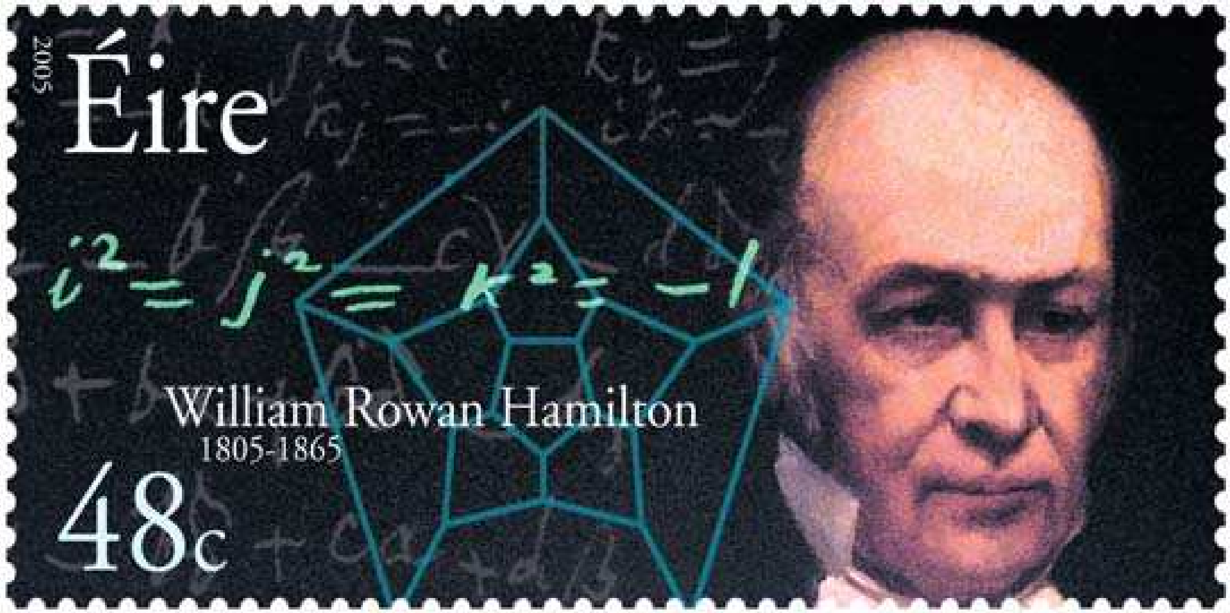
# Lecture 7: The Drazin Inverse



## Extension to matrices with index $k$

For a matrix $A \in \mathbb{C}^{n \times n}$ with index 1, the **group inverse** $A^{\#}$ is the **unique solution** of the system

$$AXA = A\,, \tag{1}$$

$$XAX = X\,, \tag{2}$$

$$AX = XA\,, \tag{5}$$

also called the $\{1, 2, 5\}$–**inverse** of $A$.

# Extension to matrices with index $k$

For a matrix $A \in \mathbb{C}^{n \times n}$ with index 1, the **group inverse** $A^{\#}$ is the **unique solution** of the system

$$AXA = A \,, \tag{1}$$

$$XAX = X \,, \tag{2}$$

$$AX = XA \,, \tag{5}$$

also called the $\{1,2,5\}$–**inverse** of $A$.

For matrices with index $k$, the appropriate system is

$$A^k X A = A^k \,, \tag{$1^k$}$$

$$XAX = X \,, \tag{2}$$

$$AX = XA \,, \tag{5}$$

and its unique solution is the **Drazin inverse**, or $\{1^k, 2, 5\}$–**inverse**, of $A$, denoted $A^D$.

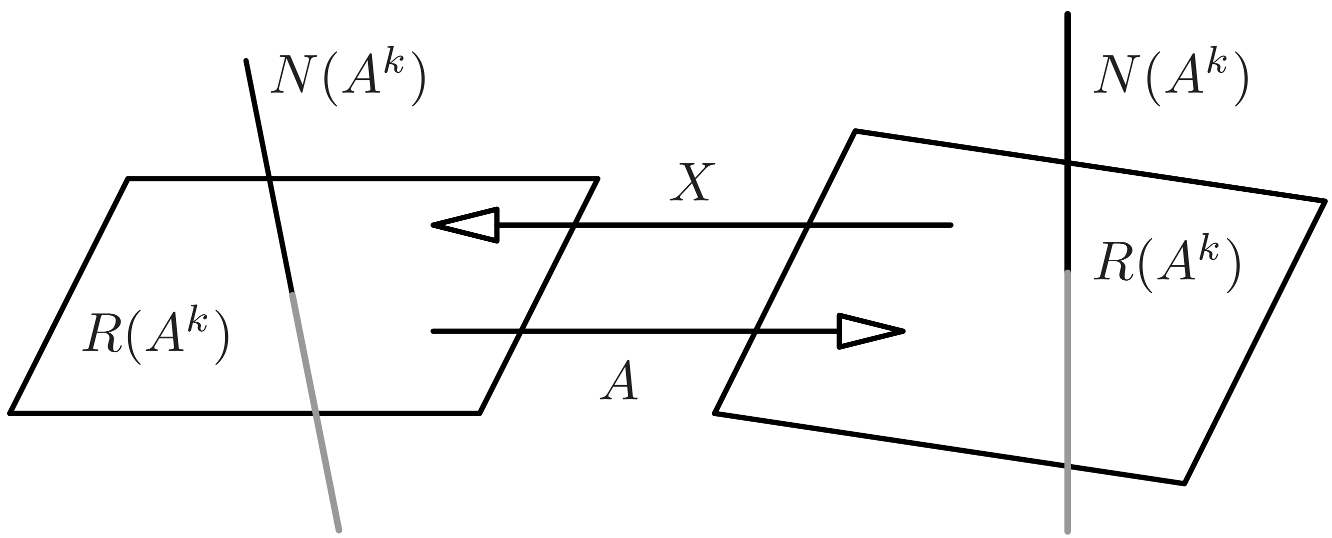
# Motivation

If $A \in \mathbb{C}^{n\times n}$ , $\operatorname{Ind} A = k$, then $C^n = R(A^k) \oplus N(A^k)$, and the **restriction** $A|_{R(A^k)} \in \mathcal{L}(R(A^k), R(A^k))$ is **invertible**.

Let $X \in \mathbb{C}^{n\times n}$ be defined by

$$X\,\mathbf{u} = \begin{cases} A|_{R(A^k)}^{-1}\,\mathbf{u}, & \text{if } \mathbf{u} \in R(A^k); \\ \mathbf{0}, & \text{if } \mathbf{u} \in N(A^k). \end{cases}$$

Then $X$ is an $\{1^k, 2, 5\}$–**inverse** of $A$.



# The Drazin inverse

**Lemma**. If $Y$ is a $\{1^\ell, 5\}$–inverse of square matrix $A$, then

$$X = A^\ell Y^{\ell+1}$$

is a $\{1^\ell, 2, 5\}$–inverse.

**Theorem** (**Drazin**). Let $A \in \mathbb{C}^{n\times n}$ have index $k$. Then $A$ has a unique $\{1^k, 2, 5\}$–inverse, which is expressible as a polynomial in $A$, and is also the unique $\{1^\ell, 2, 5\}$–inverse for every $\ell \geq k$.

**Proof**. If $A$ has index $k$, its minimum polynomial is

$$m(\lambda) = \lambda^k p(\lambda)\ , \text{ where } p(0) \neq 0.$$

$$\therefore\ m(\lambda) = c\lambda^k (1 - \lambda q(\lambda))\ , \text{ for some polynomial } q(\cdot)$$

$$\therefore\ A^{k+1} q(A) = A^k\ .$$

showing that $q(A)$ is a $\{1^k, 5\}$–inverse of $A$. Use lemma to get a $\{1^k, 2, 5\}$–inverse. Proof of uniqueness omitted. $\square$

# Jordan form

**Theorem**. Let $A \in \mathbb{C}^{n\times n}$ have the Jordan form

$$A = XJX^{-1} = X \begin{bmatrix} J_1 & O \\ O & J_0 \end{bmatrix} X^{-1}\,, \tag{1}$$

where $J_0$ and $J_1$ are the parts of $J$ corresponding to zero and non-zero eigenvalues. Then

$$A^D = X \begin{bmatrix} J_1^{-1} & O \\ O & O \end{bmatrix} X^{-1}\,. \tag{2}$$

**Proof**. Let $A$ be singular of index $k$ (i.e., the biggest block in the submatrix $J_0$ is $k \times k$). Then the matrix given by (2) is a $\{1^k, 2, 5\}$–inverse of $A$.

## Properties of the Drazin inverse

(a) If $X$ is nonsingular, then

$$A = XBX^{-1} \implies A^D = XB^DX^{-1} \, .$$

(b) $(A^*)^D = (A^D)^*$

(c) $(A^\ell)^D = (A^D)^\ell$ for $\ell = 1, 2, \ldots$

(d) $(A^D)^D = A$ if and only if $A$ has index 1.

(e) $AA^D = A^DA = P_{R(A^D),N(A^D)} = P_{R(A^\ell),N(A^\ell)}$ for all $\ell \geq \operatorname{Ind} A$.

(f) If $\operatorname{Ind} A = k$ then

$$A^D = \lim_{\alpha \to 0} (A^{k+1} + \alpha^2 I)^{-1} A^k \, , \quad \alpha \text{ real} \, ,$$

and the approximation error is

$$\| A^D - (A^{k+1} + \alpha^2 I)^{-1} A^k \| \leq \frac{\alpha^2 \| A^D \|^{k+2}}{1 - \alpha^2 \| A^D \|^{k+1}} \, ,$$

where $\| \cdot \|$ is the spectral norm

# The Wedderburn decomposition

**Theorem** (**Wedderburn**). A **square** matrix $A$ with **index** $k$ has a **unique decomposition**

$$A = C + N \, , \tag{1}$$

such that $C$ has **index 0 or 1**, $N$ is **nilpotent of index** $k$, and $NC = CN = O$ .

**Proof**. Let $A \in \mathbb{C}^{n \times n}$ be given in Jordan form

$$A = XJX^{-1} = X \begin{bmatrix} J_1 & O \\ O & J_0 \end{bmatrix} X^{-1}$$

where $J_0$ and $J_1$ are the parts of $J$ corresponding to zero and non-zero eigenvalues. Then

$$C = X \begin{bmatrix} J_1 & O \\ O & O \end{bmatrix} X^{-1} \, , \quad N = X \begin{bmatrix} O & O \\ O & J_0 \end{bmatrix} X^{-1}$$

# The Wedderburn decomposition (cont'd)

**Ex.**

$$A = X^{-1} \begin{bmatrix} J_1(5) & O & O & O & O \\ O & J_2(3) & O & O & O \\ O & O & J_3(-7) & O & O \\ O & O & O & J_4(0) & O \\ O & O & O & O & J_5(0) \end{bmatrix} X \ ,$$

$$N = X^{-1} \begin{bmatrix} O & O & O & O & O \\ O & O & O & O & O \\ O & O & O & O & O \\ O & O & O & J_4(0) & O \\ O & O & O & O & J_5(0) \end{bmatrix} X \ , \ C = A - N \ .$$

---

**Theorem**. If $A = C + N$ is a Wedderburn decomposition then $C = A^2 A^D = (A^D)^\# \ , \ N = A(I - A^D A) \ , \ A^D = C^D \in C\{1, 2\}$.

# Spectral property of the Drazin inverse

The definition of $S$–inverse is weakened in (b) below.

(a) Let $A \in \mathbb{C}^{n\times n}$. Then $X$ is an $S$–**inverse** of $A$ if they share the property that, for every $\lambda \in \mathbb{C}$ and every vector $\mathbf{x}$,

$$\left\{ \begin{array}{l} \mathbf{x} \text{ is a } \lambda\text{–vector of } A \\ \text{of grade } p \end{array} \right\} \iff \left\{ \begin{array}{l} \mathbf{x} \text{ is a } \lambda^{\dagger}\text{–vector of } X \\ \text{of grade } p \end{array} \right\}$$

(b) $X$ is an $S'$–**inverse** of $A$ if, for all $\lambda \neq 0$, and every vector $\mathbf{x}$,

$$\left\{ \begin{array}{l} \mathbf{x} \text{ is a } \lambda\text{–vector of } A \\ \text{of grade } p \end{array} \right\} \iff \left\{ \begin{array}{l} \mathbf{x} \text{ is a } \lambda^{\dagger}\text{–vector of } X \\ \text{of grade } p \end{array} \right\}$$

and $\mathbf{x}$ is a 0–vector of $X$ if and only if it is a 0–vector of $A$ (without regard to grade).

**Theorem**. For every square matrix $A$, $A$ and $A^D$ are $S'$–inverses of each other.

# Matrix functions

A **matrix function** is a mapping $\mathbf{f} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ that corresponds, in some sense, to a **scalar function** $f : \mathbb{C} \to \mathbb{C}$. For $\mathbf{f}(A)$ to be defined, the scalar function $f$ is required to be **analytic** in some open set containing $\lambda(A)$.

# Matrix functions

A **matrix function** is a mapping $\mathbf{f} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ that corresponds, in some sense, to a **scalar function** $f : \mathbb{C} \to \mathbb{C}$. For $\mathbf{f}(A)$ to be defined, the scalar function $f$ is required to be **analytic** in some open set containing $\lambda(A)$.

**Ex. Inhomogeneous linear differential equation.**

Scalar equation: $\dot{x} + ax = b(t)$
Solution: $x(t) = e^{-at}y + e^{-at}\int^t e^{as}b(s)ds\ ,\ y$ arbitrary.

Vector equation: $\dot{\mathbf{x}}(t) + A\mathbf{x}(t) = \mathbf{b}(t)$
Solution: $\mathbf{x}(t) = e^{-At}\mathbf{y} + e^{-At}\int^t e^{As}\mathbf{b}(s)ds\ ,\ \mathbf{y}$ arbitrary.

# Matrix functions

A **matrix function** is a mapping $\mathbf{f} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ that corresponds, in some sense, to a **scalar function** $f : \mathbb{C} \to \mathbb{C}$. For $\mathbf{f}(A)$ to be defined, the scalar function $f$ is required to be **analytic** in some open set containing $\lambda(A)$.

**Ex. Inhomogeneous linear differential equation.**

Scalar equation: $\dot{x} + ax = b(t)$
Solution: $x(t) = e^{-at}y + e^{-at}\int^t e^{as}b(s)ds$ , $y$ arbitrary.

Vector equation: $\dot{\mathbf{x}}(t) + A\mathbf{x}(t) = \mathbf{b}(t)$
Solution: $\mathbf{x}(t) = e^{-At}\mathbf{y} + e^{-At}\int^t e^{As}\mathbf{b}(s)ds$ , $\mathbf{y}$ arbitrary.

The **matrix function** $e^{-At}$ plays here the same role as the **scalar function** $e^{-at}$. What is the nature of the **correspondence**

$$e^{-at} \longleftrightarrow e^{-At}$$

## The Fantappiè requirements

To be useful, the **correspondence**

$$f(z) \longleftrightarrow \mathbf{f}(A)$$

must satisfy certain formal conditions. The following four conditions are due to **Fantappiè**:

# The Fantappiè requirements

To be useful, the **correspondence**

$$f(z) \longleftrightarrow \mathbf{f}(A)$$

must satisfy certain formal conditions. The following four conditions are due to **Fantappiè**:

I. If $f(z) = k$ then $\mathbf{f}(A) = kI$

## The Fantappiè requirements

To be useful, the **correspondence**

$$f(z) \longleftrightarrow \mathbf{f}(A)$$

must satisfy certain formal conditions. The following four conditions are due to **Fantappiè**:

I. If $f(z) = k$ then $\mathbf{f}(A) = kI$

II. If $f(z) = z$ then $\mathbf{f}(A) = A$

# The Fantappiè requirements

To be useful, the **correspondence**

$$f(z) \longleftrightarrow \mathbf{f}(A)$$

must satisfy certain formal conditions. The following four conditions are due to **Fantappiè**:

I. If $f(z) = k$ then $\mathbf{f}(A) = kI$

II. If $f(z) = z$ then $\mathbf{f}(A) = A$

III. If $f(z) = g(z) + h(z)$ then $\mathbf{f}(A) = \mathbf{g}(A) + \mathbf{h}(A)$

## The Fantappiè requirements

To be useful, the **correspondence**

$$f(z) \longleftrightarrow \mathbf{f}(A)$$

must satisfy certain formal conditions. The following four conditions are due to **Fantappiè**:

I. If $f(z) = k$ then $\mathbf{f}(A) = kI$

II. If $f(z) = z$ then $\mathbf{f}(A) = A$

III. If $f(z) = g(z) + h(z)$ then $\mathbf{f}(A) = \mathbf{g}(A) + \mathbf{h}(A)$

IV. If $f(z) = g(z)h(z)$ then $\mathbf{f}(A) = \mathbf{g}(A)\mathbf{h}(A)$.

## The Fantappiè requirements

To be useful, the **correspondence**

$$f(z) \longleftrightarrow \mathbf{f}(A)$$

must satisfy certain formal conditions. The following four conditions are due to **Fantappiè**:

I. If $f(z) = k$ then $\mathbf{f}(A) = kI$

II. If $f(z) = z$ then $\mathbf{f}(A) = A$

III. If $f(z) = g(z) + h(z)$ then $\mathbf{f}(A) = \mathbf{g}(A) + \mathbf{h}(A)$

IV. If $f(z) = g(z)h(z)$ then $\mathbf{f}(A) = \mathbf{g}(A)\mathbf{h}(A)$.

A fifth condition serves to assure **consistency of compositions** of matrix functions:

V. If $f(z) = h(g(z))$ then $\mathbf{f}(A) = \mathbf{h}(\mathbf{g}(A))$.

## Spectral theorem for diagonable matrices

**Theorem**. Let $A \in \mathbb{C}^{n\times n}$ with $s$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$. Then $A$ is **diagonable** if and only if there exist **projectors** $E_1, E_2, \ldots, E_s$ such that

$$E_i E_j = \delta_{ij}\, E_i \ , \tag{1}$$

$$I_n = \sum_{i=1}^{s} E_i \ , \tag{2}$$

$$A = \sum_{i=1}^{s} \lambda_i\, E_i \ . \tag{3}$$

Let $L_i$ be the $i_{\,\rm th}$ **eigenspace**, $X = (X_1\, X_2\, \cdots\, X_s) \in \mathbb{C}^{n\times n}$, cols $X_i$ = basis for $L_i$, $Y_i$ the corresponding submatrix of $X^{-1}$, then

$$E_i = X_i Y_i \ , \ \forall\ i \in \overline{1, s}$$

the **principal idempotents** (or **Frobenius invariants**) of $A$.

# Functions of diagonable matrices

**Definition**. If $A$ is **diagonable**, then

$$\mathbf{f}(A) := \sum_{i=1}^{s} f(\lambda_i)\, E_i$$

# Functions of diagonable matrices

**Definition**. If $A$ is **diagonable**, then

$$\mathbf{f}(A) := \sum_{i=1}^{s} f(\lambda_i)\, E_i$$

This definition satisfies the **Fantappiè requirements**. In particular, if $h \longleftrightarrow \mathbf{h}$ and $g \longleftrightarrow \mathbf{g}$ then

$$f := h \circ g \longleftrightarrow \mathbf{f} := \mathbf{h} \circ \mathbf{g}$$

defined as

$$(\mathbf{h} \circ \mathbf{g})(A) := \mathbf{h}(\mathbf{g}(A)) = \sum_{i=1}^{s} h(g(\lambda_i))\, E_i$$

# A polynomial in a square matrix

Let $A \in \mathbb{C}^{n \times n}$ have $s$ distinct eigenvalues, and Jordan form ($J_i$ all blocks corresponding to the $i_{\text{th}}$ eigenvalue),

$$A = XJX^{-1} = (X_1\, X_2\, \cdots\, X_s) \begin{bmatrix} J_1 & O & \cdots & O \\ O & J_2 & \cdots & O \\ O & & \ddots & \vdots \\ O & O & \cdots & J_s \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_s \end{bmatrix}$$

Then for any polynomial $p(\lambda)$,

$$p(A) = (X_1\, X_2\, \cdots\, X_s) \begin{bmatrix} p(J_1) & O & \cdots & O \\ O & p(J_2) & \cdots & O \\ O & & \ddots & \vdots \\ O & O & \cdots & p(J_s) \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_s \end{bmatrix}$$

# A power of a $k \times k$ Jordan block

$$(J_k(\lambda))^m = \begin{bmatrix} \lambda^m & m\lambda^{m-1} & \binom{m}{2}\lambda^{m-2} & \cdots & \binom{m}{k-1}\lambda^{m-k+1} \\ 0 & \lambda^m & m\lambda^{m-1} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \binom{m}{2}\lambda^{m-2} \\ \vdots & & & \lambda^m & m\lambda^{m-1} \\ 0 & \cdots & \cdots & 0 & \lambda^m \end{bmatrix}$$

$$= \sum_{j=0}^{k-1} \binom{m}{j} \lambda^{m-j} (J_k(\lambda) - \lambda I_k)^j$$

$$= \sum_{j=0}^{k-1} \frac{p^{(j)}(\lambda)}{j!} (J_k(\lambda) - \lambda I_k)^j ,$$

where $p(\lambda) = \lambda^m$, $\binom{m}{\ell}$ is interpreted as zero if $m < \ell$.

# A polynomial in a Jordan block

Let $p(\lambda)$ be a polynomial. Then

$$p(J_k(\lambda)) := \begin{bmatrix} p(\lambda) & \frac{1}{1!}p'(\lambda) & \frac{1}{2!}p''(\lambda) & \cdots & \frac{1}{(k-1)!}p^{(k-1)}(\lambda) \\ 0 & p(\lambda) & \frac{1}{1!}p'(\lambda) & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{2!}p''(\lambda) \\ \vdots & & & p(\lambda) & \frac{1}{1!}p'(\lambda) \\ 0 & \cdots & \cdots & 0 & p(\lambda) \end{bmatrix}$$

or

$$p(J_k(\lambda)) = \sum_{j=0}^{k-1} \frac{p^{(j)}(\lambda)}{j!} \left(J_k(\lambda) - \lambda\, I_k\right)^j \; .$$

## Spectral theorem for square matrices

**Theorem**. Let the matrix $A \in \mathbb{C}^{n\times n}$ have $s$ distinct eigenvalues. Then there exist $s$ unique projectors $\{E_\lambda : \lambda \in \lambda(A)\}$ such that

$$E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda \ , \tag{1}$$

$$I_n = \sum_{\lambda\in\lambda(A)} E_\lambda \ , \tag{2}$$

$$A = \sum_{\lambda\in\lambda(A)} \lambda\, E_\lambda + \sum_{\lambda\in\lambda(A)} (A - \lambda I)\, E_\lambda \ , \tag{3}$$

$$AE_\lambda = E_\lambda A \ , \ \text{for all } \lambda \in \lambda(A) \ , \tag{4}$$

$$E_\lambda (A - \lambda I)^k = O \ , \ \text{for all } \lambda \in \lambda(A) \ , \ k \geq \nu(\lambda) \ . \tag{5}$$

where $\nu(\lambda)$ denote the index of $(A - \lambda\, I)$, called the **index** of $\lambda$.

# Matrix functions on a Jordan block

Let $\lambda \in \lambda(A)$, and let $\nu(\lambda)$ denote the index of $(A - \lambda I)$, called the **index** of $\lambda$.

Let $J$ be a $\nu \times \nu$ Jordan block corresponding to $\lambda$. The matrix function $\mathbf{f}$ ($\longleftrightarrow$ a scalar function $f$) is defined on $J$ as follows

$$\mathbf{f}(J) = E_\lambda \sum_{k=0}^{\nu-1} \frac{f^{(k)}(\lambda)}{k!}(J - \lambda I_\nu)^k$$

$$= E_\lambda \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \cdots & \frac{f^{(\nu-1)}(\lambda)}{(\nu-1)!} \\ \vdots & f(\lambda) & f'(\lambda) & & \vdots \\ \vdots & & \ddots & \ddots & \frac{f''(\lambda)}{2!} \\ \vdots & & & f(\lambda) & f'(\lambda) \\ 0 & \cdots & \cdots & \cdots & f(\lambda) \end{bmatrix}$$

# Matrix functions

For any $A \in \mathbb{C}^{n \times n}$ with spectrum $\lambda(A)$, let $\mathcal{F}(A)$ denote the class of all functions $f : \mathbb{C} \to \mathbb{C}$ which are analytic in some open set containing $\lambda(A)$. For any scalar function $f \in \mathcal{F}(A)$, the **corresponding matrix function** $\mathbf{f}(A)$ is defined by

$$\mathbf{f}(A) = \sum_{\lambda \in \lambda(A)} E_\lambda \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} \left(A - \lambda\, I_n\right)^k . \tag{1}$$

# Matrix functions

For any $A \in \mathbb{C}^{n \times n}$ with spectrum $\lambda(A)$, let $\mathcal{F}(A)$ denote the class of all functions $f : \mathbb{C} \to \mathbb{C}$ which are analytic in some open set containing $\lambda(A)$. For any scalar function $f \in \mathcal{F}(A)$, the **corresponding matrix function** $\mathbf{f}(A)$ is defined by

$$\mathbf{f}(A) = \sum_{\lambda \in \lambda(A)} E_\lambda \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} \, (A - \lambda \, I_n)^k \, . \tag{1}$$

$\mathbf{f}(A)$ can be evaluated from the Jordan form of $A$

$$A = X \operatorname{diag}(J_1, J_2, \cdots) \, X^{-1} \, ,$$

as

$$\mathbf{f}(A) = X \operatorname{diag}(\mathbf{f}(J_1), \mathbf{f}(J_2), \cdots) \, X^{-1} \, .$$

# The Drazin inverse is the reciprocal function

For $A \in \mathbb{C}^{n \times n}$ with spectrum $\lambda(A)$, and a scalar function $f \in \mathcal{F}(A)$, the corresponding matrix function $f(A)$ is

$$\mathbf{f}(A) = \sum_{\lambda \in \lambda(A)} E_\lambda \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} \left(A - \lambda\, I_n\right)^k \,, \tag{1}$$

# The Drazin inverse is the reciprocal function

For $A \in \mathbb{C}^{n\times n}$ with spectrum $\lambda(A)$, and a scalar function $f \in \mathcal{F}(A)$, the corresponding matrix function $f(A)$ is

$$\mathbf{f}(A) = \sum_{\lambda \in \lambda(A)} E_\lambda \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} (A - \lambda I_n)^k , \tag{1}$$

The analogous result for the Drazin inverse is:

**Theorem**. Let $A \in \mathbb{C}^{n\times n}$

$$A^D = \sum_{0\neq\lambda \in \lambda(A)} E_\lambda \sum_{k=0}^{\nu(\lambda)-1} \frac{(-1)^k}{\lambda^{k+1}} (A - \lambda I_n)^k . \tag{2}$$

This shows that the Drazin inverse is the matrix function corresponding to the **reciprocal** $f(z) = 1/z$, **defined on nonzero eigenvalues**.

# The system $\mathbf{x}' + A\mathbf{x} = \mathbf{f}$

The **general solution** of $\mathbf{x}' + A\mathbf{x} = \mathbf{f}$

is
$$\mathbf{x} = e^{-At}\left(\int e^{At}\,\mathbf{f}(t)dt\right)$$

where
$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^k\frac{t^k}{k!} + \cdots$$

If $A$ is **nonsingular** then
$$\int e^{At}\,dt = A^{-1}\,e^{At} + C$$

where $C$ is arbitrary (the constant of integration.)

If $A$ is **singular**, of **index** $k$, then
$$\int e^{At}\,dt = A^D\,e^{At} + (I - AA^D)\left[tI + A\frac{t^2}{2!} + A^2\frac{t^3}{3!} + \cdots + A^{k-1}\frac{t^k}{k!}\right] + C$$

# The system $\mathbf{x}' + A\mathbf{x} = \mathbf{f}$ (cont'd)

If $\mathbf{f}$ is a constant vector then

$$\mathbf{x}' + A\mathbf{x} = \mathbf{f}$$

has a particular solution that is a polynomial in $t$. The general solution is

$$\mathbf{x} = e^{-At}\left(\int e^{At}\,dt\right)\mathbf{f} + C$$

and a particular solution is

$$\mathbf{x} = A^D\mathbf{f} + (I - AA^D)\left[tI - A\frac{t^2}{2!} + A^2\frac{t^3}{3!} - \cdots (-1)^{k-1}A^{k-1}\frac{t^k}{k!}\right]\mathbf{f}$$

**Ex.**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{aligned} x_1 &= 1 \\ x_2 &= t - t^2/2 \\ x_3 &= t \end{aligned}$$

# The system $A\mathbf{x}' + Bx = \mathbf{f}$ , $AB = BA$

Let $A$ be **singular**, $B$ **commute** with $A$, and consider

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{f} \tag{1}$$

and the corresponding **homogeneous equation**

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{0} \tag{2}$$

# The system $A\mathbf{x}' + Bx = \mathbf{f}$ , $AB = BA$

Let $A$ be **singular**, $B$ **commute** with $A$, and consider

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{f} \tag{1}$$

and the corresponding **homogeneous equation**

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{0} \tag{2}$$

Let $A = C + N$ be the **Wedderburn decomposition** of $A$, with $C$ of index 0 or 1, $N$ nilpotent of index $k$, $CN = NC = O$ and

$$A^D = C^D \ , \ N = A(I - A^D A) \ , \ C = A^2 A^D \ .$$

## The system $A\mathbf{x}' + Bx = \mathbf{f}$ , $AB = BA$

Let $A$ be **singular**, $B$ **commute** with $A$, and consider

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{f} \tag{1}$$

and the corresponding **homogeneous equation**

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{0} \tag{2}$$

Let $A = C + N$ be the **Wedderburn decomposition** of $A$, with $C$ of index 0 or 1, $N$ nilpotent of index $k$, $CN = NC = O$ and

$$A^D = C^D \ , \ N = A(I - A^D A) \ , \ C = A^2 A^D \ .$$

If $AB = BA$ then

$$AB^D = B^D A \ , \ A^D B = BA^D \ , \text{ and } A^D B^D = B^D A^D \ .$$

## The system $A\mathbf{x}' + Bx = \mathbf{f}$ , $AB = BA$ (cont'd)

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{f} \tag{1}$$

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{0} \tag{2}$$

**Theorem**. If $AB = BA$ then

$$\mathbf{y} = e^{-A^D B t} A^D A \mathbf{v} \ ,$$

is a solution of (2) for any vector $\mathbf{v}$.

**Proof**.
$A\mathbf{y}' = -AA^D B e^{-A^D B t} A^D A \mathbf{v} = -B e^{-A^D B t} A^D A \mathbf{v} = -B\mathbf{y}$ . $\quad \square$

## The system $A\mathbf{x}' + Bx = \mathbf{f}$ , $AB = BA$ (cont'd)

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{f} \tag{1}$$

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{0} \tag{2}$$

**Theorem**. If $AB = BA$ then

$$\mathbf{y} = e^{-A^D Bt} A^D A\mathbf{v} \ ,$$

is a solution of (2) for any vector $\mathbf{v}$.

**Proof**.
$A\mathbf{y}' = -AA^D Be^{-A^D Bt} A^D A\mathbf{v} = -Be^{-A^D Bt} A^D A\mathbf{v} = -B\mathbf{y}$ . $\square$

Let

$$\mathbf{x}_1 = A^D A\mathbf{x} \ , \ \mathbf{x}_2 := (I - A^D A)\,\mathbf{x} \tag{3}$$

Then (1) becomes

$$(C + N)(\mathbf{x}_1' + \mathbf{x}_2') + B(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f} \tag{4}$$

## The system $A\mathbf{x}' + Bx = \mathbf{f}$ , $AB = BA$ (cont'd)

The story so far

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{f} \tag{1}$$

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{0} \tag{2}$$

$$\mathbf{x}_1 = A^D A\mathbf{x} \ , \ \mathbf{x}_2 = (I - A^D A)\mathbf{x} \tag{3}$$

$$(C + N)(\mathbf{x}_1' + \mathbf{x}_2') + B(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f} \tag{4}$$

## The system $A\mathbf{x}' + Bx = \mathbf{f}$ , $AB = BA$ (cont'd)

The story so far

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{f} \tag{1}$$

$$A\mathbf{x}' + B\mathbf{x} = \mathbf{0} \tag{2}$$

$$\mathbf{x}_1 = A^D A\mathbf{x} \, , \; \mathbf{x}_2 = (I - A^D A)\mathbf{x} \tag{3}$$

$$(C + N)(\mathbf{x}_1' + \mathbf{x}_2') + B(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f} \tag{4}$$

Multiplying (4) by $C^D C$ and then by $(I - C^D C)$ it follows that (2) is equivalent to the system

$$C\mathbf{x}_1' + B\mathbf{x}_1 = \mathbf{f}_1 \tag{5}$$

$$N\mathbf{x}_2' + B\mathbf{x}_2 = \mathbf{f}_2 \tag{6}$$

where $\mathbf{f}_1 = C^D C\mathbf{f}$ and $\mathbf{f}_2 = (I - C^D C)\mathbf{f}$, etc.