Lecture 1: Preliminaries

Scalars and vectors

The **complex field** $\mathbb{C}$ is used, specialized to the **real field** $\mathbb{R}$ as necessary. A **generic field** is denoted by $\mathbb{F}$.

Scalars in $\mathbb{F}$ are denoted: $x, y, \lambda, \ldots$.

The n–**dimensional vector space** over $\mathbb{F}$ is denoted by $\mathbb{F}^n$

Vectors in $\mathbb{F}^n$ are denoted: $\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \ldots$, their **elements** recorded as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \ , \ \mathbf{x} = (x_i) \ , \ \text{or } \mathbf{x}[i] = x_i \ , \ \ i \in \overline{1, n} \ , \ x_i \in \mathbb{F} \ .$$

The vector $\mathbf{e}_i$ given by $\mathbf{e}_i[j] = \delta_{ij}$, is the $i_{\,\mathrm{th}}$–**unit vector** of $\mathbb{F}^n$.

The **standard basis** of $\mathbb{F}^n$ in $\mathcal{E}_n = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

Inner product

Let V be a complex vector space: A function : $V \times V \mapsto \mathbb{C}$, denoted $\langle \mathbf{x}, \mathbf{y} \rangle$, that satisfies $\forall\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in V,\ \alpha \in \mathbb{C}$,

(I1) $\langle \alpha \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (**linearity**),

(I2) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ (**Hermitian symmetry**),

(I3) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Longleftrightarrow \mathbf{x} = \mathbf{0}$ (**positivity**),

is called an **inner product** on V.

The vectors $\mathbf{x}, \mathbf{y}$ are **orthogonal** if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Ex: The **standard inner product** in $\mathbb{C}^n$ is $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i \, \overline{y_i}$

Ex: For nonzero $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the **angle** $\angle\{\mathbf{x}, \mathbf{y}\}$ is given by

$$\cos \angle\{\mathbf{x}, \mathbf{y}\} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}\sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}}$$

Q: What about angles in $\mathbb{C}^n$?

Norm

Let V be a complex vector space: A function $: V \mapsto \mathbb{R}$, denoted $\|\mathbf{x}\|$, that satisfies $\forall\ \mathbf{x}, \mathbf{y} \in V,\ \alpha \in \mathbb{C}$,

(N1) $\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$ (**positivity**),

(N2) $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$ (**positive homogeneity**),

(N3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (**triangle inequality**),

is called a **norm** in V.

Ex: For $p \geq 1$, the ℓ_p–**norm** in $\mathbb{C}^n$ is

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}, \quad \|\mathbf{x}\|_\infty = \max\{|x_j|\}$$

and $\|\mathbf{x}\|_2 =$ the **Euclidean norm**.

Matrices

$\mathbb{F}^{m\times n}$ = the $m \times n$ **matrices** over $\mathbb{F}$, $\quad A = (a_{ij})$, $\quad A[i,j] = a_{ij}$

The matrix A is **diagonal** if $A[i,j] = 0$ for $i \neq j$,

in particular, $A = \mathrm{diag}\,(a_{11}, \cdots, a_{pp})$ where $p = \min\{m, n\}$.

A is **upper [lower] triangular** if $A[i,j] = 0$ for $i > j$ $[i < j]$

Given a matrix $A \in \mathbb{C}^{m\times n}$, its

transpose is $A^T \in \mathbb{C}^{n\times m}$ with $A^T[i,j] = A[j,i]$

conjugate transpose is $A^* \in \mathbb{C}^{n\times m}$ with $A^*[i,j] = \overline{A[j,i]}$

$A \in \mathbb{F}^{n\times n}$ is:

Hermitian [**symmetric**] if $A = A^*$ $[A = A^T$ if A is real$]$,

normal if $AA^* = A^*A$,

unitary [**orthogonal**] if $A^* = A^{-1}$ $[A^T = A^{-1}$ if A is real$]$.

Linear transformations

Let U, V be vector spaces over $\mathbb{F}$, T a mapping: $U \to V$.

T is a **linear transformation**, if

$$T(\alpha\mathbf{x} + \mathbf{y}) = \alpha T\mathbf{x} + T\mathbf{y} \ , \ \forall \ \mathbf{x}, \mathbf{y} \in U \ , \ \alpha \in \mathbb{F}.$$

The **set of linear transformations**: $U \to V$ is denoted $\mathcal{L}(U, V)$. $\mathcal{L}(U, V)$ is a **vector space** with **vector addition** and **scalar multiplication**

$$(T_1 + T_2)\mathbf{u} = T_1\mathbf{u} + T_2\mathbf{u} \ , \ (\alpha T)\mathbf{u} = \alpha(T\mathbf{u}) \ , \ \forall \ \mathbf{u} \in U \ .$$

The **zero** element of $\mathcal{L}(U, V)$ is the transformation O,

$$O\,\mathbf{u} = \mathbf{0} \in V \ , \ \forall \ \mathbf{u} \in U$$

$\mathcal{L}(U, U)$ is a **ring** with **identity** I

$$I\,\mathbf{u} = \mathbf{u} \ , \ \forall \ \mathbf{u} \in U \ .$$

Images

Let $T \in \mathcal{L}(U, V)$, $\mathbf{u} \in U$, $\mathbf{v} \in V$. Then

the **image** of $\mathbf{u}$ is the point $T\,\mathbf{u} \in V$,

the **inverse image** of $\mathbf{v}$ is the set $T^{-1}(\mathbf{v}) = \{\mathbf{u} \in U : T\mathbf{u} = \mathbf{v}\}$.

The **range** of T, denoted $R(T)$ is the set of all its images

$$R(T) = \{\mathbf{v} \in V : \mathbf{v} = T\mathbf{u} \text{ for some } \mathbf{u} \in U\} .$$

The **null space** of T, denoted $N(T)$, is the inverse image of $\mathbf{0} \in V$,

$$N(T) = T^{-1}(\mathbf{0}) = \{\mathbf{u} \in U : T\mathbf{u} = \mathbf{0}\} .$$

$R(T)$ and $N(T)$ are **subspaces** of U.

Ex: $P_n(x) =$ the **vector space of polynomials of degree** $\leq n$.
$T = \frac{d}{dx} \in \mathcal{L}(P_n(x), P_{n-1}(x))$, $R(T) = P_{n-1}(x)$, $N(T) = P_0(x)$.

Inverse

$T \in \mathcal{L}(U, V)$ is **one–to–one** if $\forall\ \mathbf{x}, \mathbf{y} \in U$, $\mathbf{x} \neq \mathbf{y} \implies T\mathbf{x} \neq T\mathbf{y}$,

onto if $R(T) = V$.

T one–to–one and onto $\implies$ $\exists$ **inverse** $T^{-1} \in \mathcal{L}(V, U)$ such that

$$T^{-1}(T\mathbf{u}) = \mathbf{u}\ ,\ T(T^{-1}\mathbf{v})\ ,\ \forall\ \mathbf{u} \in U, \mathbf{v} \in V\ , = \mathbf{v}\ ,$$

$$\text{or equivalently}\ ,\ T^{-1}T = I_U\ ,\ TT^{-1} = I_V\ ,$$

in which case T is called **invertible**, so is T^{-1}, and

$$\left(T^{-1}\right)^{-1} = T$$

Ex: Permutations

$$\pi_3(x_1, x_2, x_3) = (x_2, x_3, x_1)\ ,\ \pi_4(x_1, x_2, x_3) = (x_3, x_1, x_2)\ .$$

$$\pi_3 = \pi_4^{-1}\ .$$

Matrix representations

Given

- linear transformation $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ and
- bases $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $\mathbb{C}^m$ and $\mathbb{C}^n$,

the **matrix representation of** A **relative to** $\{U, V\}$ is the unique

$$A_{\{\mathcal{U},\mathcal{V}\}} = \left[a_{ij}\right] \in \mathbb{C}^{m \times n} \text{ determined by } A\mathbf{v}_j = \sum_{i=1}^{m} a_{ij}\,\mathbf{u}_i\ ,\ j \in \overline{1, n}\ .$$

For given $\{\mathcal{U}, \mathcal{V}\}$, a **1:1 correspondence** $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m) \longleftrightarrow \mathbb{C}^{m \times n}$.

If $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$, $A_{\{\mathcal{V},\mathcal{V}\}}$ is denoted $A_{\{\mathcal{V}\}}$, and is the unique

$$A_{\{\mathcal{V}\}} = \left[a_{ij}\right] \in \mathbb{C}^{n \times n} \text{ determined by } A\mathbf{v}_j = \sum_{i=1}^{n} a_{ij}\,\mathbf{v}_i\ ,\ j \in \overline{1, n}\ .$$

The standard bases $\{\mathcal{E}_m, \mathcal{E}_n\}$ are the defaults and are omitted.

Similarity

Matrices $A, B \in \mathbb{C}^{n\times n}$ are **similar** if

$$A = S^{-1}BS \tag{1}$$

for some nonsingular matrix S.

Similar matrices represent the same linear transformation in $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$, say

$$A = T_{\{\mathcal{U}_1\}},\ B = T_{\{\mathcal{U}_2\}} \tag{2}$$

A **similarity invariant** is the same for all similar matrices.

Ex: **Rank**, **determinant**, **eigenvalues** (not eigenvectors), **characteristic polynomial**, **minimal polynomial**

Q: How does the matrix S depend on the bases $\mathcal{U}_1$, $\mathcal{U}_2$?

Q: What can be said about matrices in $\mathbb{C}^{m\times n}$ representing the same linear transformation $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$?

Adjoint

Let

- $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}^m}$ be inner products on C^n and $\mathbb{C}^m$, resp.,
- $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$.

The **adjoint** of T, denoted T^*, is $T^* \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ defined by

$$\langle T\mathbf{v}, \mathbf{u} \rangle_{\mathbb{C}^m} = \langle \mathbf{v}, T^*\mathbf{u} \rangle_{\mathbb{C}^n} \ , \ \forall \ \mathbf{v} \in \mathbb{C}^n, \ \mathbf{u} \in \mathbb{C}^m$$

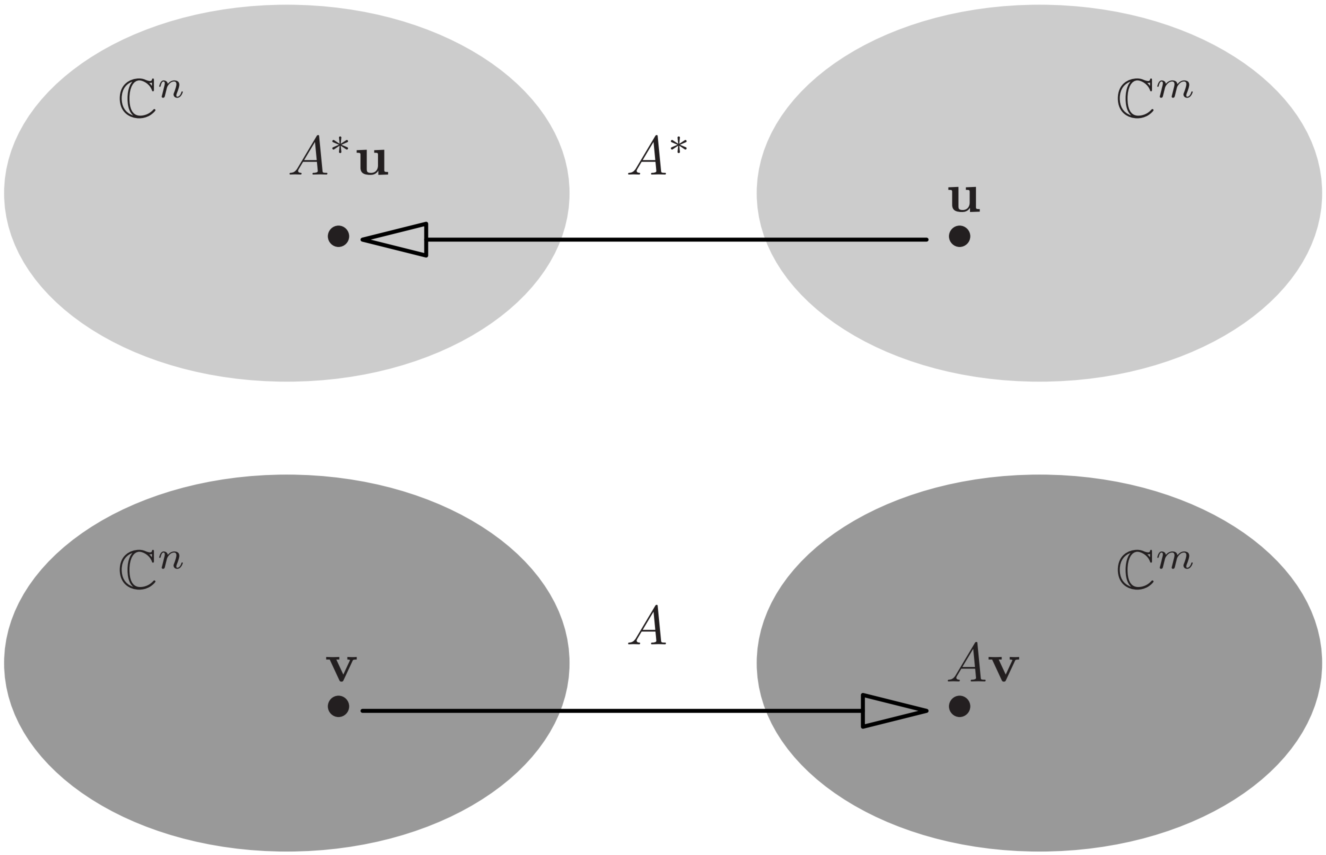
Ex: With standard inner products and bases, if $A \in \mathbb{C}^{m \times n}$ represents $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ then A^* represents T^*.

A matrix $Q \in \mathbb{C}^{n \times n}$ is **positive definite** if $Q = Q^*$ and

$$\langle \mathbf{x}, Q\mathbf{x} \rangle > 0 \ , \ \forall \ \mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$$

Q: Let $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{C}^n} = \mathbf{x}^* P \mathbf{y}$ and $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{C}^m} = \mathbf{u}^* Q \mathbf{v}$, where P, Q are positive definite. If $A \in \mathbb{C}^{m \times n}$ represents $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$, what is the matrix representation of T^* ?

A linear transformation and its adjoint



$$\langle \mathbf{u}, A\mathbf{v} \rangle = \langle A^*\mathbf{u}, \mathbf{v} \rangle \ , \ \forall \ \mathbf{v}, \mathbf{u}$$

Direct sums

The **sum** of 2 subspaces $L, M \subset \mathbb{C}^n$, denoted $L + M$, is defined as

$$L + M = \{\mathbf{y} + \mathbf{z} : \mathbf{y} \in L, \mathbf{z} \in M\} \text{ , and is also a subspace.}$$

If $L \cap M = \{\mathbf{0}\}$, then $L + M$ is denoted $L \oplus M$, a **direct sum**.

The subspaces $L, M \subset \mathbb{C}^n$ are **complementary** if

$$\mathbb{C}^n = L \oplus M \; , \tag{1}$$

or equivalently, if every $\mathbf{x} \in \mathbb{C}^n$ is expressed **uniquely** as a sum

$$\mathbf{x} = \mathbf{x}_L + \mathbf{x}_M \; , \; (\mathbf{x}_L \in L, \; \mathbf{x}_M \in M) \; . \tag{2}$$

The (**linear**) map $P_{L,M} : \mathbf{x} \to \mathbf{x}_L$ is the **projector on** L **along** M.

Let the columns of $X = [\mathbf{x}_1, \cdots, \mathbf{x}_\ell]$, $Y = [\mathbf{y}_1, \cdots, \mathbf{y}_m]$ be bases of L, M, resp. Then $P_{L,M}$ is represented by the matrix

$$P_{L,M} = [X \; O][X \; Y]^{-1}$$

Idempotents and projectors

A matrix E is **idempotent** if $E^2 = E$.

Lemma 1. Let $E \in \mathbb{C}^{n\times n}$ be idempotent. Then:

(a) E^* and $I - E$ are idempotent.

(b) $\Lambda(E) = \{0, 1\}$. The multiplicity of the eigenvalue 1 is $\operatorname{rank} E$.

(c) $\operatorname{rank} E = \operatorname{trace} E$.

(d) $E(I - E) = (I - E)E = O$.

(e) $E\mathbf{x} = \mathbf{x} \iff \mathbf{x} \in R(E)$.

(f) $E \in E\{1, 2\}$.

(g) $N(E) = R(I - E)$.

Theorem 1. For every idempotent matrix $E \in \mathbb{C}^{n\times n}$,

$$\mathbb{C}^n = R(E) \oplus N(E)\ , \text{ and } E = P_{R(E),N(E)}\ .$$

Conversely, if L and M are complementary subspaces, there is a unique idempotent $P_{L,M}$ such that $R(P_{L,M}) = L\ ,\ N(P_{L,M}) = M$.

Orthogonal direct sums

Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ are **orthogonal**, denoted $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

A set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathbb{C}^n$ is **orthonormal (o.n.)** if

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij}, \ \forall\, i, j \in \overline{1, k}.$$

The **orthogonal complement** $L^\perp$ of a subspace $L \subset \mathbb{C}^n$ is

$$L^\perp := \{\mathbf{x} \in \mathbb{C}^n : \mathbf{y} \in L \implies \mathbf{x} \perp \mathbf{y}\}, \text{ also a subspace.}$$

$\mathbb{C}^n$ is the **orthogonal direct sum** of L, $L^\perp$, a fact denoted by

$$\mathbb{C}^n = L \overset{\perp}{\oplus} L^\perp .$$

The **projector on** L **along** $L^\perp$, denoted P_L, is the **orthogonal projector** on L. It is represented by

$$P_L = [X\ O][X\ Y]^{-1} = \sum_{i=1}^{\ell} \mathbf{x}_i \mathbf{x}_i^*$$

where $X = [\mathbf{x}_1, \cdots, \mathbf{x}_\ell]$, $Y = [\mathbf{y}_1, \cdots, \mathbf{y}_m]$ are o.n. bases of L, $L^\perp$.

Hermitian idempotents and orthogonal projectors

A matrix E is **Hermitian idempotent** if $E^2 = E = E^*$.

Lemma 2. Let $\mathbb{C}^n = L \oplus M$. Then $M = L^\perp \iff P_{L,M} = P^*_{L,M}$.

Proof: Let $\mathbb{C}^n = L \overset{\perp}{\oplus} M$, and consider the matrix $P^*_{L,M}$. By Lemma 1(a), it is idempotent and by Theorem 1, a projector.

$$\therefore\ N(P^*_{L,M}) = R(P_{L,M})^\perp = L^\perp = M$$

$$\therefore\ R(P^*_{L,M}) = N(P_{L,M})^\perp = M^\perp = L$$

$$\therefore\ P^*_{L,M} = P_{L,M}\ . \qquad \square$$

Theorem 2. For every Hermitian idempotent $E \in \mathbb{C}^{n\times n}$,

$$\mathbb{C}^n = R(E) \overset{\perp}{\oplus} N(E)\ , \text{ and } E = P_{R(E)}\ .$$

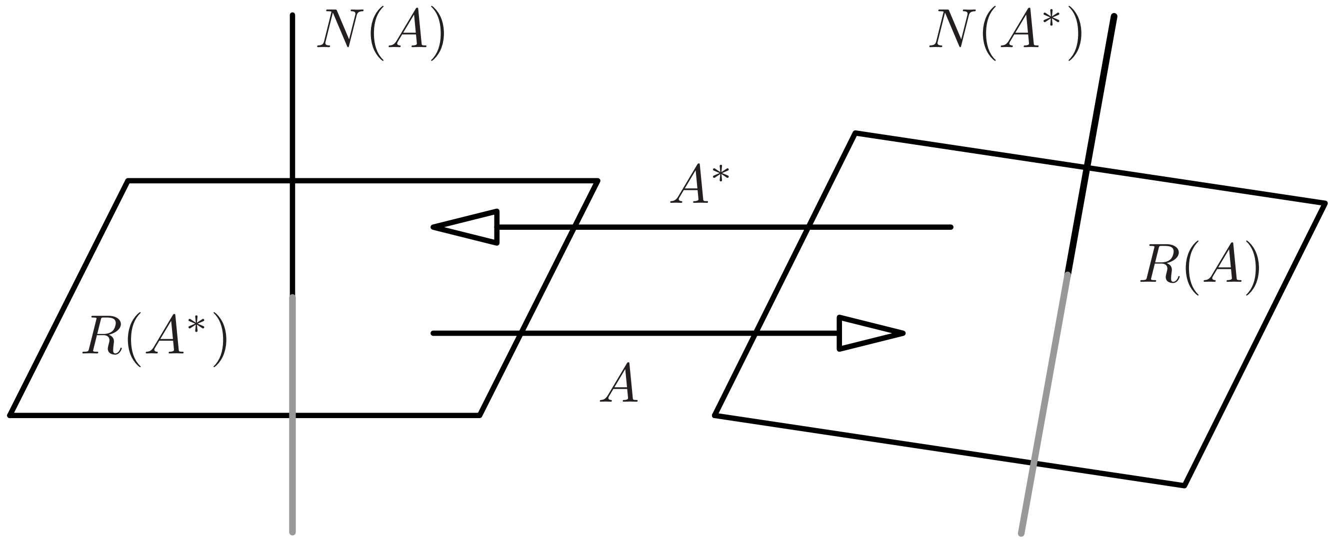
Conversely, if L is a subspace of $\mathbb{C}^n$, there is a unique Hermitian idempotent P_L such that $R(P_L) = L$, $N(P_L) = L^\perp$.

4 subspaces

If $A \in \mathbb{C}^{m \times n}$ then

$$\mathbb{C}^n = R(A^*) \overset{\perp}{\oplus} N(A) \tag{1}$$

$$\mathbb{C}^n = R(A) \overset{\perp}{\oplus} N(A^*) \tag{2}$$



The **restriction** of A to $R(A^*)$, $A|_{R(A^*)} \in \mathcal{L}(R(A^*), R(A)$, is **invertible**.

The Singular Value Decomposition (SVD)

Let $A \in \mathbb{C}^{m \times n}$, let the eigenvalues $\lambda_j(AA^*)$ of AA^* (which is PSD) be ordered by

$$\lambda_1(AA^*) \geq \cdots \geq \lambda_r(AA^*) > \lambda_{r+1}(AA^*) = \cdots = \lambda_n(AA^*) = 0 \, .$$

The **singular values** of A, denoted by $\sigma_j(A)$ or σ_j, $j \in \overline{1,r}$, are defined as

$$\sigma_j(A) = +\sqrt{\lambda_j(AA^*)} \, , \quad j \in \overline{1,r} \, ,$$

or equivalently, $\sigma_j(A) = +\sqrt{\lambda_j(A^*A)} \, , \quad j \in \overline{1,r} \, ,$

and are ordered,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \, .$$

The **set of singular values** of A is denoted by $\sigma(A)$.

The SVD (cont'd)

Let A and $\{\sigma_j\}$ be as above, let $\{\mathbf{u}_i : i \in \overline{1,m}\}$ be an o.n. basis of $\mathbb{C}^m$ made of eigenvectors of AA^*,

$$AA^*\mathbf{u}_i = \sigma_i^2\mathbf{u}_i \ , \ i \in \overline{1,r} \ ,$$

$$AA^*\mathbf{u}_i = \mathbf{0} \ , \ i \in \overline{r+1,m} \ ,$$

let

$$\mathbf{v}_j = \frac{1}{\sigma_j} A^*\mathbf{u}_j \ , \ j \in \overline{1,r} \ ,$$

and let $\{\mathbf{v}_j : j \in \overline{r+1,n}\}$ be an o.n. set of vectors, orthogonal to $\{\mathbf{v}_j : j \in \overline{1,r}\}$. Then the set $\{\mathbf{v}_j : j \in \overline{1,n}\}$ is an o.n. basis of $\mathbb{C}^n$ consisting of eigenvectors of A^*A,

$$A^*A\mathbf{v}_j = \sigma_i^2\mathbf{v}_j \ , \ j \in \overline{1,r} \ ,$$

$$A^*A\mathbf{v}_j = \mathbf{0} \ , \ j \in \overline{r+1,n} \ .$$

The SVD (cont'd)

Conversely, starting from an o.n. basis $\{\mathbf{v}_j : j \in \overline{1,n}\}$ of $\mathbb{C}^n$ satisfying

$$A^* A\mathbf{v}_j = \sigma_i^2 \mathbf{v}_j \ , \ j \in \overline{1,r} \ ,$$

$$A^* A\mathbf{v}_j = \mathbf{0} \ , \ j \in \overline{r+1,n} \ .$$

we construct an o.n. basis of $\mathbb{C}^m$ with

$$AA^* \mathbf{u}_i = \sigma_i^2 \mathbf{u}_i \ , \ i \in \overline{1,r} \ ,$$

$$AA^* \mathbf{u}_i = \mathbf{0} \ , \ i \in \overline{r+1,m} \ ,$$

by

$$\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_j \ , \ i \in \overline{1,r} \ ,$$

and completing to an o.n. set.

The SVD (cont'd)

Let A, $\{\mathbf{u}_i : i \in \overline{1,m}\}$ and $\{\mathbf{v}_j : j \in \overline{1,n}\}$ be as above. Then

$$\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_j \ , \ i \in \overline{1,r} \ , \text{ can be written as}$$

$$AV = U\,\Sigma \ , \text{ where } \Sigma = \begin{bmatrix} \sigma_1 & & & \vdots & \\ & \ddots & & \vdots & O \\ & & \sigma_r & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & O & & \vdots & O \end{bmatrix} ,$$

$$U = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix} \ , \ V = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \ .$$

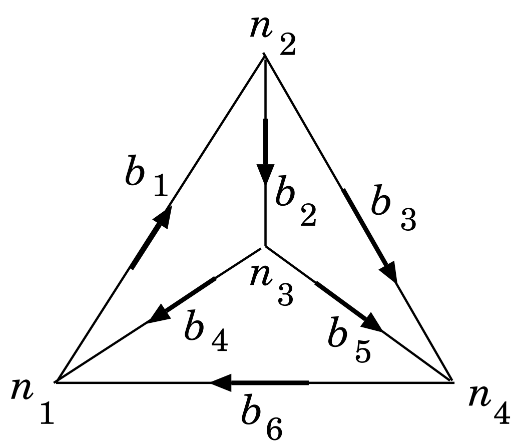
Since V is unitary,

$$A = U\,\Sigma\,V^* \ ,$$

called a **singular value decomposition** (abbreviated **SVD**) of A.

An application to electrical networks

Consider a network, such as



represented by a **node–branch incidence matrix**

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

Electrical networks (cont'd)

Let

M = the **node–branch incidence matrix**,

A = the diagonal matrix of **branch conductances**,

$\mathbf{y}$ = **branch currents**,

$\mathbf{p}$ = **node potentials**,

$\mathbf{x}$ = **potential difference across branches**, e.g., $x_3 = p_2 - p_4$, or

$$\mathbf{x} = M^T \mathbf{p}$$

Therefore,

$R(M^T)$ = the **voltages** satisfying **Kirchhoff voltage law**,

$N(M)$ = the **currents** satisfying **Kirchhoff current law**,

and the network satisfies **Ohm's law**,

$$A\mathbf{x} + \mathbf{y} = A\mathbf{v} + \mathbf{w} \ , \ \mathbf{x} \in R(M^T), \mathbf{y} \in N(M) \tag{1}$$

$\mathbf{v}, \mathbf{w}$ = the **voltages** and **currents** generated in the branches.

The Bott–Duffin inverse

Consider the constrained system, arising in electrical network theory,

$$A\mathbf{x} + \mathbf{y} = \mathbf{b}\ ,\ \ x \in L\ ,\ \ \mathbf{y} \in L^{\perp}\ , \tag{1}$$

or, equivalently,

$$(AP_L + P_{L^\perp})\,\mathbf{z} = \mathbf{b} \tag{2}$$

where

$$\mathbf{x} = P_L\mathbf{z}, \quad \mathbf{y} = P_{L^\perp}\mathbf{z} = \mathbf{b} - AP_L\mathbf{z}\ . \tag{3}$$

If $(AP_L + P_{L^\perp})$ is nonsingular, then (1) has a unique solution for all $\mathbf{b} \in \mathbb{C}^m$.

The operator

$$A^{(-1)}_{(L)} := P_L(AP_L + P_{L^\perp})^{-1}$$

is the **Bott–Duffin inverse** of A w.r.t. L.

The Bott–Duffin inverse (cont'd)

Theorem (**Bott and Duffin**). Let $(AP_L + P_{L^\perp})$ be nonsingular. Then the equation

$$A\mathbf{x} + \mathbf{y} = \mathbf{b}\ ,\ \ x \in L\ ,\ \ \mathbf{y} \in L^\perp \tag{1}$$

has for every $\mathbf{b}$, the unique solution

$$\mathbf{x} = A_{(L)}^{(-1)}\mathbf{b}\ ,$$

$$\mathbf{y} = (I - AA_{(L)}^{(-1)})\mathbf{b}\ .$$

Ex. If $A = (a_{ij})$, $A_{(L)}^{(-1)} = (t_{ij})$, $d_{A,L} = \det(AP_L + P_{L^\perp})$, and $\psi_{A,L} = \log d_{A,L}$ then

$$t_{ji} = \frac{\partial \psi_{A,L}}{\partial a_{ij}}$$

Ex. If the columns of K are a basis for L, then

$$A_{(L)}^{(-1)} = K(K^*AK)^{-1}K^*\ ,\ (\exists\, A_{(L)}^{(-1)} \Longleftrightarrow \exists\,(K^*AK)^{-1})$$

A generalized inverse of any $A \in \mathbb{C}^{m\times n}$

Recall that $A|_{R(A^*)}$, is **invertible** as a map: $R(A^*) \mapsto R(A)$.

Let $M \subset \mathbb{C}^m$ be a subspace such that

$$\mathbb{C}^m = R(A) \oplus M$$

and define

$$A_M^{-1} := \begin{cases} A|_{R(A^*)}^{-1}\mathbf{y}, & \mathbf{y} \in R(A); \\ \mathbf{0}, & \mathbf{y} \in M. \end{cases}$$

For $M = N(A^*)$, A_M^{-1} is the **Moore–Penrose inverse**.