a particular solution is

$$x = A^D f + (I - AA^D) \left[ tI - A\frac{t^2}{2!} + A^2\frac{t^3}{3!} - \cdots (-1)^{k-1} A^{k-1}\frac{t^k}{k!} \right] f$$

Ex.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_1 = 1, \quad x_2 = t - \frac{t^2}{2}, \quad x_3 = t$$
The system $Ax' + Bx = f$, $AB = BA$

Let $A$ be singular, $B$ commute with $A$, and consider

$$Ax' + Bx = f \quad (1)$$

and the corresponding homogeneous equation

$$Ax' + Bx = 0 \quad (2)$$
The system $Ax' + Bx = f$, $AB = BA$

Let $A$ be singular, $B$ commute with $A$, and consider

$$Ax' + Bx = f$$

and the corresponding homogeneous equation

$$Ax' + Bx = 0$$

Let $A = C + N$ be the Wedderburn decomposition of $A$, with $C$ of index 0 or 1, $N$ nilpotent of index $k$, $CN = NC = O$ and

$$A^D = C^D, \quad N = A(I - A^D A), \quad C = A^2 A^D.$$
The system \( Ax' + Bx = f \), \( AB = BA \)

Let \( A \) be singular, \( B \) commute with \( A \), and consider

\[
Ax' + Bx = f \tag{1}
\]

and the corresponding homogeneous equation

\[
Ax' + Bx = 0 \tag{2}
\]

Let \( A = C + N \) be the **Wedderburn decomposition** of \( A \), with \( C \) of index 0 or 1, \( N \) nilpotent of index \( k \), \( CN = NC = O \) and

\[
A^D = C^D , \ N = A(I - A^D A) , \ C = A^2 A^D .
\]

If \( AB = BA \) then

\[
AB^D = B^D A , \ A^D B = BA^D , \text{ and } A^D B^D = B^D A^D .
\]
The system $Ax' + Bx = f$, $AB = BA$ (cont’d)

$$Ax' + Bx = f$$  \hspace{1cm} (1)

$$Ax' + Bx = 0$$  \hspace{1cm} (2)

**Theorem.** If $AB = BA$ then

$$y = e^{-AD}BtADAv,$$

is a solution of (2) for any vector $v$.

**Proof.**

$$Ay' = -AA^DBe^{-AD}BtADAv = -Be^{-AD}BtADAv = -By.$$  \hspace{1cm} $\square$
The system $Ax' + Bx = f$, $AB = BA$ (cont’d)

$$Ax' + Bx = f$$  \hspace{1cm} (1)

$$Ax' + Bx = 0$$  \hspace{1cm} (2)

**Theorem.** If $AB = BA$ then

$$y = e^{-AD}Bt ADv,$$

is a solution of (2) for any vector $v$.

**Proof.**

$$Ay' = -AADBe^{-AD}Bt ADv = -Be^{-AD}Bt ADv = -By.$$ □

Let

$$x_1 = ADAx, \quad x_2 := (I - AD^2)x$$  \hspace{1cm} (3)

Then (1) becomes

$$(C + N)(x_1' + x_2') + B(x_1 + x_2) = f$$  \hspace{1cm} (4)
The system $Ax' + Bx = f$, $AB = BA$ (cont’d)

The story so far

\[ Ax' + Bx = f \quad (1) \]
\[ Ax' + Bx = 0 \quad (2) \]
\[ x_1 = A^D Ax, \ x_2 = (I - A^D A)x \quad (3) \]
\[ (C + N)(x'_1 + x'_2) + B(x_1 + x_2) = f \quad (4) \]
The system $Ax' + Bx = f$, $AB = BA$ (cont’d)

The story so far

$$Ax' + Bx = f \quad (1)$$

$$Ax' + Bx = 0 \quad (2)$$

$$x_1 = A^D Ax, \ x_2 = (I - A^D A)x \quad (3)$$

$$(C + N)(x'_1 + x'_2) + B(x_1 + x_2) = f \quad (4)$$

Multiplying (4) by $C^DC$ and then by $(I - C^DC)$ it follows that (2) is equivalent to the system

$$Cx'_1 + Bx_1 = f_1 \quad (5)$$

$$Nx'_2 + Bx_2 = f_2 \quad (6)$$

where $f_1 = C^DCf$ and $f_2 = (I - C^DC)f$, etc.