A Simple Insurance Model: Optimal Coverage and Deductible

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Abstract An insurance model, with realistic assumptions about coverage, deductible and premium, is studied. Insurance is shown to decrease the variance of the cost to the insured, but increase the expected cost, a tradeoff that places our model in the Markowitz mean–variance model.

Keywords Insurance · Optimal coverage · Deductible · Markowitz mean–variance model

1 Introduction

In an ever tightening economy, uncertainty plays an increasing role. This led companies and researchers to a systematic study of risk and its impact, an effort that comes under the title of **Enterprise Risk Management** (**ERM**). There are several definitions of ERM (e.g., [3], [4], [14] and [21]), representing a wide spectrum of theory and applications. In this paper we concentrate on one aspect of ERM, namely the mitigation of risk by means of two basic parameters of insurance, coverage and deductible.

Insurance, in particular the problem of optimal coverage and deductible, has been widely studied in the economic literature, mainly through expected–utility (e.g., [1], [15], [19], [16], [17], [10]) and stochastic dominance ([7], [18]). For a detailed exposition see [6] and references therein.

A common feature in the literature is that insurance coverage is unbounded, so the purchase of insurance will fully cover any loss above a deductible. Recent works ([5], [22]) consider insurance problems with upper bounds on coverage [5], or reimbursement [22]. Both of these papers use the expected utility criterion.

A purchase of insurance must increase the expected cost to the insurance buyer (the **insured**), for otherwise the insurance seller (the **insurer**) cannot remain solvent. This expected cost increase should be compensated by a decrease in risk, that we establish here directly without using expected utility.

We consider an insurance model, suggested by automobile insurance, where coverage is bounded, and the problem of the insured is to determine the optimal levels of **coverage** C and **deductible** D. Our model, described in § 2, assumes the premium to be a function of C and D, and that loss can exceed the coverage. The premium function is based on real–world insurance listings. We establish a lower bound (17) on the premium, under the assumption that the insurer will not incur an expected loss. The insurance budget, and its implication for the coverage and deductible, are discussed in § 3.

First–order optimality conditions for the insured are derived in § 4, and show a monotone dependence of the optimal deductible on the coverage. In particular, a zero deductible may be optimal. The optimality conditions of the insurer are the same, but what is best to the former is worst for the latter, and consequently the insurer may not offer the policy desired by the insured, who must then settle for a non–optimal plan.

Our main results, Theorem 1 and Corollary 2 in § 5, show that the variance of the insured cost, as a function of C and D, is decreasing in C and increasing in D. This tradeoff between the expected cost increase and the variance decrease places our model within the framework of the Markowitz mean–variance model, see § 6.

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Coverage (C)	Premium (%)	Deductible (D)	Premium (%)
\$1 million	134%		
\$250,000	100%	\$250	100%
\$150,000	97%	\$500	87%
\$75,000	95%	\$1,000	80%
\$50,000	92%	\$2,000	77%
\$15,000	87%	\$2,500	74%

Table 1 PIP premium as function of coverage and deductible, Source: NJ Auto Insurance Buyer's Guide, [13]

2 The model

Insurance is a way of mitigating the effects of a **loss**, an adverse event such as accident, fire, illness, etc. We model loss by a random variable $\mathbf{X}$ with a distribution function $F(x) = \text{Prob}\{\mathbf{X} \le x\}$, and a density function $f(x) = F'(x)$. We assume that $F(0) = 0$ (i.e. the possible losses are nonnegative), and $\mathbf{X}$ has finite moments of all orders, in particular expected value $\mathrm{E}\,\mathbf{X}$, and variance $\mathrm{Var}\,\mathbf{X}$.

A loss x induces a **cost** $L(x)$, where $L(\cdot)$ is a nondecreasing function. In the uninsured case we assume

$$L(x) = x, \;\; x \ge 0. \tag{1}$$

Alternatively, the **insured** (a person, or a company) may buy from the **insurer** (an insurance company) an insurance **policy** in order to control the cost of $\mathbf{X}$, or make it more predictable. Indeed, the uninsured case (1) does not exist for automobile drivers, who are required by law to buy some insurance.

A policy is characterized by the **reimbursement** $I(x)$ from the insurer for each possible loss x, and the **premium** p paid to the insurer. In turn, the reimbursement $I(x)$ and the premium p depend, as follows, on two parameters, the **deductible** $D \ge 0$ and the **coverage** $C \ge D$.

Reimbursement. For each possible loss x the reimbursement $I(x|C, D)$ is given by,

$$I(x|C,D) = \max\{x - D, 0\} - \max\{x - C, 0\} = \begin{cases} 0, & \text{if } 0 \le x < D; \\ x - D, & \text{if } D \le x < C; \\ C - D, & \text{if } C \le x; \end{cases} \tag{2}$$

see also [5, Proposition 1] and [22, eq. (3)]. Such an insurance policy is called here a $\{C, D\}$–**policy**. The uninsured case $I(x) = 0$ corresponds to $C = D$.

Premium. The premium $p(C, D)$ is assumed increasing in the coverage C and decreasing in the deductible D (policies with higher deductible are cheaper). The following example from auto–insurance in New Jersey shows the premium as a function of C and D, note that not all values are permissible, in particular there are minimum coverage and deductible.

Example 1 (**Auto–insurance**). Standard auto insurance policies in the State of New Jersey contain Personal Injury Protection (PIP), with premiums depending on the deductible D and coverage C as shown in Table 1. The table lists 6 possible coverages and 5 deductibles; the required minimums are $C = \$15{,}000$, and $D = \$250$. The standard premium (100% in Table 1) is for $C = \$250{,}000$ and $D = \$250$, and changes depending on the insurance company.

There is also a 20% co–payment for losses between the deductible selected (by the buyer) and \$5,000. This effectively raises the deductible.

For example, Jane chose the minimum coverage $C = \$15{,}000$ (resulting in a reduction of 13% from the premium for the standard coverage $C = \$250{,}000$) and a deductible $D = \$2{,}500$ (getting a 26% reduction from the standard premium for $D = \$250$). She pays $0.87 \cdot 0.74 = 0.64$ times the standard premium.

If Jane has an accident resulting in \$10,000 of medical expenses, she pays the first \$2,500 as deductible, and \$500 (20% of the \$2,500 that is left of the first \$5,000), and the insurance pays the remaining \$7,000.

Analyzing the data of Table 1, that is typical for car–insurance, we approximate the premium $p(C, D)$ as a product of two functions, each depending on one of the variables,

$$p(C, D) = \alpha(D)\,\beta(C), \tag{3}$$

where $\alpha(D)$ is a positive, monotonely decreasing function of D, and the function $\beta(C)$ is positive and monotonely increasing. The data suggests that the cost of coverage $\beta(C)$ is affine in C, say

$$\beta(C) = \beta(0) + m\,C, \;\; \text{for some } \beta(0) \ge 0,\; m > 0, \tag{4}$$

and the premium is therefore

$$p(C, D) = \alpha(D)\,(\beta(0) + m\,C). \tag{5}$$

This is a departure from the common assumption, in the insurance literature, that the premium is proportional to the expected reimbursement, see, e.g., [5].

Zero coverage is typically not allowed, (for example, in New Jersey the minimum coverage for auto–insurance is \$15,000, see Table 1), and therefore the term $\beta(0)$ in (4) is just a device for expressing $\beta(C)$ in its domain.

Example 2 The data of Table 1 is interpolated by (3) with

$$\alpha(D) = a\,D^{-b},\ a = 1.94,\ b = 0.123,\ (250 \le D \le 2500), \tag{6a}$$

$$\beta(C) = \beta(0) + m\,C,\ \beta(0) = 0.9,\ m = 4.5 \cdot 10^{-7},\ (15,000 \le C \le 10^6), \tag{6b}$$

up to a multiplicative constant, and plotted in Figure 1, where darker color indicates higher premium. The error of the interpolations (6a)–(6b) at the given points is $O(10^{-2})$.

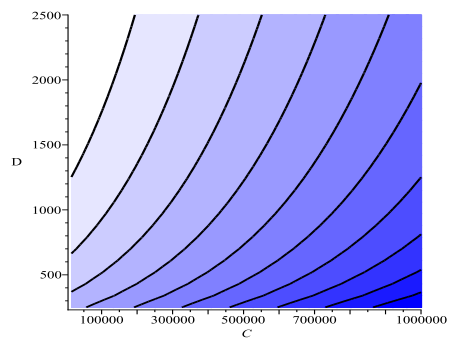


Fig. 1 Contour plot of the PIP premium in Table 1

Example 3 ($\alpha(\cdot)$ **exponential**). The monotone dependence of the premium on the deductible can be modeled by the exponential function,

$$\alpha(D) = \alpha(0)\,e^{-\delta D},\ D \ge 0, \tag{7}$$

for some $\delta > 0$ and $0 < \alpha(0) < 1$.

Example 4 (**Life insurance**). In life insurance there is no deductible ($D = 0$), the premium depends on the insured's age and other factors, and is linear in the coverage C, which usually can be bought in multiples of a standard amount, say \$10,000. The underlying random variable $\mathbf{X}$ is discrete with two values (survival, say $x = 0$, or death, $x = 1$) and the reimbursement (2) simplifies to

$$I(x) = \begin{cases} 0, & \text{if } x = 0; \\ C, & \text{if } x = 1. \end{cases}$$

Cost and profit. For each possible loss x, the **cost** to the insured is

$$L(x|C, D) = p(C, D) + x - I(x|C, D) = p(C, D) + \begin{cases} x, & \text{if } 0 \le x < D; \\ D, & \text{if } D \le x < C; \\ x + D - C, & \text{if } C \le x; \end{cases} \tag{8}$$

see Fig. 2(a). The corresponding **profit** of the insurer is

$$R(x|C, D) = p(C, D) - I(x|C, D) = p(C, D) - \begin{cases} 0, & \text{if } x < D; \\ x - D, & \text{if } D \le x < C; \\ C - D, & \text{if } C \le x; \end{cases} \tag{9}$$

see Fig. 2(b) which shows the **loss** $-R(x|C, D)$ of the insurer.

The expected insured cost is by (8) and (3),

$$\mathrm{E}\,L(\mathbf{X}|C, D) = \alpha(D)\,\beta(C) + \mathrm{E}\,\mathbf{X} + \int_D^C (D - x)\,f(x)\,dx + (D - C)\int_C^\infty f(x)\,dx. \tag{10}$$

Similarly, the expected profit of the $\{C, D\}$–insurer is,

$$\mathrm{E}\, R(\mathbf{X}|C, D) = \alpha(D)\, \beta(C) - \int_D^C (x - D)\, f(x)\, dx - (C - D) \int_C^\infty f(x)\, dx. \tag{11}$$

It follows from (8) and (9) that

$$L(x|C, D) = x + R(x|C, D), \text{ for all } x \geq 0,$$

and consequently, the expected cost of the insured equals the (uninsured) expected loss plus the expected profit of the insurance company,

$$\mathrm{E}\, L(\mathbf{X}|C, D) = \mathrm{E}\, \mathbf{X} + \mathrm{E}\, R(\mathbf{X}|C, D), \tag{12}$$

as can be seen also from (10)–(11).

If the insurance company is profitable, as is generally the case, an insurance policy as above does not reduce the expected cost to the insured – if it did, it would qualify as a "free lunch".

A $\{C, D\}$–policy is called **actuarially fair** if the expected profit $\mathrm{E}\, R(\mathbf{X}|C, D)$ is zero, i.e. if

$$\mathrm{E}\, I(\mathbf{X}|C, D) = p(C, D). \tag{13}$$

Let $\bar{F}(x) = 1 - F(x) = \int_x^\infty f(u)du$. Then,

$$\begin{aligned}
\int_D^C \bar{F}(x)dx &= \int_D^C (1 - F(x))dx \\
&= C - D - \int_D^C F(x)dx \\
&= C - D + \int_D^C xf(x)dx - xF(x)\Big]_D^C \quad \text{(using integration by parts)} \\
&= \int_D^C xf(x)dx + C(1 - F(C)) - D(1 - F(D)) \\
&= \int_D^C (x - D)f(x)dx + \int_C^\infty (C - D)f(x)dx. \qquad (14)
\end{aligned}$$

Proposition 1 *For any pair (C, D), an insurance plan is actuarially fair if the premium $\alpha(D)\, \beta(C)$ satisfies*

$$\alpha(D)\, \beta(C) = \int_D^C \bar{F}(x)\, dx. \tag{15}$$

Proof For $\mathrm{E}\, R(\mathbf{X}|C, D) = 0$ we need, by (11),

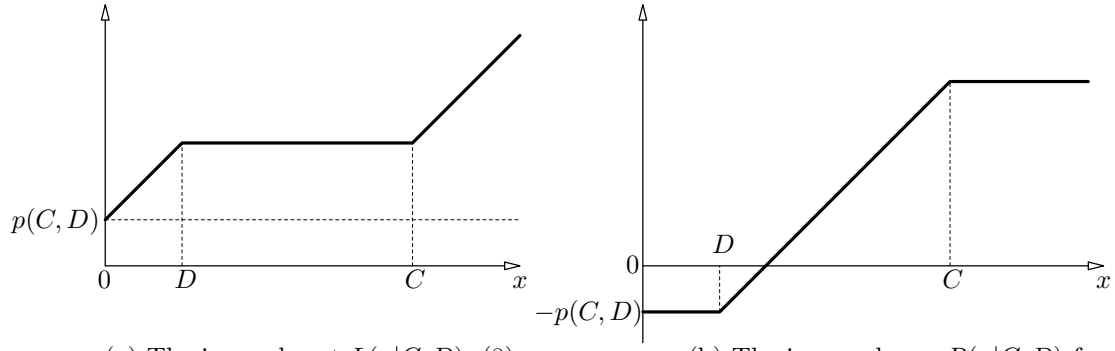
$$\alpha(D)\, \beta(C) = \int_D^C (x - D)f(x)dx + \int_C^\infty (C - D)f(x)dx, \tag{16}$$

and the proof follows by noting that the right sides of (14) and (16) are equal.

Remark 1 Assuming the expected profit to the insurer is nonnegative, it follows from Proposition 1 that, for any pair (C, D), the insurance premium satisfies

$$\alpha(D)\, \beta(C) \geq \int_D^C \bar{F}(x)\, dx, \tag{17}$$

a condition that the premium must satisfy in order to guarantee an expected profit (or break-even) for the insurer.



(a) The insured cost $L(x|C,D)$, (8) (b) The insurer loss $-R(x|C,D)$ from (9)

Fig. 2 The costs of the insured and insurer for given C, D

3 The insurance budget

The **budget** B available for buying insurance imposes conditions on the coverage C and deductible D. Assume first that all the budget is spent. Then, by (3),

$$\alpha(D)\,\beta(C) = B, \tag{18}$$

an equation that can be solved for D as a function of C,

$$D = \alpha^{-1}(B/\beta(C)), \tag{19}$$

an increasing function, i.e. buying, with a fixed budget, more coverage makes it necessary to increase the deductible. Because D is nonnegative, the smallest possible coverage corresponds to $D = 0$ in (18), i.e., C must satisfy,

$$C \geq \beta^{-1}(B/\alpha(0)), \tag{20}$$

the right–hand side is the smallest possible coverage (corresponding to zero deductible) for the given budget B.

Example 5 ($\boldsymbol{\alpha}(\cdot)$ **exponential**). In the exponential case (7), the deductible (19) becomes

$$D = -\frac{1}{\delta}\,\log\left(\frac{B}{\alpha(0)\,\beta(C)}\right), \tag{21}$$

where $D \geq 0$ if (20) holds, with $D = 0$ if $B = \alpha(0)\,\beta(C)$.

Some of the curves (21) are shown in Fig. 3 for $\beta(C) = C$, $\alpha(0) = 0.6$ and $\delta = 0.005$. Higher curves (i.e. greater deductible) correspond to lower budgets B.

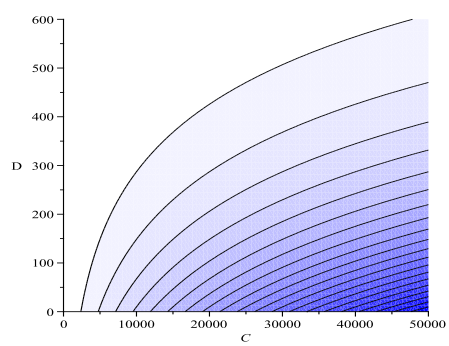


Fig. 3 Illustration of Example 5

4 First order optimality conditions

We assume throughout that the support of the distribution F is not contained in $[D, C]$, i.e.,

$$\int_D^C f(x)\,dx < 1. \tag{22}$$

Indeed if $\int_D^C f(x)\,dx = 1$ then by (8) the insured cost $L(\mathbf{X}|C,D) = p(C,D) + D$ with certainty. We also assume the premium is given by (5).

The expected value (10) is unchanged if there is a positive probability that no loss occurs (i.e. $x = 0$). Indeed, $\mathrm{E}\,\mathbf{X}$ requires then a Stieltjes integral, but the contribution of the value $x = 0$ is zero.

Denote $\mathrm{E}\,L(\mathbf{X}|C,D)$ by $U(C,D)$. Its respective derivatives give rise to the first order optimality conditions

$$\begin{aligned}\frac{\partial U}{\partial C} &= \alpha(D)\,\beta'(C) - \int_C^\infty f(x)\,dx = 0\\ &= m\,\alpha(D) - \int_C^\infty f(x)\,dx = 0,\ \text{by (4)}, \qquad (23a)\\ \frac{\partial U}{\partial D} &= \alpha'(D)\,\beta(C) + \int_D^\infty f(x)\,dx = 0,\ \text{if } D > 0, \qquad (23b)\\ \frac{\partial U}{\partial D} &= \alpha'(D)\,\beta(C) + \int_D^\infty f(x)\,dx \geq 0,\ \text{if } D = 0. \qquad (23c)\end{aligned}$$

Remark 2 Writing (23a) as

$$\int_C^\infty f(x)\,dx = m\,\alpha(D), \qquad (24)$$

we note that the left side of (24) is decreasing in C, and for fixed C, the right side is decreasing in D, by the assumption on $\alpha(\cdot)$. It follows that C and D move in the same direction, a higher deductible D corresponds to a higher coverage C.

If D is a differentiable function of C we get from (23a)

$$\frac{dD}{dC} = -\frac{\dfrac{\partial^2 U}{\partial C^2}}{\dfrac{\partial^2 U}{\partial D\,\partial C}} = -\frac{f(C)}{m\,\alpha'(D)}, \qquad (25)$$

which is positive by the assumptions on the premium function $\alpha(\cdot)$.

Remark 3 The first–order optimality conditions (23a)–(23b),

$$\mathrm{Prob}\,\{\mathbf{X} \geq C\} = \int_C^\infty f(x)\,dx = m\,\alpha(D) \qquad (26a)$$

$$\mathrm{Prob}\,\{\mathbf{X} \geq D\} = \int_D^\infty f(x)\,dx = -\,\alpha'(D)\,\beta(C) \qquad (26b)$$

can be solved for D and C. For $D = 0$ the optimality condition (23c) gives the inequality

$$\alpha'(0)\,\beta(C) + 1 \geq 0,\ \text{or by (4)},$$

$$C \leq -\frac{1}{m}\left(\frac{1}{\alpha'(0)} + \beta(0)\right), \qquad (27)$$

an upper bound on the coverage, corresponding to a zero deductible. There may be external upper bound $C_{\max}$ on coverage, say

$$C \leq C_{\max}. \qquad (28)$$

For example, in the auto–insurance data of Example 1, $C_{\max} = \$1$ million.

Corollary 1 *A sufficient condition for a positive deductible is*

$$\int_{-\frac{1}{m}\left(\frac{1}{\alpha'(0)}+\beta(0)\right)}^\infty f(x)\,dx > m\,\alpha(0). \qquad (29)$$

Proof It follows from Remark 2 that the lowest possibly optimal coverage $C(0)$ corresponds to $D = 0$. $C(0)$ is determined from (26a),

$$\int_{C(0)}^\infty f(x)\,dx = m\,\alpha(0),$$

and in addition must satisfy (27). These two conditions are satisfied only if

$$\int_{-\frac{1}{m}\left(\frac{1}{\alpha'(0)}+\beta(0)\right)}^\infty f(x)\,dx \leq m\,\alpha(0), \qquad (30)$$

which is then a necessary condition for $D = 0$. The reverse inequality, (29), is therefore sufficient for a positive deductible.

Optimality conditions for the insurer. It follows from (12) that the first order optimality conditions are the same as those of the insured, which is to be expected since this is a zero–sum game, and the best outcome for one player is the worst for the other. It follows that the insurer may not offer the optimal policy desired by the insured, who must then settle for a non–optimal plan.

The next example illustrates, for the commonly occurring Gamma distribution (see [2], [8], [9], and [20]), the calculation of C and D satisfying the necessary optimality conditions.

Example 6 **The Gamma Distribution**. Let the random variable $\mathbf{X}$ have the **Gamma distribution** with **scale** $1/\lambda$ and **shape** $k \geq 1$,

$$f(x) = \frac{\lambda^k}{\Gamma(k)}\, x^{k-1}\, e^{-\lambda x},\ x \geq 0, \tag{31}$$

in particular,

$$\Gamma(k) = (k-1)!, \tag{32}$$

if k is integer, in which case (31) is the **Erlang distribution**. Let $\alpha(D)$ be given by (7),

$$\alpha(D) = \alpha(0)\, e^{-\delta D},\ D \geq 0.$$

Zero deductible. For $D = 0$ to be optimal we must have, by (30),

$$\int_S^\infty f(x)\, dx = \frac{\lambda^k}{\Gamma(k)} \int_S^\infty x^{k-1}\, e^{-\lambda x}\, dx = e^{-\lambda S} \sum_{i=0}^{k-1} (i!)^{-1} (\lambda S)^i \leq m\, \alpha(0),$$

where

$$S = -\frac{1}{m}\left(\frac{1}{\alpha'(0) + \beta(0)}\right). \tag{33}$$

Positive deductible. Equations (26a)–(26b) become

$$e^{-\lambda C} \sum_{i=0}^{k-1} (i!)^{-1} (\lambda C)^i = m\, \alpha(0)\, e^{-\delta D}, \tag{34a}$$

$$e^{-\lambda D} \sum_{i=0}^{k-1} (i!)^{-1} (\lambda D)^i = \alpha(0)\, \beta(C)\, \delta\, e^{-\delta D} \tag{34b}$$

or,

$$e^{\delta D - \lambda C} = \frac{m\, \alpha(0)}{\sum_{i=0}^{k-1} (i!)^{-1} (\lambda C)^i}, \tag{35a}$$

$$e^{(\delta - \lambda) D} = \frac{\alpha(0)\, \beta(C)\, \delta}{\sum_{i=0}^{k-1} (i!)^{-1} (\lambda D)^i}. \tag{35b}$$

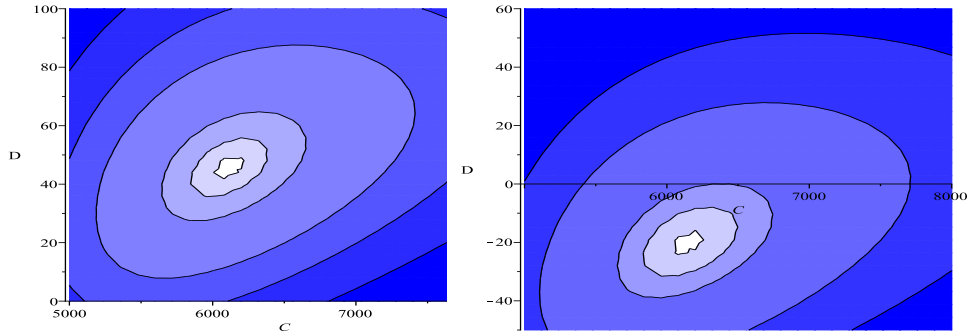
From (35a) we get D as a function of C,

$$D = \frac{1}{\delta}\left(\log\left(\frac{m\, \alpha(0)}{\sum_{i=0}^{k-1} (i!)^{-1} (\lambda C)^i}\right) + \lambda C\right) \tag{36}$$

which can be substituted in (35b) to give an equation for C.

Figure 4 gives some contour lines of the expected cost (10) for the Gamma distribution with $k = 2$ and $\lambda = 10^{-3}$. For the premium (7) we use $\delta = 10^{-2}$ and $\alpha(0) = 0.5$, and for (4) we use $\beta(0) = 50$. In Figure 4(a) the slope m in (4) is $m = 0.25$, and in Figure 4(b), $m = 0.125$.

In these figures, lighter shades correspond to lower costs, and the optimal pair $\{C, D\}$ is indicated by the brightest spot. Figure 4(b) illustrates that a zero deductible may be optimal, since the unconstrained minimum gives a negative deductible. The optimal deductible $D = 0$ lies on the closest contour (where $D \geq 0$) to the theoretical optimum.



(a) Example where the optimal deductible is positive (b) Example where the optimal deductible is zero

Fig. 4 Contour lines of (10), the expected cost

5 Variance

The insured cannot expect to find an insurance plan that will lower his expected cost, because by (12) this would entail an expected loss for the insurer. In this section we establish that insurance lowers the variance of the cost, which is a rationale for buying insurance.

We compute the variance of loss when insured, and compare to the variance when there is no insurance. Let $\mathbf{X}$ be the loss when there is no insurance, so we have

$$\operatorname{Var} \mathbf{X} = \mathrm{E}\mathbf{X}^2 - (\mathrm{E}\mathbf{X})^2. \tag{37}$$

For convenience we write (8) as

$$L(x|C,D) = p(C,D) + x + \phi(x|C,D), \tag{38}$$

where

$$\phi(x|C,D) = \begin{cases} 0, & \text{if } 0 \le x < D; \\ D - x, & \text{if } D \le x < C; \\ D - C, & \text{if } C \le x. \end{cases} \tag{39}$$

In particular, if $C = D$,

$$\phi(x|C,C) = 0, \;\; x \ge 0. \tag{40}$$

We assume that C, D are given, and do not write them if they are not needed explicitly, thus $p(C,D)$ is abbreviated p, $\phi(\mathbf{X}|C,D)$ abbreviated $\phi(\mathbf{X})$, $L(\mathbf{X}|C,D)$ is written $L(\mathbf{X})$, etc.

The variance of $L(\mathbf{X})$ is therefore

$$\begin{aligned} \operatorname{Var} L(\mathbf{X}) &= \operatorname{Var} \mathbf{X} + \operatorname{Var} \phi(\mathbf{X}) + 2 \operatorname{Cov}(\mathbf{X}, \phi(\mathbf{X})) \\ &= \operatorname{Var} \mathbf{X} + \operatorname{Var} \phi(\mathbf{X}) + 2 \left[\mathrm{E}\left(\mathbf{X}\, \phi(\mathbf{X})\right) - \mathrm{E}\, \mathbf{X}\, \mathrm{E}\, \phi(\mathbf{X})\right]. \end{aligned} \tag{41}$$

The following theorem relates the variance $\operatorname{Var} L(\mathbf{X}|C,D)$ to the coverage C and deductible D.

Theorem 1 *Let F, C, D satisfy (22),*

$$\int_D^C f(x)\, dx < 1.$$

Then

$$\frac{\partial}{\partial C} \mathit{Var}\; L(\mathbf{X}|C,D) < 0, \tag{42}$$

and

$$\frac{\partial}{\partial D} \mathit{Var}\; L(\mathbf{X}|C,D) > 0. \tag{43}$$

The proof is given in Appendix A.

The variance $\operatorname{Var} L(\mathbf{X}|C,D)$ is thus a decreasing function of C (for fixed D) and an increasing function of D (for fixed C):

Corollary 2
(a) Let $0 < D < C_1 < C_2$*, and let*

$$\int_D^{C_2} f(x)\,dx < 1.$$

Then

$$Var\ L(\mathbf{X}|C_2, D) < Var\ L(\mathbf{X}|C_1, D) < Var\ \mathbf{X}. \tag{44}$$

(b) Let $0 < D_1 < D_2 < C$*, and let*

$$\int_{D_1}^{C} f(x)\,dx < 1.$$

Then

$$Var\ L(\mathbf{X}|C, D_1) < Var\ L(\mathbf{X}|C, D_2) < Var\ \mathbf{X}. \tag{45}$$

Proof It follows from (40) that for $C = D$,

$$\text{Var } L(\mathbf{X}|C, C) = \text{Var } \mathbf{X}. \tag{46}$$

(a) The left inequality in (44) follows from (42) and the right inequality from (46), writing Var $\mathbf{X}$ as Var $L(\mathbf{X}|D, D)$. (b) is similarly proved.

Using (A9) in Appendix A,

$$\frac{1}{2}\frac{\partial}{\partial C}\text{Var L}(\mathbf{X}) = \int_C^{\infty} f(x)\,dx \left[C\int_0^C f(x)\,dx - D\int_0^D f(x)\,dx + \int_0^D xf(x)\,dx + \int_C^{\infty} xf(x)\,dx\right] - \int_C^{\infty} xf(x)\,dx,$$

approaching zero as $C \to \infty$ provided

$$\lim_{C\to\infty}\int_C^{\infty} xf(x)\,dx = 0 \tag{47}$$

holds. Our assumption that $\mathrm{E}\,\mathbf{X}$ is finite precludes heavy–tailed distributions such as the log–Cauchy distribution, where (47) does not hold.

Example 7 Consider the Gamma distribution (31) with $k = 2$ and $\lambda = 10^{-3}$. Figure 5 shows contour lines of Var $L(\mathbf{X})$ in the (C, D)–plane. For convenience we represent these contour lines as the curves $\beta - \text{Var}\,L(\mathbf{X}) = 0$ for several values of β. The red line is $D = C$, corresponding to the uninsured case (where the variance is Var $\mathbf{X}$). Lower curves correspond to lower values of Var $L(\mathbf{X})$. These contour lines illustrate the flattening out of the variance, and the fact that above a certain threshold value, increasing the coverage C has a negligible effect on the variance.

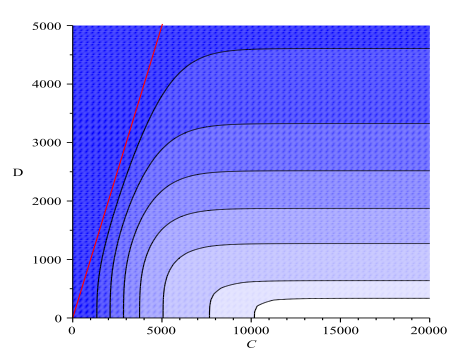


Fig. 5 Contour lines of (41), the variance

6 Mean-Variance Analysis

A $\{C, D\}$–insurance plan was shown to reduce the variance of the costs incurred by the insured, while increasing their expected value. This leads naturally to the **Markowitz mean–variance model**, [11]–[12], that is adapted to give the objective

$$\min_{C,D}(\mathrm{E}\mathbf{Z} + \lambda\,\mathrm{Var}\mathbf{Z}), \tag{48}$$

where $\mathbf{Z}$ is the random cost in question. The parameter λ expresses the tradeoff between the mean (expected cost) and variance, and represents the decision maker's attitude towards risk.

In (48), $\mathbf{Z} = L(\mathbf{X}|C, D)$ if insurance is purchased, and $\mathbf{Z} = \mathbf{X}$ in the uninsured case. There is a critical value λ^* where a Markowitzian decision maker would be indifferent between buying or not buying insurance, and that value is given by

$$\mathrm{E}\,L(\mathbf{X}|C, D) + \lambda^*\,\mathrm{Var}\,L(\mathbf{X}|C, D) = \mathrm{E}\,\mathbf{X} + \lambda^*\,\mathrm{Var}\,\mathbf{X}$$

or

$$\lambda^* = \frac{\mathrm{E}\,L(\mathbf{X}|C, D) - \mathrm{E}\,\mathbf{X}}{\mathrm{Var}\,\mathbf{X} - \mathrm{Var}\,L(\mathbf{X}|C, D)}. \tag{49}$$

For $\lambda > \lambda^*$ it is optimal to buy insurance, and for $\lambda < \lambda^*$ insurance cannot be justified in the Markowitz model, although - in the case of automobile insurance - it is required by law.

7 Conclusion

We studied an insurance model, suggested by automobile insurance, and showed that the variance of the cost to the insured is a decreasing function of the coverage and an increasing function of the deductible. Since insurance increases the expected cost, there is a trade–off between the cost increase and the risk decrease, suggesting a Markowitzian approach to modeling the problem of optimal insurance coverage – minimizing a weighted sum of expectation and variance.

Our model considered a single ("typical") customer, but in practice an insurer has many customers with different loss distributions, and their totality determines the expected value and variance of the insurer profit. The insurer thus has an analogous trade–off between expected profit and variance, and he can aim for the "right" customers by designing policies that will attract them. We defer this for future study.

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Appendix A: Proof of Theorem 1

We prove (42)

$$\frac{\partial}{\partial C}\text{Var } L(\mathbf{X}|C,D) < 0$$

given that (22) holds,

$$\int_D^C f(x)\,dx < 1,$$

which implies that either

$$\int_C^\infty f(x)dx > 0, \tag{A1}$$

or

$$\int_0^D f(x)dx > 0, \tag{A2}$$

hold.

From (39),

$$\text{E}\phi(\mathbf{X}|C,D) = \int_D^C (D-x)f(x)\,dx + (D-C)\int_C^\infty f(x)\,dx, \tag{A3}$$

and

$$\begin{aligned}\text{Var } \phi(\mathbf{X}|C,D) &= \text{E}\phi^2(\mathbf{X}|C,D) - (\text{E}\phi(\mathbf{X}|C,D))^2\\ &= \int_D^C (D-x)^2 f(x)\,dx + (D-C)^2\int_C^\infty f(x)\,dx - \left(\int_D^C (D-x)f(x)\,dx + (D-C)\int_C^\infty f(x)\,dx\right)^2.\end{aligned} \tag{A4}$$

By (41) the partial derivative of Var $L(\mathbf{X})$ w.r.t. C is

$$\frac{\partial}{\partial C}\text{Var } L(\mathbf{X}) = \frac{\partial}{\partial C}\text{ Var } \mathbf{X} + \frac{\partial}{\partial C}\text{ Var } \phi(\mathbf{X}) + 2\,\frac{\partial}{\partial C}\Big(\text{E}[\mathbf{X}\phi(\mathbf{X})] - \text{E}\mathbf{X}\,\text{E}\phi(\mathbf{X})\Big). \tag{A5}$$

We calculate next the three derivatives on the right side.

(a) Clearly,

$$\frac{\partial}{\partial C}\text{ Var } \mathbf{X} = 0. \tag{A6}$$

(b)

$$\begin{aligned}\frac{\partial}{\partial C}\text{ Var } \phi(\mathbf{X}) &= (2C-2D)\int_C^\infty f(x)\,dx + 2\int_D^C (D-x)f(x)\,dx\int_C^\infty f(x)\,dx - 2(C-D)\left(\int_C^\infty f(x)\,dx\right)^2\\ &= 2\int_C^\infty f(x)\,dx\left[C\int_0^C f(x)\,dx - D\int_0^D f(x)\,dx - \int_D^C xf(x)\,dx\right]\end{aligned} \tag{A7}$$

(c) From (39),

$$\begin{aligned}\text{E}(\mathbf{X}\phi(\mathbf{X})) - \text{E}\mathbf{X}\,\text{E}\phi(\mathbf{X}) &= D\left[\int_D^\infty xf(x)dx - (\text{E}\mathbf{X})\int_D^\infty f(x)dx\right]\\ &\quad - C\left[\int_C^\infty xf(x)dx - (\text{E}\mathbf{X})\int_C^\infty f(x)dx\right]\\ &\quad - \left[\int_D^C x^2 f(x)dx - (\text{E}\mathbf{X})\int_D^C xf(x)dx\right].\end{aligned}$$

Therefore

$$2\,\frac{\partial}{\partial C}\Big(\text{E}[\mathbf{X}\phi(\mathbf{X})] - \text{E}\mathbf{X}\,\text{E}\phi(\mathbf{X})\Big) = 2\left[\text{E}\mathbf{X}\int_C^\infty f(x)\,dx - \int_C^\infty xf(x)\,dx\right]. \tag{A8}$$

Substituting (A6), (A7), and (A8) in (A5) we get

$$\begin{aligned}\frac{1}{2}\,\frac{\partial}{\partial C}\text{Var L}(\mathbf{X}) &= \int_C^\infty f(x)\,dx\left[C\int_0^C f(x)\,dx - D\int_0^D f(x)\,dx - \int_D^C xf(x)\,dx + \text{E}\mathbf{X}\right] - \int_C^\infty xf(x)\,dx\\ &= \int_C^\infty f(x)\,dx\left[C\int_0^C f(x)\,dx - D\int_0^D f(x)\,dx + \int_0^D xf(x)\,dx + \int_C^\infty xf(x)\,dx\right] - \int_C^\infty xf(x)\,dx\end{aligned} \tag{A9}$$

Rearrange (A9) to get

$$\frac{1}{2}\frac{\partial}{\partial C}\text{Var L}(\mathbf{X}) = C\int_C^\infty f(x)\,dx\int_0^C f(x)\,dx + \int_C^\infty f(x)\,dx\int_C^\infty xf(x)\,dx - \int_C^\infty xf(x)\,dx \tag{A10}$$

$$+\int_C^\infty f(x)\,dx\left(\int_0^D xf(x)\,dx - D\int_0^D f(x)\,dx\right) \tag{A11}$$

The right side of (A10) can be written as

$$\begin{aligned}
&= C\int_C^\infty f(x)\,dx\int_0^C f(x)\,dx + \int_C^\infty xf(x)\,dx\left(\int_C^\infty f(x)\,dx\ - 1\right)\\
&= C\int_C^\infty f(x)\,dx\int_0^C f(x)\,dx - \int_C^\infty xf(x)\,dx\int_0^C f(x)\,dx\\
&= \int_0^C f(x)\,dx\left(C\int_C^\infty f(x)\,dx - \int_C^\infty xf(x)\,dx\right)\\
&< 0 \text{ if (A1) holds, because then } \int_C^\infty xf(x)\,dx > C\int_C^\infty f(x)\,dx.
\end{aligned}$$

Similarly, (A11) is negative if (A2) holds, because then

$$D\int_0^D f(x)\,dx > \int_0^D xf(x)\,dx.$$

This completes the proof of (42) if any of the conditions (A1)–(A2) holds.

The inequality (43) is similarly proved. □