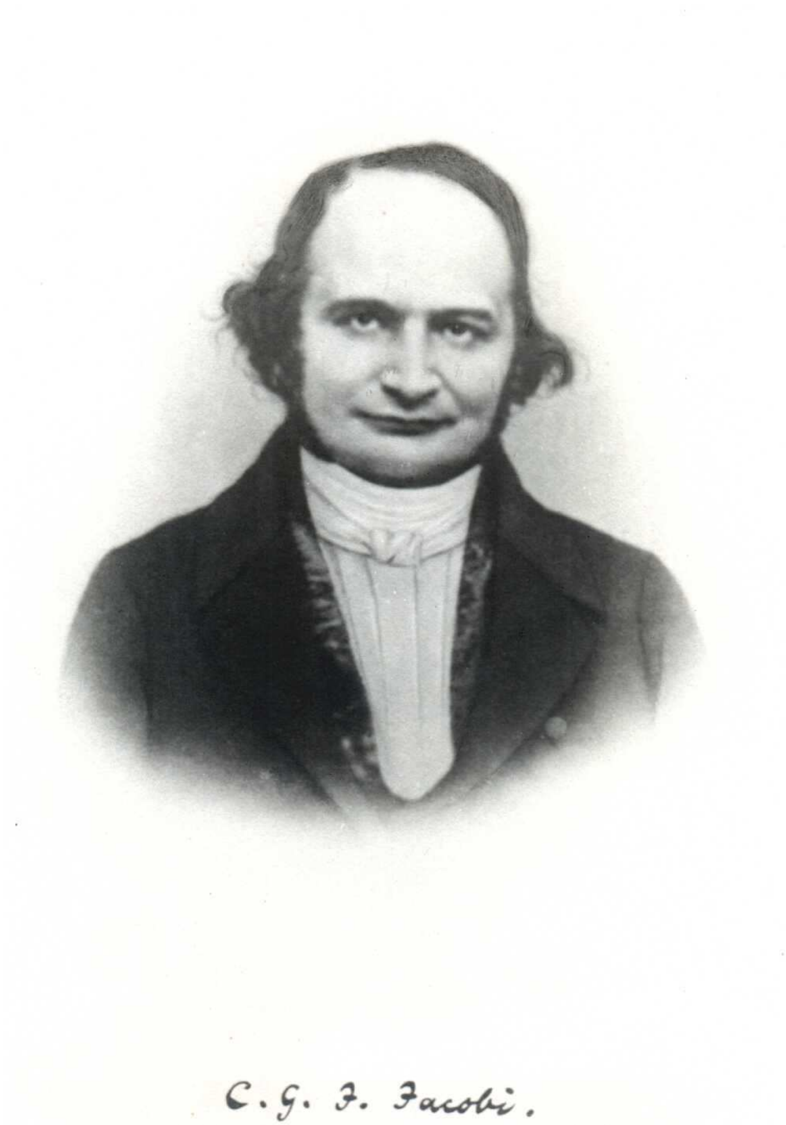
Lecture 5: Matrix Volume



Index sets of maximal full–rank submatrices

Given $A \in \mathbb{C}_r^{m\times n}$, we denote

$$\mathcal{I}(A) = \{I \in Q_{r,m} : \ \mathrm{rank}\, A[I,*] = r\} \ ,$$

$$\mathcal{J}(A) = \{J \in Q_{r,n} : \ \mathrm{rank}\, A[*,J] = r\} \ ,$$

$$\mathcal{N}(A) = \{(I,J) : \ I \in Q_{r,m}, J \in Q_{r,n}, \ A[I,J] \text{ is nonsingular}\} \ ,$$

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If

$$A = CR$$

is a **full rank factorization**, then

$$\mathcal{I}(A) = \mathcal{I}(C)$$
$$\mathcal{J}(A) = \mathcal{J}(R)$$
$$\mathcal{N}(A) = \mathcal{I}(A) \times \mathcal{J}(A)$$

Note: $I \in \mathcal{I}(A)$ and $J \in \mathcal{J}(A) \implies A[I,J]$ nonsingular (!?)

The Gramian

The **Gram matrix** of a set $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ is the $k \times k$ matrix $G(S)$, or $G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$, of **inner products**

$$G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)[i, j] := \langle \mathbf{x}_i, \mathbf{x}_j \rangle \ .$$

$\det G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$ is the **Gramian** of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$.

If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ are columns of $X \in \mathbb{R}^{n \times k}$, then

$$\det G(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \det X^T X = \sum_{I \in Q_{k,n}} \det{}^2 X_{I*} \ , \ \text{(by C–B)}$$

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(a) The set S is **linearly dependent** iff $G(S) = 0$.

(b) The **volume** of the **parallelepiped** generated by the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is

$$\mathrm{vol}\,\square\,\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} = \sqrt{\det G(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k)}$$

(in the subspace span $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$, not in $\mathbb{R}^n$ if $k < n$)

Volume

The **volume** of a matrix $A \in \mathbb{R}_r^{m \times n}$, denoted $\operatorname{vol} A$, is defined as 0 if $r = 0$, and otherwise

$$\operatorname{vol} A := \sqrt{\sum_{(I,J) \in \mathcal{N}(A)} \det^2 A_{IJ}}\ . \tag{1}$$

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(a) If $C \in \mathbb{R}_n^{m\times n}$ (**full row rank**) then $\operatorname{vol} C = \sqrt{\det C^T C}$.
(b) If $R \in \mathbb{R}_m^{m\times n}$ (**full column rank**) then $\operatorname{vol} R = \sqrt{\det RR^T}$.
(c) If $A = CR$ is a **full rank factorization** then

$$\operatorname{vol}^2(A) = \sum_{I\in\mathcal{I}(A)} \operatorname{vol}^2(A_{I*}) \tag{2}$$

$$= \sum_{J\in\mathcal{J}(A)} \operatorname{vol}^2(A_{*J}) \tag{3}$$

$$= \operatorname{vol}^2(C)\operatorname{vol}^2(R) \tag{4}$$

Proof. Each nonsingular A_{IJ} is $C_{I*}R_{*J},\ I \in \mathcal{I}(A), J \in \mathcal{J}(A)$.

Singular values

Theorem. The **volume** of $A \in \mathbb{R}_r^{m\times n}$ is the **product** of its **singular values** $\{\sigma_1, \cdots, \sigma_r\}$

$$\operatorname{vol} A = \prod_{\sigma_i \in \sigma(A)} \sigma_i \tag{1}$$

Proof. The SVD of A,

$$A = U\Sigma V^* = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_{11} & O \\ O & O \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} , \ \Sigma_{11} \in \mathbb{R}_r^{r\times r} ,$$

is a full rank factorization

$$A = U_1 \Sigma_{11} V_1^* \ , \ \text{with } U_1^* U_1 = V_1^* V_1 = I_r \ .$$

$$\therefore \ \operatorname{vol} A = \operatorname{vol} \Sigma_{11} = \prod_{\sigma_i \in \sigma(A)} \sigma_i$$

Why "volume"?

Let $\{\mathbf{v}_j : j \in \overline{1,r}\}$ be an o.n. basis of $R(A^*A) = R(A^*)$ such that

$$A^*A\mathbf{v}_j = \sigma_i^2 \mathbf{v}_j \ , \ j \in \overline{1,r} \ ,$$

Then an o.n. basis $\{\mathbf{u}_j : j \in \overline{1,r}\}$ of $R(A)$ is given by

$$A\mathbf{v}_j = \sigma_j \, \mathbf{u}_j$$

showing that A maps the **unit cube** $\Box\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ into the **cube** $\Box\{\sigma_1\mathbf{u}_1, \sigma_2\mathbf{u}_2, \cdots, \sigma_r\mathbf{u}_r\}$ of **volume** $\sigma_1\sigma_2\cdots\sigma_r$. But the singular values are **orthogonally invariant**, and therefore **every** unit cube in $\mathbb{R}(A^*)$ is mapped into a cube of volume $\sigma_1\sigma_2\cdots\sigma_r$.

Given $A \in \mathbb{R}^{m\times n}$, the **volume** of A is an **intrinsic property** of the **linear transformation** in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ represented by A.

LSS

Let $A \in \mathbb{R}_n^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$. Then the LSS of $A\mathbf{x} = \mathbf{b}$ is

$$A^\dagger \mathbf{b} = \sum_{I \in \mathcal{I}(A)} \lambda_I \, \widehat{A_{I*}^{-1} \mathbf{b}_I} \; , \quad \lambda_I = \frac{\det^2 A_{I*}}{\sum\limits_{K \in \mathcal{I}(A)} \det^2 A_{K*}}$$

Ex. $n = 1$.

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} x = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} .$$

The LSS is

$$x = \frac{\sum\limits_{i=1}^m a_i b_i}{\sum\limits_{k=1}^m a_k^2} = \sum_{i=1}^m \left(\frac{a_i^2}{\sum\limits_{k=1}^m a_k^2} \right) a_i^{-1} b_i = \sum_{i=1}^m \lambda_i \, a_i^{-1} b_i$$

This trivial example proves the general case.

MNLSS in 2 stages

For $A \in \mathbb{R}_r^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, the MNLSS of the linear equation

$$A\mathbf{x} = \mathbf{b} \ ,$$

is the solution of a **two-stage minimization problem**:

Stage 1: minimize $\|A\mathbf{x} - \mathbf{b}\|$ (1)

Stage 2: minimize $\|\mathbf{x}\|$ among all solutions of Stage 1 (2)

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Stage 2: minimize $\|\mathbf{x}\|$ among all solutions of Stage 1 (2)

Let

$$A = CR$$

be a **full rank factorization**. Then:

Stage 1: minimize $\|C\mathbf{y} - \mathbf{b}\|$ (1a)

Stage 2: minimize $\|\mathbf{x}\|$ among all solutions of $R\mathbf{x} = \mathbf{y}$ (2a)

The problem (1a), unlike (1), has a **unique solution $\mathbf{y} = C^\dagger\mathbf{b}$.**

Full rank systems

Lemma.

(a) Let $C \in \mathbb{R}_r^{m\times r}$, $\mathbf{b} \in \mathbb{R}^m$. Then the LSS of

$$C\mathbf{y} = \mathbf{b} \ , \tag{1}$$

is

$$\mathbf{y} = \sum_{I \in \mathcal{I}(C)} \mu_{I*} \, C_{I*}^{-1}\mathbf{b}_I \ , \quad \mu_{I*} = \frac{\mathrm{vol}^2 C_{I*}}{\mathrm{vol}^2 C} \ . \tag{2}$$

Full rank systems

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(b) Let $R \in \mathbb{R}_r^{r\times n}$, $\mathbf{y} \in \mathbb{R}^r$. Then the MNS of

$$R\mathbf{x} = \mathbf{y} \ , \tag{3}$$

is

$$\mathbf{x} = \sum_{J\in\mathcal{J}(R)} \nu_{*J}\ \widehat{R_{*J}^{-1}\mathbf{y}} \ , \quad \nu_{*J} = \frac{\mathrm{vol}^2 R_{*J}}{\mathrm{vol}^2 R} \ . \tag{4}$$

MNLSS as convex combination

Theorem (Berg). Let $A \in \mathbb{R}_r^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$. Then the MNLSS of

$$A\mathbf{x} = \mathbf{b} \ , \tag{1}$$

is the convex combination

$$\mathbf{x} = \sum_{(I,J) \in \mathcal{N}(A)} \lambda_{IJ} \, \widehat{A_{IJ}^{-1} \, \mathbf{b}_I} \ , \tag{2}$$

with weights given by

$$\lambda_{IJ} = \frac{\det^2 A_{IJ}}{\sum\limits_{(K,L) \in \mathcal{N}(A)} \det^2 A_{KL}} \ , \quad (I,J) \in \mathcal{N}(A) \ . \tag{3}$$

Proof. Let $A = CR$ be a full rank factorization, and use previous lemma and the fact

$$A_{IJ} = C_{I*} R_{*J} \ , \ \forall \ I \in \mathcal{I}(A), J \in \mathcal{J}(A) \ .$$

Weighted LS

Consider next a **weighted** least squares problem

$$\min \|D^{1/2}(A\mathbf{x} - \mathbf{b})\| \ , \ D = \mathrm{diag}\,(d_i) \ , \ d_i > 0 \ . \tag{1}$$

Theorem (**Ben–Tal and Teboulle**). The solutions of (1), i.e., the least–squares solutions of $D^{1/2}A\mathbf{x} = D^{1/2}\mathbf{b}$, satisfy the normal equation, $A^T DA\mathbf{x} = A^T D\mathbf{b}$. The **MN** (weighted) **LSS** is

$$\mathbf{x}(D) = \sum_{(I,J)\in\mathcal{N}(A)} \lambda_{IJ}(D)\, \widehat{A_{IJ}^{-1}\mathbf{b}_I} \ , \tag{2}$$

with weights

$$\lambda_{IJ}(D) = \frac{\left(\prod_{i\in I} d_i\right)\, \det^2 A_{IJ}}{\sum\limits_{(I,J)\in\mathcal{N}(A)} \left(\prod_{i\in K} d_i\right)\, \det^2 A_{KL}} \ . \tag{3}$$

Note: Only the weights λ_{IJ} depend on D.

Bordered matrices

Let $A \in \mathbb{R}_r^{m\times n}$, and let

$$U \in \mathbb{R}^{m\times(m-r)}\ ,\ \operatorname{cols} U = \text{o.n. basis}\, N(A^T)\ ,$$

$$V \in \mathbb{R}^{n\times(n-r)}\ ,\ \operatorname{cols} V = \text{o.n. basis}\, N(A)\ .$$

Consider the **bordered matrix**,

$$\mathbf{B}(A) := \begin{bmatrix} A & U \\ V^T & O \end{bmatrix}\ , \tag{1}$$

and if $m = n$, the **complemented matrix**,

$$\mathbf{C}(A) := A + UV^T\ . \tag{2}$$

Then

$$\operatorname{vol} A = |\det \mathbf{B}(A)|\ , \tag{3}$$

$$= |\det \mathbf{C}(A)|\ . \tag{4}$$

The change-of-variables formula in integration

Theorem (**Jacobi** general n, **Euler** $n = 2$, **Lagrange** $n = 3$)

$$\int_{\mathcal{V}} f(\mathbf{v})\, d\mathbf{v} = \int_{\mathcal{U}} (f \circ \phi)(\mathbf{u}) \; |\det J_\phi(\mathbf{u})| \; d\mathbf{u} \qquad (1)$$

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$\mathcal{U}, \mathcal{V}$ sets in $\mathbb{R}^n$,

$\phi : \mathcal{U} \to \mathcal{V}$ a sufficiently well-behaved function,

f is integrable on $\mathcal{V}$,

$d\mathbf{x}$ denotes the **volume element** $|\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|$,

J_ϕ is the **Jacobian matrix** (or **Jacobian**)

$$J_\phi \;:=\; \left(\frac{\partial \phi_i}{\partial u_j}\right) \;, \quad \text{also denoted} \quad \frac{\partial(v_1, v_2, \cdots, v_n)}{\partial(u_1, u_2, \cdots, u_n)} \;, \tag{2}$$

representing the **derivative** of ϕ.

An **advantage** of (1) is that integration on $\mathcal{V}$ is translated to (perhaps **simpler**) integration on $\mathcal{U}$.

The change-of-variable formula (cont'd)

$$\int_{\mathcal{V}} f(\mathbf{v})\, d\mathbf{v} \;=\; \int_{\mathcal{U}} (f \circ \phi)(\mathbf{u})\; |\det J_\phi(\mathbf{u})|\; d\mathbf{u} \tag{1}$$

If $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^m$ with $n > m$, formula (1) cannot be applied.

The change-of-variable formula (cont'd)

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If $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^m$ with $n > m$, formula (1) cannot be applied.

However, if the Jacobian J_ϕ is of **full column rank** throughout $\mathcal{U}$, we can replace $|\det J_\phi|$ by the **volume** $\operatorname{vol} J_\phi$ of the **Jacobian** to get

$$\int_{\mathcal{V}} f(\mathbf{v})\, d\mathbf{v} \;=\; \int_{\mathcal{U}} (f \circ \phi)(\mathbf{u}) \operatorname{vol} J_\phi(\mathbf{u})\, d\mathbf{u}\,. \tag{2}$$

Here,

$$\operatorname{vol} J_\phi = \sqrt{\det\left(J_\phi^T J_\phi\right)} \tag{3}$$

since J_ϕ is assumed of full rank, and (2) reduces to (1) if $m = n$.

Example in surface integration

Let $\mathcal{S}$ be a subset of a **surface** in $\mathbb{R}^3$ represented by

$$z = g(x, y) \ , \tag{1}$$

and let $f(x, y, z)$ be a function integrable on $\mathcal{S}$. Let $\mathcal{A}$ be the **projection** of $\mathcal{S}$ on the xy–plane. Then

$$\mathcal{S} = \phi(\mathcal{A}) \ , \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ g(x, y) \end{bmatrix} = \phi \begin{bmatrix} x \\ y \end{bmatrix} \ , \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{A} \ . \tag{2}$$

The Jacobi matrix of ϕ is the 3×2 matrix

$$J_\phi(x, y) = \frac{\partial(x, y, z)}{\partial(x, y)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{bmatrix} \ , \tag{3}$$

where $g_x = \dfrac{\partial g}{\partial x}$, $g_y = \dfrac{\partial g}{\partial y}$.

Example (cont'd)

$$\operatorname{vol} J_\phi = \operatorname{vol} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{bmatrix} = \sqrt{1 + g_x^2 + g_y^2}\ . \qquad (4)$$

$$\therefore \int_{\mathcal{S}} f(x, y, z)\, ds = \int_{\mathcal{A}} f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2}\; dx\, dy\ .$$

Example (cont'd)

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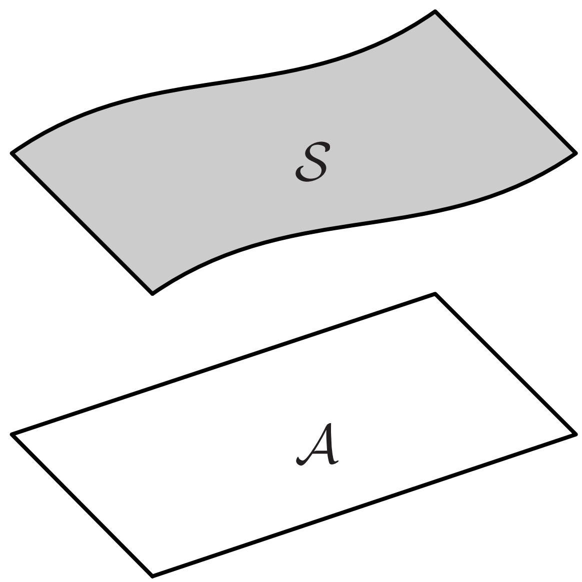

Cylindrical coordinates

Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$, represented by

$$z = z(r, \theta)$$

where r, θ are **cylindrical coordinates**,

$$\begin{aligned} x &= r\,\cos\theta \\ y &= r\,\sin\theta \\ z &= z(r, \theta) \end{aligned}$$

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The **Jacobi matrix** is

$$J_\phi = \frac{\partial(x, y, z)}{\partial(r, \theta)} = \begin{bmatrix} \cos\theta & -r\ \sin\theta \\ \sin\theta & r\ \cos\theta \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{bmatrix}$$

and its **volume** is

$$\operatorname{vol} J_\phi = \sqrt{r^2 + r^2 \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r \sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2}$$

Cylindrical coordinates (cont'd)

An integral over a domain $\mathcal{V} \subset \mathcal{S}$ is

$$\int_{\mathcal{V}} f(x, y, z)\, dV =$$

$$\int_{\mathcal{U}} f(r\ \cos\theta, r\ \sin\theta, z(r, \theta))\, \mathrm{vol} J_{\phi}(r, \theta)\, dr\, d\theta$$

Cylindrical coordinates (cont'd)

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$$\int_{\mathcal{U}} f(r\, \cos\theta, r\, \sin\theta, z(r, \theta))\, \mathrm{vol} J_{\phi}(r, \theta)\, dr\, d\theta$$

If $\mathcal{S}$ is **symmetric** about the z–axis, then

$$\frac{\partial z}{\partial \theta} = 0\ , \text{ i.e. } z = z(r),$$

and

$$\int_{\mathcal{V}} f(x, y, z)\, dV =$$

$$\int_{\mathcal{U}} f(r\, \cos\theta, r\, \sin\theta, z(r))\, r\, \sqrt{1 + z'(r)^2}\, dr\, d\theta$$

Surface integral in $\mathbb{R}^n$

Let $\mathbf{x} = (x_i)$, and $\mathbf{x}[\overline{1,k}] := \{(x_i) : i \in \overline{1,k}\}$.

A **surface** $\mathcal{S}$ in $\mathbb{R}^n$ is given by given by

$$x_n := g(x_1, \cdots, x_{n-1}) = g(\mathbf{x}[\overline{1,(n-1)}]) \ , \tag{1}$$

Let

$\mathcal{V} \subset \mathcal{S}$, f integrable on $\mathcal{V}$, and it is required to calculate

$$\int_{\mathcal{V}} f(\mathbf{x})\, dS \tag{2}$$

$\mathcal{U}$ the **projection** of $\mathcal{V}$ on $\mathbb{R}^{n-1}$, the space of variables $\mathbf{x}[1\!:\!(n-1)]$,
$\phi : \mathcal{U} \to \mathcal{V}$, the **mapping** given by its components
$\phi \ := \ (\phi_1, \phi_2, \ldots, \phi_n) \ ,$

$$\begin{aligned} \phi_i(\mathbf{x}[\overline{1,(n-1)}]) \quad &:= \quad x_i \ , \quad i = 1, \ldots, n-1 \\ \phi_n(\mathbf{x}[\overline{1,(n-1)}]) \quad &:= \quad g(\mathbf{x}[\overline{1,(n-1)}]) \end{aligned}$$

Surface integral in $\mathbb{R}^n$ (cont'd)

The Jacobi matrix of ϕ is

$$J_\phi = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} & \cdots & \dfrac{\partial g}{\partial x_{n-2}} & \dfrac{\partial g}{\partial x_{n-1}} \end{bmatrix}$$

and its volume is

$$\operatorname{vol} J_\phi = \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial g}{\partial x_i}\right)^2}$$

$$\therefore \int_{\mathcal{V}} f(\mathbf{x})\, dS = \int_{\mathcal{U}} f(\mathbf{x}[\overline{1,(n-1)}], g(\mathbf{x}[\overline{1,(n-1)}]))\operatorname{vol} J_\phi \, dx_1 \cdots dx_{n-1}$$

Radon transform

Let $\mathcal{H}_{\boldsymbol{\xi},p}$ be a hyperplane in $\mathbb{R}^n$ represented by

$$\mathcal{H}_{\boldsymbol{\xi},p} := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} \xi_i \, x_i \ = \ p \right\} = \{ \mathbf{x} : < \boldsymbol{\xi}, \mathbf{x} >= p \} \tag{1}$$

where the **normal vector** $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ of $\mathcal{H}_{\boldsymbol{\xi},p}$ has $\xi_n \neq 0$.

Then $\mathcal{H}_{\boldsymbol{\xi},p}$ is given as

$$\mathcal{H}_{\boldsymbol{\xi},p} = \phi(\mathbb{R}^{n-1}) \tag{2}$$

with

$$\phi_i(\mathbf{x}[\overline{1,(n-1)}]) := x_i \ , \ \ i \in \overline{1,(n-1)}$$

$$\phi_n(\mathbf{x}[\overline{1,(n-1)}]) := \frac{p}{\xi_n} - \sum_{i=1}^{n-1} \frac{\xi_i}{\xi_n} \, x_i \tag{3}$$

Radon transform (cont'd)

The volume of J_ϕ is here

$$\mathrm{vol}\, J_\phi \;=\; \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\xi_i}{\xi_n}\right)^2} \;=\; \frac{\|\boldsymbol{\xi}\|}{|\xi_n|} \tag{4}$$

The **Radon transform** $(\mathbf{R}f)(\boldsymbol{\xi}, p)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is its integral over the hyperplane $\mathcal{H}_{\boldsymbol{\xi},p}$,

$$(\mathbf{R}f)(\boldsymbol{\xi}, p) \;:=\; \int_{\{\mathbf{x}:\, <\boldsymbol{\xi},\mathbf{x}> \;=\; p\}} f(\mathbf{x})\, d\mathbf{x}\;. \tag{5}$$

The Radon transform can be computed as an integral in $\mathbb{R}^{n-1}$

$$(\mathbf{R}f)(\boldsymbol{\xi}, p) = \frac{\|\boldsymbol{\xi}\|}{|\xi_n|} \int_{\mathbb{R}^{n-1}} f\left(\mathbf{x}[\overline{1,(n-1)}], \frac{p}{\xi_n} - \sum_{i=1}^{n-1} \frac{\xi_i}{\xi_n}\, x_i\right) dx_1 \cdots dx_{n-1}\;. \tag{6}$$

Integrals over $\mathbb{R}^n$

Consider an integral over $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^n} f(x_1, x_2, \cdots, x_n)\, dx_1 \cdots dx_n \tag{1}$$

Since $\mathbb{R}^n$ is a **union** of (**parallel**) **hyperplanes**,

$$\mathbb{R}^n = \bigcup_{p=-\infty}^{\infty} \{\mathbf{x} : <\boldsymbol{\xi}, \mathbf{x}>= p\}\ , \quad \text{where } \boldsymbol{\xi} \neq \mathbf{0}\ , \tag{2}$$

Compute (1) iteratively: an integral over $\mathbb{R}^{n-1}$ (**Radon transform**), followed by an integral on $\mathbb{R}$,

$$\int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} = \int_{-\infty}^{\infty} \frac{dp}{\|\boldsymbol{\xi}\|}\, (\mathbf{R}f)(\boldsymbol{\xi}, p) \tag{3}$$

Integrals over $\mathbb{R}^n$ (cont'd)

$$\int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} = \int_{-\infty}^{\infty} \frac{dp}{\|\boldsymbol{\xi}\|} (\mathbf{R}f)(\boldsymbol{\xi}, p) \tag{3}$$

Here $dp/\|\boldsymbol{\xi}\|$ is the **differential of the distance** along $\boldsymbol{\xi}$ (i.e. dp times the distance between the parallel hyperplanes $\mathcal{H}_{\boldsymbol{\xi},p}$ and $\mathcal{H}_{\boldsymbol{\xi},p+1}$).

Then by the result for the Radon transform,

$$\int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} =$$

$$\frac{1}{|\xi_n|} \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^{n-1}} f\left(\mathbf{x}[\overline{1, (n-1)}], \frac{p}{\xi_n} - \sum_{i=1}^{n-1} \frac{\xi_i}{\xi_n} x_i \right) dx_1 \cdots dx_{n-1} \right) dp\ . \tag{4}$$

Application in Probability

Let $\mathbf{X} = (\mathbf{X}_1, \cdots, \mathbf{X}_n)$ have **joint density** $f_{\mathbf{X}}(x_1, \cdots, x_n)$ and let

$$y = h(x_1, \cdots, x_n) \qquad (1)$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is sufficiently well-behaved, in particular

$$\frac{\partial h}{\partial x_n} \neq 0,$$

and (1) can be solved for x_n,

$$x_n = h^{-1}(y|x_1, \cdots, x_{n-1}) \qquad (2)$$

with $x_1, \cdots, x_{n-1}$ as parameters.

It is required to find the **density** $f_{\mathbf{Y}}(y)$ of $\mathbf{Y}$.

Method 1

Change variables from $\{x_1, \cdots, x_n\}$ to $\{x_1, \cdots, x_{n-1}, y\}$, and use the fact

$$\det\left(\frac{\partial(x_1, \cdots, x_n)}{\partial(x_1, \cdots, x_{n-1}, y)}\right) = \frac{\partial h^{-1}}{\partial y} \tag{3}$$

to write the density of $\mathbf{Y} = h(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ as

$$f_{\mathbf{Y}}(y) = \tag{4}$$

$$\int\limits_{\mathbb{R}^{n-1}} f_{\mathbf{X}}(x_1, \cdots, x_{n-1}, h^{-1}(y|x_1, \cdots, x_{n-1})) \left|\frac{\partial h^{-1}}{\partial y}\right| dx_1 \cdots dx_{n-1}$$

Method 2

Let $\mathcal{V}(y)$ be the surface given by (1), represented as

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ h^{-1}(y|x_1, \cdots, x_{n-1}) \end{pmatrix} = \phi \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \tag{5}$$

Then the surface integral of $f_{\mathbf{X}}$ over $\mathcal{V}(y)$ is

$$\int\limits_{\mathcal{V}(y)} f_{\mathbf{X}} = \tag{6}$$

$$\int\limits_{\mathbb{R}^{n-1}} f_{\mathbf{X}}(x_1, \cdots, x_{n-1}, h^{-1}(y|\, x_1, \cdots, x_{n-1}))$$

$$\sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial h^{-1}}{\partial x_i} \right)^2} \, dx_1 \cdots dx_{n-1}$$

The density of $\mathbf{Y} = h(\mathbf{X}_1, \cdots, \mathbf{X}_n)$

Theorem. If the ratio

$$\frac{\frac{\partial h^{-1}}{\partial y}}{\sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial h^{-1}}{\partial x_i}\right)^2}} \quad \text{does not depend on } x_1, \cdots, x_{n-1} \,, \qquad (\mathbf{X})$$

then

$$f_{\mathbf{Y}}(y) = \frac{\left|\frac{\partial h^{-1}}{\partial y}\right|}{\sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial h^{-1}}{\partial x_i}\right)^2}} \int_{\mathcal{V}(y)} f_{\mathbf{X}}$$

Proof. Compare (4) and (6).

Hyperplanes

Condition (**X**) holds for hyperplanes. Let

$$y \;=\; h(x_1, \cdots, x_n) \;:=\; \sum_{i=1}^{n} \xi_i \, x_i$$

where $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ is a given vector with $\xi_n \neq 0$. Then (2) is

$$x_n = h^{-1}(y|\, x_1, \cdots, x_{n-1}) := \frac{y}{\xi_n} - \sum_{i=1}^{n-1} \frac{\xi_i}{\xi_n}\, x_i$$

$$\text{and} \quad \frac{\partial h^{-1}}{\partial y} \;=\; \frac{1}{\xi_n}\,,$$

$$\sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial h^{-1}}{\partial x_i} \right)^2} = \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\xi_i}{\xi_n} \right)^2} \;=\; \frac{\|\boldsymbol{\xi}\|}{|\xi_n|}$$

Hyperplanes (cont'd)

Corollary. Let $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n)$ be random variables with joint density $f_{\mathbf{X}}(x_1, x_2, \cdots, x_n)$, and let $\mathbf{0} \neq \boldsymbol{\xi} \in \mathbb{R}^n$.

The random variable

$$\mathbf{Y} := \sum_{i=1}^{n} \xi_i \, \mathbf{X}_i$$

has the density

$$f_{\mathbf{Y}}(y) = \frac{(\mathbf{R}f_{\mathbf{X}})(\boldsymbol{\xi}, y)}{\|\boldsymbol{\xi}\|} .$$

where $(\mathbf{R}f_{\mathbf{X}})(\boldsymbol{\xi}, y)$ is the **Radon transform** of $f_{\mathbf{X}}$,

$$(\mathbf{R}f_{\mathbf{X}})(\boldsymbol{\xi}, y) =$$

$$\frac{\|\boldsymbol{\xi}\|}{|\xi_n|} \int_{\mathbb{R}^{n-1}} f_{\mathbf{X}} \left(x_1, \cdots, x_{n-1}, \frac{y}{\xi_n} - \sum_{i=1}^{n-1} \frac{\xi_i}{\xi_n} x_i \right) dx_1 dx_2 \cdots dx_{n-1} .$$

Spheres

Condition (**X**) holds for spheres. Let

$$y = h(x_1, \cdots, x_n) := \sum_{i=1}^{n} x_i^2$$

which has two solutions for x_n, representing the upper and lower hemispheres,

$$x_n = h^{-1}(y|\, x_1, \cdots, x_{n-1}) := \pm\sqrt{y - \sum_{i=1}^{n-1} x_i^2}$$

$$\text{with} \quad \frac{\partial h^{-1}}{\partial y} = \pm \frac{1}{2\sqrt{y - \sum_{i=1}^{n-1} x_i^2}},$$

$$\sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial h^{-1}}{\partial x_i}\right)^2} = \frac{\sqrt{y}}{\sqrt{y - \sum_{i=1}^{n-1} x_i^2}}$$

Spheres (cont'd)

Corollary. Let $\mathbf{X} = (\mathbf{X}_1, \cdots, \mathbf{X}_n)$ have joint density $f_{\mathbf{X}}(x_1, \cdots, x_n)$. The density of

$$Y = \sum_{i=1}^{n} \mathbf{X}_i^2$$

is

$$f_{\mathbf{Y}}(y) = \frac{1}{2\sqrt{y}} \int_{\mathcal{S}_n(\sqrt{y})} f_{\mathbf{X}}$$

where the integral is over the sphere $\mathcal{S}_n(\sqrt{y})$ of radius $\sqrt{y}$,

Spheres (cont'd)

computed as an integral over the ball $\mathcal{B}_{n-1}(\sqrt{y})$,

$$\int_{\mathcal{S}_n(\sqrt{y})} f_{\mathbf{X}} =$$

$$\int_{\mathcal{B}_{n-1}(\sqrt{y})} \left[f_{\mathbf{X}} \left(x_1, \cdots, x_{n-1}, \sqrt{y - \sum_{i=1}^{n-1} x_i^2} \right) + \right.$$

$$\left. + f_{\mathbf{X}} \left(x_1, \cdots, x_{n-1}, -\sqrt{y - \sum_{i=1}^{n-1} x_i^2} \right) \right] \frac{\sqrt{y}\ dx_1 \cdots dx_{n-1}}{\sqrt{y - \sum_{i=1}^{n-1} x_i^2}}$$