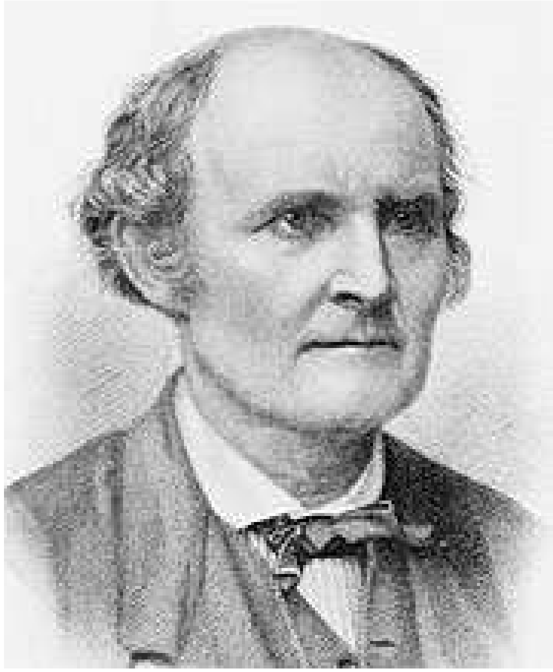
Lecture 6: The Group Inverse



The matrix index

Let $A \in \mathbb{C}^{n \times n}$, k positive integer. Then $R(A^{k+1}) \subset R(A^k)$. The **index** of A, denoted $\operatorname{Ind} A$, is the smallest integer k such that

$$R(A^k) = R(A^{k+1}) \ , \ \text{or equivalently} \ , \ \operatorname{rank} A^k = \operatorname{rank} A^{k+1} \qquad (1)$$

holds. Then

$$R(A^j) = R(A^{\operatorname{Ind} A}) \ , \ \forall \ j > \operatorname{Ind} A \ .$$

A matrix A is **range Hermitian** (or **EP matrix**) if

$$R(A) = R(A^*)$$

Special cases: (a) Nonsingular matrices, (b) normal matrices, in particular (c) $A = O$.

If A is range Hermitian, then $\operatorname{Ind} A = 1$.

The converse is not true, e.g., $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Index and complementary subspaces

Theorem. Let $A \in \mathbb{C}^{n\times n}$ have **index** k, and let ℓ be a positive integer. Then $R(A^\ell)$ and $N(A^\ell)$ are **complementary subspaces** if and only if $\ell \geq k$.

Proof. The theorem is obvious for nonsingular matrices.

Let $A \in \mathbb{C}^{n\times n}$. Then, for any positive integer ℓ,

$$\dim R(A^\ell) + \dim N(A^\ell) = \operatorname{rank} A^\ell + \operatorname{nullity} A^\ell = n \ .$$

$$\therefore \ \mathbb{C}^n = R(A^\ell) \oplus N(A^\ell) \iff R(A^\ell) \cap N(A^\ell) = \{\mathbf{0}\} \ .$$

From

$$R(A^{\ell+1}) \subset R(A^\ell) \text{ and } N(A^\ell) \subset N(A^{\ell+1}) \ ,$$

it follows that

$$\mathbb{C}^n = R(A^\ell) \oplus N(A^\ell) \iff R(A^\ell) = \dim R(A^{\ell+1}) \ . \ \square$$

Nilpotent matrices

A matrix N is **nilpotent** if $N^k = O$ for some integer $k \geq 0$. The smallest such k is called the **index of nilpotency** of N.

Let $J_k(\lambda)$ be a $k \times k$ Jordan block,

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & \cdots & \cdots & 0 \\ \vdots & \lambda & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \lambda & 1 \\ 0 & \cdots & \cdots & \cdots & \lambda \end{bmatrix}$$

Then $J_k(\lambda)$ is nilpotent if $\lambda = 0$,

$$\operatorname{ind} J_k(\lambda) = \begin{cases} 1, & \text{if } \lambda \neq 0; \\ k, & \text{if } \lambda = 0. \end{cases}$$

Jordan blocks

Let $A \in \mathbb{C}^{n \times n}$, $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_k \end{bmatrix} \in \mathbb{C}_k^{n \times k}$ and $\lambda \in \mathbb{C}$ satisfy

$$AX = X J_k(\lambda) \ , \quad \text{where} \quad J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix} \in \mathbb{C}^{k \times k} \ ,$$

$$\therefore \ A\mathbf{x}_1 = \lambda \, \mathbf{x}_1 \ ,$$

$$A\mathbf{x}_j = \lambda \, x_j + \mathbf{x}_{j-1} \ , \ j = 2, \ldots, k \ .$$

$$\therefore \ \forall \ j \in \overline{1, k} : \ (A - \lambda I)^j \, \mathbf{x}_j = \mathbf{0} \ , \quad (A - \lambda I)^{j-1} \, \mathbf{x}_j = \mathbf{x}_1 \neq \mathbf{0} \ ,$$

$\mathbf{x}_1$ is an **eigenvector** of A corresponding to λ

$\mathbf{x}_j$ is a λ–**vector** (**principal vector**, **generalized eigenvector**) of A of **grade** j

Powers of the matrix $(J_k(\lambda) - \lambda I) \in \mathbb{C}^{k\times k}$

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}, \; J_k(\lambda) - \lambda I = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

$$(J_k(\lambda) - \lambda I)^2 = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \; (J_k(\lambda) - \lambda I)^k = O \,.$$

The Jordan normal form

Theorem. Let $A \in \mathbb{C}^{n\times n}$. Then A is similar to a block diagonal matrix J with Jordan blocks on its diagonal, i.e. $\exists$ nonsingular $X \ni$

$$X^{-1}AX = J = \begin{bmatrix} J_{k_1}(\lambda_1) & O & \cdots & O \\ O & J_{k_2}(\lambda_2) & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & J_{k_p}(\lambda_p) \end{bmatrix} . \qquad (1)$$

The matrix J, called the **Jordan normal form** of A, is unique up to a rearrangement of its blocks. □

The scalars $\{\lambda_1, \ldots, \lambda_p\}$ in (1) are the eigenvalues of A. The set of eigenvalues, or **spectrum** of A, is denoted $\Lambda(A)$.

For A as above,

$$(A - \lambda_1 I)^{k_1}(A - \lambda_2 I)^{k_2} \cdots (A - \lambda_p I)^{k_p} = O . \qquad (2)$$

Vanishing polynomials

The polynomial

$$c(z) = (z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - \lambda_p)^{k_p}$$

is the **characteristic polynomial** of A. An eigenvalue λ may be repeated, say

$$c(z) = p(z)(z - \lambda)^{k_1}(z - \lambda)^{k_1} \cdots (z - \lambda)^{k_m} \ , \ p(\lambda) \neq 0$$

The **algebraic multiplicity** of the eigenvalue λ is then

$$k_1 + k_2 + \cdots + k_m$$

and $\max\{k_i : i \in \overline{1, m}\}$ is the **geometric multiplicity**, or **index** of the eigenvalue λ, denoted $\nu(\lambda)$. The polynomial

$$m(z) = \prod_{\lambda \in \Lambda(A)} (z - \lambda)^{\nu(\lambda)}$$

is the **minimal polynomial** of A.

Spectral inverses

Let $A \in \mathbb{C}^{n\times n}$. Then X is an S–**inverse** of A if they share the property that, for every $\lambda \in \mathbb{C}$ and every vector $\mathbf{x}$,

$$\left\{ \begin{array}{l} \mathbf{x} \text{ is a } \lambda\text{–vector of } A \\ \text{of grade } p \end{array} \right\} \iff \left\{ \begin{array}{l} \mathbf{x} \text{ is a } \lambda^\dagger\text{–vector of } X \\ \text{of grade } p \end{array} \right\}$$

where $\lambda^\dagger = 0$ if $\lambda = 0$, and otherwise $1/\lambda$.

Ex. If $A \in \mathbb{C}^{n\times n}$ is **diagonable**,

$$A = PJP^{-1}\ ,\ \ J = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

then

$$X = PJ^\dagger P^{-1}\ ,\ \ J^\dagger = \operatorname{diag}(\lambda_1^\dagger, \lambda_2^\dagger, \cdots, \lambda_n^\dagger)$$

is an S–inverse of A. It is a $\{1, 2\}$–inverse, and **commutes** with A,

$$AX = XA\ .$$

The group inverse

Theorem. Let $A \in \mathbb{C}^{n\times n}$. Then the system

$$AXA = A\,, \tag{1}$$

$$XAX = X\,, \tag{2}$$

$$AX = XA\,, \tag{5}$$

has a solution X iff

$$\operatorname{Ind} A = 1$$

and the solution is unique. It is called the **group inverse**, or $\{1,2,5\}$–**inverse**, of A, and is denoted $A^{\#}$.

Proof.

$$X \in A\{1,2\} \iff AX = P_{R(A),N(X)},\; XA = P_{R(X),N(A)}$$

$$\therefore\; X \in A\{1,2,5\} \iff AX = P_{R(A),N(A)}$$

$$\mathbb{C}^n = R(A) \oplus N(A) \iff \operatorname{Ind} A = 1 \quad \square$$

The group inverse (cont'd)

Theorem. A is range–Hermitian if, and only if, $A^{\#} = A^{\dagger}$.

Proof. $A^{\dagger} = A^{(1,2)}_{R(A),N(A)}$, $A^{\#} = A^{(1,2)}_{R(A^*),N(A^*)}$. □

Theorem (Erdélyi). Let A have index 1 and Jordan form

$$A = PJP^{-1} ,$$

Then

$$A^{\#} = PJ^{\dagger}P^{-1} .$$

Proof. The relations (1) $AXA = A$, (2) $XAX = X$, and (5) $AX = XA$, are similarity invariants. Therefore

$$J^{\#} = P^{-1}A^{\#}P$$

and since $\operatorname{Ind} J = 1$,

$$J^{\#} = J^{\dagger} . \quad \square$$

The group inverse (cont'd)

Theorem (**Cline**). Let a square matrix A have the FRF

$$A = CR\ .$$

Then A has group inverse if and only if RC is nonsingular, and

$$A^{\#} = C(RC)^{-2}R\ .$$

Ex. Let $A \in \mathbb{C}^{n \times n}$. Then A has **index 1** if and only if the **limit**

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} A$$

exists, in which case

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} A = AA^{\#}\ .$$

Proof. A full rank factorization $A = CR$ gives

$$(\lambda I_n + A)^{-1} A = C(\lambda I_r + RC)^{-1} R\ ,\ \text{etc.}$$

Properties of the group inverse

(a) If A is nonsingular, $A^{\#} = A^{-1}$.

(b) $A^{\#\#} = A$.

(c) $A^{*\#} = A^{\#*}$.

(d) $A^{T\#} = A^{\#T}$.

(e) $(A^{\ell})^{\#} = (A^{\#})^{\ell}$ for every positive integer ℓ.

(f) Let A have index 1 and denote

$$A^{-j} := (A^{\#})^{j} \ , \ \forall \ j = 1, 2, \dots \ ,$$
$$A^{0} := AA^{\#} \ .$$

Then,

$$A^{\ell}A^{m} = A^{\ell+m} \ , \ \forall \ \ell, m \ .$$

The "powers" of A, positive, negative and zero, constitute an **Abelian group** under matrix multiplication ($\therefore$ "**group inverse**")

Lemma on λ–vectors

Lemma 1. Let $\mathbf{x}$ be a λ–vector of A with $\lambda \neq 0$. Then $\mathbf{x} \in R(A^\ell)$ where ℓ is an arbitrary positive integer.

Proof. Let $(A - \lambda I)^p \mathbf{x} = \mathbf{0}$ for some positive integer p, or, using the binomial expapsion,

$$\mathbf{x} = c_1 A\mathbf{x} + c_2 A^2 \mathbf{x} + \cdots + c_p A^p \mathbf{x} \ , \ \ c_i = (-1)^{i-1} \lambda^{-i} \binom{p}{i} \ . \qquad (1)$$

$$\begin{aligned} \therefore \ \ A\mathbf{x} &= c_1 A^2 \mathbf{x} + c_2 A^3 \mathbf{x} + \cdots + c_p A^{p+1} \mathbf{x} \ , \\ A^2 \mathbf{x} &= c_1 A^3 \mathbf{x} + c_2 A^4 \mathbf{x} + \cdots + c_p A^{p+2} \mathbf{x} \ , \\ \cdots &= \cdots\cdots\cdots\cdots \qquad (2) \\ A^{\ell-1} \mathbf{x} &= c_1 A^\ell \mathbf{x} + c_2 A^{\ell+1} \mathbf{x} + \cdots + c_p A^{p+\ell-1} \mathbf{x} \ , \end{aligned}$$

Successive substitutions of (2) in (1) give

$$\mathbf{x} = A^\ell q(A) \mathbf{x} \ , \ \text{ where } q \text{ is some polynomial.}$$

Lemma on spectral inverses

Lemma 2. Let A be a square matrix and let

$$XA^{\ell+1} = A^{\ell} \tag{1}$$

for some positive integer ℓ. Then every λ–vector of A of grade p for $\lambda \neq 0$ is a λ^{-1}–vector of X of grade p.

Proof by induction on the grade p.

$\underline{p = 1}$. Let $A\mathbf{x} = \lambda\mathbf{x}$, $\lambda \neq 0$. Then

$$A^{\ell+1}\mathbf{x} = \lambda^{\ell+1}\mathbf{x}$$

$$\therefore\ \mathbf{x} = \lambda^{-\ell-1}A^{\ell+1}\mathbf{x}$$

$$\therefore\ X\mathbf{x} = \lambda^{-\ell-1}XA^{\ell+1}\mathbf{x} = \lambda^{-1}\mathbf{x}\ .$$

$\underline{\text{Assume true for } p \in \overline{1, r}}$. Let $\mathbf{x}$ be a λ–vector of A of grade $r + 1$.

Lemma 2 (cont'd)

Then, by Lemma 1, $\mathbf{x} = A^\ell \mathbf{y}$, for some $\mathbf{y}$.

$$\begin{aligned} \therefore\ (X - \lambda^{-1} I)\,\mathbf{x} &= (X - \lambda^{-1} I) A^\ell\,\mathbf{y} = X(A^\ell - \lambda^{-1} A^{\ell+1})\,\mathbf{y} \\ &= X(I - \lambda^{-1} A) A^\ell\,\mathbf{y} = -\lambda^{-1} X (A - \lambda I)\,\mathbf{x}\ . \end{aligned}$$

By the induction hypothesis, $(A - \lambda I)\,\mathbf{x}$ is a λ^{-1}–vector of X of grade r.

$$\therefore\ (X - \lambda^{-1} I)^r (A - \lambda I)\,\mathbf{x} = \mathbf{0}\ ,$$

$$\mathbf{z} = (X - \lambda^{-1} I)^{r-1} (A - \lambda I)\,\mathbf{x} \neq \mathbf{0}$$

$$X\mathbf{z} = \lambda^{-1}\mathbf{z}\ .$$

$$\therefore\ (X - \lambda^{-1} I)^{r+1}\,\mathbf{x} = -\lambda^{-1} X (X - \lambda^{-1} I)^r (A - \lambda I)\,\mathbf{x} = \mathbf{0}\ ,$$

$$(X - \lambda^{-1} I)^r\,\mathbf{x} = -\lambda^{-1} X\,\mathbf{z} = -\lambda^{-2}\mathbf{z} \neq \mathbf{0}\ .$$

$\therefore$ $\mathbf{x}$ is a λ^{-1}–vector of X of grade $r + 1$. $\square$

The group inverse is spectral

Recall: Let $A \in \mathbb{C}^{n\times n}$. Then X is an S–**inverse** of A if they share the property that, for every $\lambda \in \mathbb{C}$ and every vector $\mathbf{x}$,

$$\left\{ \begin{array}{l} \mathbf{x} \text{ is a } \lambda\text{–vector of } A \\ \text{of grade } p \end{array} \right\} \iff \left\{ \begin{array}{l} \mathbf{x} \text{ is a } \lambda^{\dagger}\text{–vector of } X \\ \text{of grade } p \end{array} \right\}$$

where $\lambda^{\dagger} = 0$ if $\lambda = 0$, and otherwise $1/\lambda$.

Theorem. Let $A \in \mathbb{C}^{n\times n}$ have index 1. Then $A^{\#}$ is the unique S–inverse of A in $A\{1\} \cup A\{2\}$. If A is diagonable, $A^{\#}$ is the only S–inverse of A.

Proof. <u>$A^{\#}$ is an S–inverse of A</u>. Since $X = A^{\#}$ satisfies $XA^{\ell+1} = A^{\ell}$ with $\ell = 1$, it follows from Lemma 2 that $A^{\#}$ satisfies the $\Longrightarrow$ part of the definition of S–inverse for $\lambda \neq 0$. Replacing A by $A^{\#}$ establishes the $\Longleftarrow$ part for $\lambda \neq 0$, since $A^{\#\#} = A$.

The rest of the proof omitted.

Application to finite Markov chains

A **system** is observed at times $t = 0, 1, 2, \ldots$. The system has N **states**, denoted $\{1, 2, \ldots, N\}$, the **state at time** t is denoted $\mathbf{X}_t$

The system is a **(finite) Markov chain** (or **chain**) if $\exists$ a matrix $P = (p_{ij}) \in \mathbb{R}^{N \times N}$ such that

$$\text{Prob}\{\mathbf{X}_{t+1} = j | \mathbf{X}_t = i\} = p_{ij} \ , \quad \forall\, i, j \in \overline{1, N} \ , \ \forall\, t = 0, 1, \ldots \ .$$

The numbers p_{ij} are called the **transition probabilities**, and the matrix P is called the **transition matrix**.

$$\therefore \ \sum_{j=1}^{N} p_{ij} = 1 \ , \quad i \in \overline{1, N} \ , \tag{a}$$

$$p_{ij} \geq 0 \ , \quad i, j \in \overline{1, N} \ . \tag{b}$$

A square matrix $P = [p_{ij}]$ satisfying (a),(b) is called **stochastic**.

Condition (a) is $P\mathbf{e} = \mathbf{e}$, i.e., $1 \in \Lambda(P)$.

Markov chains (cont'd)

Let $p_{ij}^{(n)}$ denote the n–**step transition probability**

$$p_{ij}^{(n)} = \text{Prob}\left\{\mathbf{X}_{t+n} = j \,|\, \mathbf{X}_t = i\right\} \,,\; p_{ij}^{(0)} := \delta_{ij} \;.$$

The **Chapman–Kolmogorov equations**

$$p_{ij}^{(m+n)} = \sum_{k=1}^{N} p_{ik}^{(m)}\, p_{kj}^{(n)} \;,\; \forall\, m, n \in \mathbb{Z}_+ \;. \tag{1}$$

$\therefore\; p_{ij}^{(n)}$ is the (i, j)–th element of P^n .

More terminology:

(a) A state i **leads** to state j, denoted by $i \rightsquigarrow j$, if $p_{ij}^{(n)} > 0$ for some $n \geq 1$.

(b) Two states i, j **communicate**, denoted by $i \leftrightsquigarrow j$ if each state leads to the other.

(c) A set S of states is **closed** if $i, j \in S \implies i \leftrightsquigarrow j$

Terminology (cont'd)

(d) A chain is **irreducible** if its only closed set is the set of all states $\overline{1,N}$, and is **reducible** otherwise.

(e) A single state forming a closed set is an **absorbing state**. A reducible chain is **absorbing** if each of its closed sets consists of a single (necessarily absorbing) state.

(f) A chain is **regular** if for some k, P^k is a positive matrix.

(g) A state i has **period** τ if $p_{ii}^{(n)} = 0$ except when $n = \tau, 2\tau, 3\tau, \ldots$. The period of i is denoted $\tau(i)$. If $\tau(i) = 1$, i is **aperiodic**.

(h) Let $f_{ij}^{(n)}$ = **probability that starting from i, state j is reached for 1st time at step** n, and let

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} \quad \textbf{probability that } j \textbf{ eventually reached from } i\,,$$

$$\mu_{ij} = \sum_{n=1}^{\infty} n\, f_{ij}^{(n)} \quad \textbf{the mean 1st passage time from } i \textbf{ to } j\,.$$

Terminology (cont'd)

(h) $\mu_{kk} := \sum_{n=1}^{\infty} n\, f_{kk}^{(n)}$ is the **mean return time of state** k.

(i) A state i is **recurrent** if $f_{ii} = 1$, and is **transient** otherwise. A chain is **recurrent** if all its states are recurrent.

(j) If i is recurrent and $\mu_{ii} = \infty$, i is called a **null state**. All states in a closed set are of the same type.

(k) A state is **ergodic** if **recurrent** and **aperiodic**, but not a **null state**. If a chain is irreducible, all its states have the same period. An irreducible aperiodic chain is called **ergodic**.

Theorem (**Chung**). For any states i, j of an irreducible recurrent chain

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ij}^{(k)} = \frac{1}{\mu_{jj}} \,. \tag{1}$$

Proof. Both sides of (1) give the expected number of returns to state j in unit time.

Stationary probabilities

The probabilities $\{\pi_1, \pi_2, \ldots, \pi_N\}$ are **stationary** if

$$\pi_k = \sum_{i=1}^{N} p_{ik}\, \pi_i \;, \quad k \in \overline{1, N} \;,$$

or, in terms of vector of stationary probabilities,

$$P^T \boldsymbol{\pi} = \boldsymbol{\pi} \;.$$

$\boldsymbol{\pi}$ is called a **stationary distribution** or **steady state**. In an irreducible recurrent chain, the stationary distribution is simply,

$$\pi_k = \frac{1}{\mu_{kk}} \;, \; k \in \overline{1, N} \;.$$

If a chain is **ergodic**, the system converges to its stationary distribution from **any** initial distribution.

In terms of P, every row of P^n converges to $\boldsymbol{\pi}^T$.

Summary for ergodic chains

Theorem (**Feller**). Let $P \in \mathbb{R}^{N \times N}$ be the transition matrix of an ergodic chain. Then:

(a) $\forall\ j, k \in \overline{1, N}$ the following limit exists, independent of j,

$$\lim_{n \to \infty} p_{jk}^{(n)} = \pi_k > 0 \tag{1}$$

(b) $\pi_k = \dfrac{1}{\mu_{kk}}$

(c) The numbers $\{\pi_k \,:\, k \in \overline{1, N}\}$ are probabilities,

$$\pi_k \geq 0 \ , \ \sum_{k=1}^{N} \pi_k = 1 \ , \tag{2}$$

and are a stationary distribution of the chain:

$$\pi_k = \sum_{i=1}^{N} p_{ik}\, \pi_i \ , \quad k \in \overline{1, N} \ . \tag{3}$$

The distribution $\{\pi_k\}$ is uniquely determined by (2) and (3). $\square$

Solution using the group inverse

Theorem (Meyer). Let P be the transition matrix of a **finite Markov chain**, and let $Q = I - P$.

Then Q has a **group inverse**, and:

$$I - QQ^{\#} = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P^k \; , \qquad (1)$$

$$= \lim_{n\to\infty} \left(\alpha I + (1-\alpha)P\right)^n \; , \text{ for any } 0 < \alpha < 1 \; , \qquad (2)$$

and if the chain is **regular**, or **absorbing**,

$$I - QQ^{\#} = \lim_{n\to\infty} P^n \; . \qquad (3)$$

For proofs and other details see original work.