NUMERICAL LINEAR ALGEBRA WITH APPLICATIONS
Numer. Linear Algebra Appl. 2006; **13**:589–598
Published online 19 June 2006 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nla.491

Jordan's principal angles in complex vector spaces

A. Galántai[1,*,†] and Cs. J. Hegedűs[2,‡]

[1] *University of Miskolc, 3515 Miskolc, Hungary*
[2] *Eötvös Loránd University, 1117 Budapest, Pázmány Péter sétány 1/c, Hungary*

SUMMARY

We analyse the possible recursive definitions of principal angles and vectors in complex vector spaces and give a new projector based definition. This enables us to derive important properties of the principal vectors and to generalize a result of Björck and Golub (*Math. Comput.* 1973; **27**(123):579–594), which is the basis of today's computational procedures in real vector spaces. We discuss other angle definitions and concepts in the last section. Copyright © 2006 John Wiley & Sons, Ltd.

Received 8 September 2005; Revised 14 April 2006; Accepted 2 May 2006

KEY WORDS: principal angles; subspaces; projections

1. INTRODUCTION

The principal or canonical angles (and the related principal vectors) between two subspaces provide the best available characterization of the relative subspace positions. Although Jordan [1] introduced the concept of principal angles (and vectors) in 1875 (see also References [2, 3]), the principal angles were rediscovered several times. Davis and Kahan [4] include an interesting account of these works to which we can add Reference [5] as the most recent contribution.

Jordan's recursive definition of the principal angles was formalized by Hotelling [6] in his theory of canonical correlations. The importance of the matter and Hotelling's paper [6] initiated many further investigations such as References [7–9].

Definition 1
Let $x, y \in \mathbb{R}^n$, $x, y \neq 0$. Then the (real) angle $\Theta(x, y)$ between x and y is defined by

$$\cos\Theta(x, y) = \frac{x^{\mathrm{T}} y}{\|x\|\|y\|} \quad (0 \leqslant \Theta \leqslant \pi) \tag{1}$$

where $\|x\| = (x^{\mathrm{T}} x)^{1/2}$.

*Correspondence to: A. Galántai, University of Miskolc, 3515 Miskolc, Hungary.
†E-mail: matgal@uni-miskolc.hu
‡E-mail: hegedus@numanal.inf.elte.hu

Definition 2
Let $M_1, M_2 \subset \mathbb{R}^n$ be subspaces with $p_1 = \dim(M_1) \geqslant \dim(M_2) = p_2 \geqslant 1$. The principal angles $\theta_k \in [0, \pi/2]$ between M_1 and M_2 are recursively defined for $k = 1, \ldots, p_2$ by

$$\cos\theta_k = \max_{\substack{u \in M_1, v \in M_2, \|u\| = \|v\| = 1 \\ u_i^{\mathrm{T}} u = 0, v_i^{\mathrm{T}} v = 0, i = 1, \ldots, k-1}} u^{\mathrm{T}} v = u_k^{\mathrm{T}} v_k \tag{2}$$

The vectors $\{u_1, \ldots, u_{p_2}\}$, $\{v_1, \ldots, v_{p_2}\}$ are called principal vectors of the pair of spaces.

Notice that $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{p_2} \leqslant \pi/2$. The principal angles are uniquely defined, while the principal vectors are not.

Definition 2 is based on the concept of angles between two real vectors and the standard inner product. When looking for similar definitions in complex vector spaces we encountered the following problems:

1. We did not find a generally accepted definition that includes both the principal angles and the principal vectors.
2. The available definitions give only the principal angles and use CS decomposition [3, 10] or eigenvalues (see, e.g. References [4, 5, 8, 10–13]). Hence there is some loss of the geometric character.
3. There is some ambiguity in the definition of angle between complex vectors.

Scharnhorst [14] enlists six angle concepts between complex vectors (Euclidean (imbedded) angle, complex-valued angle, Hermitian angle, real-part angle, Kasner's pseudo angle, Kähler angle) that have different geometric properties. For example, if one defines $\cos\phi = |\langle x, y\rangle| / (\|x\|\|y\|)$, then $\phi = \pi/2$ if and only if $\langle x, y\rangle = 0$, but the law of cosines does not hold. If one uses $\cos\phi = \mathrm{Re}\langle x, y\rangle / (\|x\|\|y\|)$, then the law of cosines holds, but $\phi = \pi/2$ may hold for $\langle x, y\rangle \neq 0$ (see also References [4, 15]). The variety of angle definitions and properties clearly requires a thorough extension of Jordan's original concept [1, 2] to the complex case.

Here we first analyse the possible recursive definitions in complex vector spaces and give a new projector based definition as well. The new definition enables us to derive important properties of the principal vectors and to generalize a result of Björck and Golub [9], which is the basis of today's computational procedures in real vector spaces (see, e.g. References [16–19]). We also discuss other definitions in the last section.

2. RECURSIVE DEFINITIONS IN COMPLEX VECTOR SPACES

We first consider those angle definitions between vectors that are related to the complex inner product. If not otherwise stated, a general complex valued inner product and the induced norm will be assumed in the following considerations.

Definition 3
Let $x, y \in \mathbb{C}^n$, $x, y \neq 0$. Then the complex (-valued) angle $\Theta_c(x, y)$ between x and y is defined by

$$\cos\Theta_c(x, y) = \frac{\langle x, y\rangle}{\|x\|\|y\|} \tag{3}$$

The Hermitian angle $\Theta_{\mathrm{H}}(x, y)$ between vectors x and y is defined by

$$\cos\Theta_{\mathrm{H}}(x, y) = \frac{|\langle x, y\rangle|}{\|x\|\,\|y\|} \quad (0\leqslant\Theta_{\mathrm{H}}(x, y)\leqslant\pi/2) \tag{4}$$

The real-part angle is defined by

$$\cos\Theta_r(x, y) = \frac{\mathrm{Re}\langle x, y\rangle}{\|x\|\,\|y\|} \tag{5}$$

Scharnhorst [14] also deals with another possibility, i.e. the imbedding of the complex n-space into a real $2n$-space (Euclidean angle).

Since the complex numbers are not ordered, we can use only the Hermitian angle or the real-part angle to define the principal angles between subspaces.

Definition 4
Let $M_1, M_2\subset\mathbb{C}^n$ be subspaces with $p_1 = \dim(M_1)\geqslant\dim(M_2) = p_2\geqslant 1$. The principal angles $\theta_k\in[0, \pi/2]$ between M_1 and M_2 are recursively defined for $k = 1, \ldots, p_2$ by

$$\cos\theta_k = \max_{\substack{u\in M_1, v\in M_2, \|u\|=\|v\|=1\\ \langle u_i, u\rangle=0, \langle v_i, v\rangle=0, i=1,\ldots,k-1}} |\langle u, v\rangle| = \langle u_k, v_k\rangle \tag{6}$$

The vectors $\{u_1, \ldots, u_{p_2}\}$, $\{v_1, \ldots, v_{p_2}\}$ are called principal vectors of the pair of spaces.

Definition 5
Let $M_1, M_2\subset\mathbb{C}^n$ be subspaces with $p_1 = \dim(M_1)\geqslant\dim(M_2) = p_2\geqslant 1$. The principal angles $\widehat{\theta}_k\in[0, \pi/2]$ between M_1 and M_2 are recursively defined for $k = 1, \ldots, p_2$ by

$$\cos\widehat{\theta}_k = \max_{\substack{u\in M_1, v\in M_2, \|u\|=\|v\|=1\\ \langle \widehat{u}_i, u\rangle=0, \langle \widehat{v}_i, v\rangle=0, i=1,\ldots,k-1}} \mathrm{Re}\langle u, v\rangle = \langle \widehat{u}_k, \widehat{v}_k\rangle \tag{7}$$

The vectors $\{\widehat{u}_1, \ldots, \widehat{u}_{p_2}\}$, $\{\widehat{v}_1, \ldots, \widehat{v}_{p_2}\}$ are called principal vectors of the pair of spaces.

Definition 4 can also be found in Reference [20] without the restriction $\cos\theta_k = \langle u_k, v_k\rangle$. It also corresponds to Dixmier's minimal angle definition [21] repeated p_2-times (see also Reference [22]). It is easy to prove

Lemma 6
The angles $\widehat{\theta}_k$ defined in (7) are the same as the angles θ_k of (6), and $\{\widehat{u}_k, \widehat{v}_k\}$ and $\{u_k, v_k\}$ are both corresponding principal vector pairs.

Proof
Consider the case when $k = 1$. Then

$$\cos\theta_1 = \max_{u\in M_1, v\in M_2, \|u\|=\|v\|=1} |\langle u, v\rangle| = \langle u_1, v_1\rangle$$

and

$$\cos\widehat{\theta}_1 = \max_{u\in M_1, v\in M_2, \|u\|=\|v\|=1} \mathrm{Re}\langle u, v\rangle = \langle \widehat{u}_1, \widehat{v}_1\rangle$$

DOI: 10.1002/nla

Since $\cos\theta_1$, $\cos\widehat{\theta}_1 \geqslant 0$, we have $0 \leqslant \langle u_1, v_1\rangle = \mathrm{Re}\langle u_1, v_1\rangle \leqslant \cos\widehat{\theta}_1$. Hence $\cos\theta_1 \leqslant \cos\widehat{\theta}_1$. In turn, $\cos\widehat{\theta}_1 = \langle \widehat{u}_1, \widehat{v}_1\rangle = |\langle \widehat{u}_1, \widehat{v}_1\rangle| \leqslant \cos\theta_1$. Hence $\theta_1 = \widehat{\theta}_1$ and $\langle u_1, v_1\rangle = \cos\theta_1 = \cos\widehat{\theta}_1 = \langle \widehat{u}_1, \widehat{v}_1\rangle$. Clearly we can apply the same argument for $k > 1$. □

The third definition of principal angles and vectors exploits basic properties of orthogonal projections and seems to be new. We use the following notations: $M_1^{(1)} = M_1$, $M_2^{(1)} = M_2$ and

$$M_1^{(k)} = M_1 \cap \mathscr{R}^{\perp}(u_1, \ldots, u_{k-1}), \quad M_2^{(k)} = M_2 \cap \mathscr{R}^{\perp}(v_1, \ldots, v_{k-1}) \tag{8}$$

Definition 7
Let $M_1, M_2 \subset \mathbb{C}^n$ be subspaces with $p_1 = \dim(M_1) \geqslant \dim(M_2) = p_2 \geqslant 1$ and let P_1 be the orthogonal projection onto M_1 and P_2 the orthogonal projection onto M_2. Then the principal angles $\theta_k \in [0, \pi/2]$ between M_1 and M_2 are recursively defined for $k = 1, \ldots, p_2$ by

$$\cos\theta_k = \max_{\substack{v \in M_2^{(k)} \\ \|v\|=1}} \|P_1 v\| = \|P_1 v_k\| = \max_{\substack{u \in M_1^{(k)} \\ \|u\|=1}} \|P_2 u\| = \|P_2 u_k\| \tag{9}$$

The vectors $\{u_1, \ldots, u_{p_2}\}$, $\{v_1, \ldots, v_{p_2}\}$ are called principal vectors of the pair of spaces.

Theorem 8
The principal angles given by Definition 7 are identical with those of Definition 4.

Proof
First we assume that $k = 1$. For any $u \in M_1$ and $v \in M_2$ of unit norm we have

$$|\langle u, v\rangle| = |\langle P_1 u, v\rangle| = |\langle u, P_1 v\rangle| \leqslant \|u\| \|P_1 v\| \leqslant \max_{\substack{v \in M_2 \\ \|v\|=1}} \|P_1 v\| = \|P_1 v_1\|$$

There is equality if $u = \lambda P_1 v$. If $P_1 v_1 \neq 0$, then condition $\|u\| = 1$ gives $u_1 = P_1 v_1 / \|P_1 v_1\| \in M_1$ and for this u_1 we have $\cos\theta_1 = |\langle u_1, v_1\rangle| = \langle u_1, v_1\rangle = \|P_1 v_1\|$. Hence u_1 and v_1 are corresponding principal vectors so that $P_1 v_1 = (\cos\theta_1) u_1$. If $\|P_1 v_1\| = 0$, then $\langle u, v\rangle = 0$ holds for any $u \in M_1$. Thus $\theta_1 = \pi/2$ and we can select any $u_1 \in M_1$ ($\|u_1\| = 1$) as first principal vector. We also have the relation $P_1 v_1 = (\cos\theta_1) u_1$ ($= 0$). Since the role of M_1 and M_2 is symmetric, the other relation of (9) with P_2 clearly holds. So does the equality $P_2 u_1 = (\cos\theta_1) v_1$.

For any $u \in M_1^{(2)}$, $\langle u, v_1\rangle = \langle P_1 u, v_1\rangle = \langle u, P_1 v_1\rangle = (\cos\theta_1)\langle u, u_1\rangle = 0$. Similarly, if $v \in M_2^{(2)}$, then $\langle v, u_1\rangle = \langle P_2 v, u_1\rangle = \langle v, P_2 u_1\rangle = (\cos\theta_1)\langle v, v_1\rangle = 0$. Hence $\mathscr{R}(u_1) \perp M_2 \cap \mathscr{R}^{\perp}(v_1) = M_2^{(2)}$ and $\mathscr{R}(v_1) \perp M_1 \cap \mathscr{R}^{\perp}(u_1) = M_1^{(2)}$. Assume now that

$$\mathscr{R}(u_1, \ldots, u_{k-1}) \perp M_2^{(k)}, \quad \mathscr{R}(v_1, \ldots, v_{k-1}) \perp M_1^{(k)} \tag{10}$$

and

$$P_1 v_j = (\cos\theta_j) u_j, \quad P_2 u_j = (\cos\theta_j) v_j \tag{11}$$

hold for $j \leqslant k - 1$.

For any $k > 1$, $u \in M_1^{(k)} \subset M_1$ ($\|u\| = 1$) and $v \in M_2^{(k)}$ ($\|v\| = 1$) we obtain

$$|\langle u, v\rangle| = |\langle P_1 u, v\rangle| = |\langle u, P_1 v\rangle| \leqslant \|u\| \|P_1 v\| \leqslant \max_{\substack{v \in M_2^{(k)} \\ \|v\|=1}} \|P_1 v\| = \|P_1 v_k\|$$

DOI: 10.1002/nla

We show that $P_1 v \in M_1^{(k)}$ for any $v \in M_2^{(k)}$. Since $P_1 v \in M_1$, $P_1 v \in M_1^{(k)}$ if and only if $P_1 v \perp \mathscr{R}(u_1, \ldots, u_{k-1})$. This however follows from $\langle P_1 v, u_i \rangle = \langle v, P_1 u_i \rangle = \langle v, u_i \rangle = 0$ for $i \leqslant k-1$. If $P_1 v_k \neq 0$, then for $u_k = P_1 v_k / \| P_1 v_k \|$, $\cos\theta_k = |\langle u_k, v_k \rangle| = \langle u_k, v_k \rangle = \| P_1 v_k \|$ and $P_1 v_k = (\cos\theta_k) u_k$ hold. If $P_1 v_k = 0$, then for any $u \in M_1^{(k)}$ we have $\langle u, v \rangle = 0$. Hence $\theta_k = \pi/2$, $u_k \in M_1^{(k)}$ is any unit vector and $P_1 v_k = (\cos\theta_k) u_k$ $(=0)$. Again applying the symmetry principle we obtain the same for P_2 and also the relation $P_2 u_k = (\cos\theta_k) v_k$.

Assume that (11) holds for $j \leqslant k < p_2$. Let $u \in M_1^{(k+1)}$. For any v_j $(j \leqslant k)$,

$$\langle u, v_j \rangle = \langle P_1 u, v_j \rangle = \langle u, P_1 v_j \rangle = (\cos\theta_j) \langle u, u_j \rangle = 0$$

Similarly, if $v \in M_2^{(k+1)}$, then

$$\langle v, u_j \rangle = \langle P_2 v, u_j \rangle = \langle v, P_2 u_j \rangle = (\cos\theta_j) \langle v, v_j \rangle = 0$$

Hence we proved relation (10) for $k+1$ and the equivalence of Definitions 7 and 4. □

At the end of the process (recursive definition) we obtain the orthonormal vectors $\{u_i\}_{i=1}^{p_2} \subset M_1$ and $\{v_i\}_{i=1}^{p_2} \subset M_2$ such that

$$\mathscr{R}(v_1, \ldots, v_{p_2}) = M_2 \perp M_1 \cap \mathscr{R}^{\perp}(u_1, \ldots, u_{p_2}) \tag{12}$$

$\langle u_i, v_i \rangle = \cos\theta_i$ and

$$P_1 v_i = (\cos\theta_i) u_i, \quad P_2 u_i = (\cos\theta_i) v_i \quad (i = 1, \ldots, p_2) \tag{13}$$

The biorthogonality relation $\langle u_i, v_j \rangle = (\cos\theta_j)\delta_{ij}$ follows from the relation

$$\langle u_i, v_j \rangle = \langle P_1 u_i, v_j \rangle = \langle u_i, P_1 v_j \rangle = (\cos\theta_j) \langle u_i, u_j \rangle = (\cos\theta_j)\delta_{ij}$$

Property (13) and the following consequence also appear in Reference [7] for real vector spaces. Afriat called the pairs (u_i, v_i) reciprocal.

Corollary 9
$P_1 P_2 u_i = (\cos\theta_i) P_1 v_i = (\cos^2\theta_i) u_i$ and $P_2 P_1 v_i = (\cos\theta_i) P_2 u_i = (\cos^2\theta_i) v_i$.

It is possible to give another characterization of the canonical angles using the projection definition. Recall that the distance of a vector x from the subspace M_1 can be expressed by $\|(I - P_1)x\|$, where P_1 is the orthogonal projection onto M_1. Observe that $P_1 x$ and $(I - P_1)x$ are mutually orthogonal vectors and the distance comes from the Pythagorean theorem in the inner product norm metric. In the view of this distance function, we can interpret $\cos\theta_k$ in Definition 7 as the distance of the unit ball in $M_2^{(k)}$ from $M_1^{\perp}$. We get the same result from the dual relation: it is the distance of the unit ball in $M_1^{(k)}$ from $M_2^{\perp}$.

The given recursive definitions are quite analogous to the variational characterization of the eigenvalues of a Hermitian matrix as given by the Courant–Fischer theorem. The connection is more apparent later, when the principal angles are given by singular values.

3. THE PRINCIPAL ANGLES AND THE SVD

Here we analyse the connection of the principal angles and the singular value decomposition that was first exploited in Reference [9].

DOI: 10.1002/nla

Let $X=[x_1,\ldots,x_{n_1}]$, $Y=[y_1,\ldots,y_{n_2}]$ and

$$G(X,Y)=[\langle y_j,x_i\rangle]_{i,j=1}^{n_1,n_2} \tag{14}$$

Clearly, $G(X,X)$ is the classical Gram matrix. For any $A\in\mathbb{C}^{n_1\times n_1}$ and $B\in\mathbb{C}^{n_2\times n_2}$ we have

$$G(XA,YB)=A^{\mathrm{H}}G(X,Y)B \tag{15}$$

where A^{H} stands for the Hermitian adjoint or transpose of matrix A. By definition $G(XA,YB)=[\langle YBe_j,XAe_i\rangle]_{i,j=1}^{n_1,n_2}$ and

$$\begin{aligned}\langle YBe_j,XAe_i\rangle &= \left\langle \sum_{k=1}^{n_2} b_{kj}y_k, \sum_{l=1}^{n_1} a_{li}x_l\right\rangle\\ &= \sum_{l=1}^{n_1}\overline{a}_{li}\left(\sum_{k=1}^{n_2} b_{kj}\langle y_k,x_l\rangle\right)\\ &= e_i^{\mathrm{T}}A^{\mathrm{H}}G(X,Y)Be_j\end{aligned}$$

which proves the claim. Observe that the vectors $X=[x_1,\ldots,x_{n_1}]$ are orthonormal if and only if $G(X,X)=I$. Similarly, $X=[x_1,\ldots,x_{n_1}]$ and $Y=[y_1,\ldots,y_{n_2}]$ are biorthogonal ($n_1\geqslant n_2$) if and only if

$$G(X,Y)=\begin{bmatrix}\Gamma\\0\end{bmatrix}\quad(\Gamma=\mathrm{diag}(\langle y_i,x_i\rangle)) \tag{16}$$

Let $U=[u_1,\ldots,u_{p_2}]$ and $V=[v_1,\ldots,v_{p_2}]$. Since $\mathscr{R}(V)=M_2$, any orthonormal basis of $M_1\cap\mathscr{R}^{\perp}(U)$, say $\widehat{U}$, is orthogonal to M_2. Then the columns of $U_1=[U,\widehat{U}]$ are orthonormal and span M_1. The columns of U_1 and V are biorthogonal, that is

$$G(U_1,V)=\begin{bmatrix}\Gamma\\0\end{bmatrix}\quad(\Gamma=\mathrm{diag}(\cos\theta_i)) \tag{17}$$

Remark 10
It is the restriction $\cos\theta_k=\langle u_k,v_k\rangle$ of Definition 4 or 5 that guarantees that Γ has non-negative real entries. Without this restriction Γ may become complex.

From relation (17) we immediately obtain

Theorem 11 (*Afriat* [*7, 23*], *Gutmann and Shepp* [*24*])
In any pair of subspaces M_1 and M_2 there exist orthonormal bases $\{u_i\}_{i=1}^{p_1}$ and $\{v_i\}_{i=1}^{p_2}$ such that $\langle u_i,v_i\rangle\geqslant 0$ and $\langle u_i,v_j\rangle=0$ if $i\neq j$.

Following Watkins [17] we show that the SVD approach of Björck and Golub [9] is a direct consequence of the recursive definitions.

Lemma 12
Let $Q_i\in\mathbb{C}^{n\times p_i}$ be any matrix having orthonormal columns that span M_i $(i=1,2)$. Then there exists unitary matrices $F_i\in\mathbb{C}^{p_i\times p_i}$ $(i=1,2)$ such that $Q_1=U_1F_1$ and $Q_2=VF_2$.

DOI: 10.1002/nla

Proof
There is a non-singular matrix $F_1 \in \mathbb{C}^{p_1 \times p_1}$ such that $Q_1 = U_1 F_1$. Since the columns of Q_1 are orthonormal,

$$G(Q_1, Q_1) = G(U_1 F_1, U_1 F_1) = F_1^{\mathrm{H}} G(U_1, U_1) F = F_1^{\mathrm{H}} F_1 = I$$

The rest of claim follows similarly. □

We can observe now that

$$G(Q_1, Q_2) = F_1^{\mathrm{H}} G(U_1, V) F_2 = F_1^{\mathrm{H}} \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} F_2 \tag{18}$$

is a singular value decomposition with the singular values $\{\cos \theta_i\}_{i=1}^{p_2}$. Since Γ is independent of the choice of Q_1 and Q_2, we can determine the principal angles and vectors between M_1 and M_2 as follows.

Theorem 13
Let the columns of $Q_1 \in \mathbb{C}^{n \times p_1}$ and $Q_2 \in \mathbb{C}^{n \times p_2}$ be orthonormal bases for M_1 and M_2, respectively. Let

$$G(Q_1, Q_2) = Y \Sigma Z^{\mathrm{H}}, \quad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{p_2}) \in \mathbb{R}^{p_1 \times p_2} \tag{19}$$

be a singular value decomposition, where $Y \in \mathbb{C}^{p_1 \times p_1}$ and $Z \in \mathbb{C}^{p_2 \times p_2}$ are unitary. Then the principal angles and principal vectors between the subspaces M_1 and M_2 are given by

$$\cos(\theta_i) = \sigma_i, \quad u_i = Q_1 Y e_i, \quad v_i = Q_2 Z e_i \quad (i = 1, \ldots, p_2) \tag{20}$$

For real vector spaces with the standard inner product and subspaces of the same dimension Watkins [17] derived the latter result from Definition 2 with a different technique.

If we take the inner product $\langle x, y \rangle_W = y^{\mathrm{H}} W x$, where W is Hermitian and positive definite, then $G(Q_1, Q_2) = Q_1^{\mathrm{H}} W Q_2$. This gives an easy extension of Theorem 2.8 of Argentati [18], which is already a generalization of Björck and Golub [9] (see also Reference [16]). Stability analysis of SVD based algorithms for computing the principal angles are given by Björck and Golub [9], Argentati [18], Knyazev and Argentati [19].

4. OTHER ANGLE DEFINITIONS AND CONCEPTS

There are other concepts for the angle between subspaces. The product angle (cosine) ϕ between M_1 and M_2 is defined by

$$\cos \phi = \prod_{i=1}^{p_2} \cos \theta_i \tag{21}$$

where θ_i's are the principal angles (see, e.g. References [2, 5, 25]). The product cosine and the similarly defined product sine are intensively studied by Miao and Ben-Israel [26, 27].

The following two angle concepts are defined for Hilbert spaces (see Reference [22]).

DOI: 10.1002/nla

Definition 14 (*Friedrichs*)
The angle between the subspaces M_1 and M_2 of a Hilbert space H is the angle $\alpha(M_1, M_2)$ in $[0, \pi/2]$ whose cosine is given by

$$c(M_1, M_2) = \sup\{|\langle x, y\rangle| \; |x \in M_1 \cap (M_1 \cap M_2)^{\perp}, \|x\| \leqslant 1, \\ y \in M_2 \cap (M_1 \cap M_2)^{\perp}, \|y\| \leqslant 1\} \tag{22}$$

Definition 15 (*Dixmier*)
The minimal angle between the subspaces M_1 and M_2 is the angle $\alpha_0(M_1, M_2)$ in $[0, \pi/2]$ whose cosine is defined by

$$c_0(M_1, M_2) = \sup\{|\langle x, y\rangle| \; |x \in M_1, \|x\| \leqslant 1, y \in M_2, \|y\| \leqslant 1\} \tag{23}$$

The two definitions are clearly different if $M_1 \cap M_2 \neq \{0\}$ and agree, otherwise. If $H = \mathbb{C}^n$, then $\alpha_0 = \theta_1$ and $\alpha = \theta_{k+1}$ provided that $\dim(M_1 \cap M_2) = k$. Ipsen and Meyer [28] give interesting characterizations of the minimal angle for complementary subspaces of $\mathbb{R}^n$. An interesting application of the minimal angle is given in Reference [29] (see also Reference [22]).

The principal angles themselves can be defined with eigenvalues as well. The following theorem is an extension of Zassenhaus' classical result [8] (see also References [5, 30]).

Theorem 16 (*Ben-Israel* [*12*])
Let $M_1, M_2 \subset \mathbb{C}^n$ be subspaces with $p_i = \dim(M_i)$ $(i = 1, 2)$ and let U and V be two matrices whose columns span M_1 and M_2, respectively. Then the first $\min\{p_1, p_2\}$ eigenvalues of the matrix

$$U^+VV^+U \tag{24}$$

are the squares of the cosines of the principal angles between the subspaces M_1 and M_2 provided that the eigenvalues are given in descending order.

Here the non-zero eigenvalues of U^+VV^+U and VV^+UU^+ are the same and one can identify $P_2 = VV^+$ as the orthogonal projection onto M_2 and $P_1 = UU^+$ as the orthogonal projection onto M_1 according to the Penrose conditions on the pseudoinverse. Ben-Israel's theorem states

$$P_2P_1v_i = (\cos^2\theta_i)v_i \quad \text{or} \quad P_1P_2u_i = (\cos^2\theta_i)u_i \tag{25}$$

and these relations were already stated by Corollary 9 in a more general setting. As $v_i \in M_2$, another equivalent form of the first equation is $P_2P_1P_2v_i = (\cos^2\theta_i)v_i$ because $P_2v_i = v_i$. The other equation can be handled similarly. These relations show the connection to the next definition (see, e.g. Reference [13, Problems 559, 560] or Bhatia [11, Exercise VII.1.10, p. 201]):

Definition 17
If P_1 and P_2 are finite dimensional orthogonal projections, then the principal angles between them (or, equivalently, between their ranges as subspaces) is defined as the arccos of the square root of the eigenvalues (counted according to multiplicity) of the positive (self-adjoint) finite rank operator $P_2P_1P_2$.

In general, formulas based on eigenvalues instead of singular values should be avoided in computations, since otherwise accuracy may be lost. An exception may exists however. It is shown

DOI: 10.1002/nla

in Reference [31] that the principal angles can be recovered from the spectrum of $P_1 + P_2$, where P_1 and P_2 denote the orthogonal projection onto M_1 and M_2, respectively.

ACKNOWLEDGEMENTS

The authors are indebted to the referees for their useful observations and suggestions.

REFERENCES

1. Jordan C. Essai sur la géométrie à n dimensions. *Buletin de la Société Mathématique de France* 1875; **3**:103–174.
2. Sommerville D. *An Introduction to the Geometry of N Dimensions*. Dover: New York, 1958.
3. Paige C, Wei M. History and generality of the CS decomposition. *Linear Algebra and its Applications* 1994; **208/209**:303–326.
4. Davis C, Kahan W. The rotation of eigenvectors by a perturbation. III. *SIAM Journal on Numerical Analysis* 1970; **7**(1):1–46.
5. Risteski I, Trencevski K. Principal values and principal subspaces of two subspaces of vector spaces with inner product. *Beiträge zur Algebra und Geometrie* 2001; **42**(1):289–300.
6. Hotelling H. Relations between two sets of variates. *Biometrika* 1936; **28**:321–377.
7. Afriat S. Orthogonal and oblique projectors and the characteristics of pairs of vector spaces. *Proceedings of the Cambridge Philosophical Society* 1957; **53**:800–816.
8. Zassenhaus H. Angles of inclination in correlation theory. *The American Mathematical Monthly* 1964; **71**:218–219.
9. Björck A, Golub G. Numerical methods for computing angles between linear subspaces. *Mathematics of Computation* 1973; **27**(123):579–594.
10. Stewart G, Sun J. *Matrix Perturbation Theory*. Academic Press: New York, 1990.
11. Bhatia R. *Matrix Analysis*. Springer: New York, 1997.
12. Ben-Israel A. On the geometry of subspaces in Euclidean n-spaces. *SIAM Journal on Applied Mathematics* 1967; **15**(5):1184–1198.
13. Kirillov A, Gvishiani A. *Theorems and Problems in Functional Analysis*. Springer: Berlin, 1982.
14. Scharnhorst K. Angles in complex vector spaces. *Acta Applicandae Mathematicae* 2001; **69**(1):95–103.
15. Davis P. *Interpolation and Approximation*. Dover: New York, 1975.
16. Golub G, Van Loan C. *Matrix Computations*. John Hopkins University Press: Baltimore, 1983.
17. Watkins D. *Fundamentals of Matrix Computations*. Wiley: New York, 1991.
18. Argentati M. Principal angles between subspaces as related to Rayleigh quotient and Rayleigh Ritz inequalities with applications to eigenvalue accuracy and an eigenvalue solver. *Ph.D. Thesis*, University of Colorado, Denver, 2003.
19. Knyazev A, Argentati M. Principal angles between subspaces in an A-based scalar product: algorithms and perturbation estimates. *SIAM Journal on Scientific Computing* 2002; **23**(6):2009–2041.
20. Rakocevic V, Wimmer H. A variational characterization of canonical angles between subspaces. *Journal of Geometry* 2003; **78**:122–124.
21. Dixmier J. Position relative de deux variétés linéaires fermées dans un espace de Hilbert. *Revue Scientifique* 1948; **86**:387–399.
22. Deutsch F. The angle between subspaces of a Hilbert space. In *Approximation Theory*, *Wavelets and Applications*, Singh S (ed.). Kluwer: Dordrecht, 1995; 107–130.
23. Afriat S. On the latent vectors and characteristic values of products of pairs of symmetric idempotents. *The Quarterly Journal of Mathematics*, *Oxford Second Series* 1956; **7**:76–78.
24. Gutmann S, Shepp L. Orthogonal bases for two subspaces with all mutual angles acute. *Indiana University Mathematics Journal* 1978; **27**(1):79–90.
25. Gluck H. Higher curvatures of curves in Euclidean space, II. *The American Mathematical Monthly* 1967; **74**:1049–1056.
26. Miao J, Ben-Israel A. On principal angles between subspaces in $\mathbb{R}^n$. *Linear Algebra and its Applications* 1992; **171**:81–98.
27. Miao J, Ben-Israel A. Product cosines of angles between subspaces. *Linear Algebra and its Applications* 1996; **237/238**:71–81.

DOI: 10.1002/nla

28. Ipsen I, Meyer C. The angle between complementary subspaces. *The American Mathematical Monthly* 1995; **102**:904–911.
29. Achchab B, Axelsson O, Laayouni L, Souissi A. Strengthened Cauchy–Bunyakowski–Schwarz inequality for a three-dimensional elasticity system. *Numerical Linear Algebra with Applications* 2001; **8**:191–205.
30. Galántai A. *Projectors and Projection Methods*. Kluwer: Dordrecht, 2004.
31. Galántai A. Subspaces, angles and pairs of orthogonal projections. *Linear and Multilinear Algebra*, to appear.

DOI: 10.1002/nla