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INDEX TWO LINEAR TIME-VARYING SINGULAR SYSTEMS OF DIFFERENTIAL EQUATIONS*

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Abstract. An analytic method of solution is given for systems of differential equations of the form $A(t)x'(t)+B(t)x=f(t)$, where $A(t)$ may be singular and the system has index at most two.

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1. Introduction. In the last ten years the singular system of differential equations

$$Ax'+Bx=f, \tag{1.1}$$

where A, B are $n\times n$ constant matrices with A and possibly B singular has received a great deal of study. Applications have been given in economics (the Leontief model) [3], cheap control problems [1] and singular perturbations, where (1.1) is the reduced order model [11]. The early results of [9], [10] on (1.1) have also proved useful in studying nonlinear circuits with implicit models [4], [5], [7] and higher order singular arcs in control problems [8]. While some questions remain, the theory for (1.1) has reached a fairly mature level. For a detailed development and reasonably complete bibliography see [6], [7], [9] which contain discussions of most of the previously mentioned work.

The situation for the time varying case is quite different. While there do exist procedures for solving

$$A(t)x'(t)+B(t)x(t)=f(t) \tag{1.2}$$

in some cases [6], [7], [12] no analogue of the explicit formula given in [10], (or [9], [6]) has been derived except for special cases. This has complicated the numerical and analytical analysis of a variety of problems which contain (1.2) as a subsystem. This paper provides a solution of (1.2) under assumptions that include prior results and important new special cases. Examples will be given that show the difficulty in extending these results.

2. Preliminaries. All matrices are taken to be complex matrices. For a square matrix E, there exists a nonsingular matrix R so that

$$E=R\begin{bmatrix} C & 0\\ 0 & N\end{bmatrix}R^{-1}, \tag{2.1}$$

where C is invertible, N is nilpotent of index k, that is, $N^k=0$, $N^{k-1}\neq 0$ and either C or N may be absent. The index of E, Index(E), is k. If E is invertible, then Index$(E)=0$. The rank of C is called the core-rank of E [9]. The Drazin inverse of E, denoted E^D, is given by

$$E^D=R\begin{bmatrix} C^{-1} & 0\\ 0 & 0\end{bmatrix}R^{-1} \tag{2.2}$$

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if C is present and by $E^D=0$ if E is nilpotent. The Drazin inverse has the properties:

$$E^DEE^D=E^D, \quad EE^D=E^DE, \quad E^{k+1}E^D=E^k.$$

Further properties of the Drazin inverse may be found in [9]. The range and nullspace of E are denoted $\mathcal{R}(E)$, $\mathcal{N}(E)$.

The system (1.2) is said to be *solvable* at t if there exists a scalar $\lambda(t)$ so that $\lambda(t)A(t)+B(t)$ is invertible. If the system (1.2) is solvable for all t of interest, it is said to be a solvable system. For (1.1), solvability is directly related to the uniqueness of solutions. This is not true for (1.2), [6], [7]. Most attempts to numerically solve (1.2) or (1.1) have involved backward difference schemes [7]. Solvability is needed to insure that the resulting linear systems will be consistent and have a unique solution for all but a finite number of possible time step sizes. Since in most cases (1.2) will in fact be solved numerically, solvability, where possible, is a natural assumption to make.

If (1.2) is solvable at t, then the index of (1.2) at t is Index $((\lambda(t)A(t)+B(t))^{-1}A(t))$. The index depends only on $A(t)$, $B(t)$ and not $\lambda(t)$ [10], [6], [9]. For (1.2), the property that the index is identically zero, identically one, or greater than or equal to two on an interval $[0\ T]$ is invariant under "most" coordinate changes of the form $x=P(t)y$ [7].

By a *solution* of (1.2) on $[0\ T]$ we mean a differentiable function of t on $[0\ T]$. An initial condition x_0 is called *consistent* at t_0 if there is a solution $x(t)$ so that $x(t_0)=x_0$. Even for (1.1), the consistent initial conditions form a Core-rank $[(\lambda A+B)^{-1}A]$-dimensional linear manifold.

There are, in general, other kinds of solutions to (1.2) or (1.1) which are impulsive or distributional. However, the derivation of functional solutions and the consistent initial conditions is of interest. Knowing the consistent initial conditions tells when impulses may be present. If a singular perturbation approach is being used, the functional solution is all that is used from the reduced order problem. In many singular nonlinear systems, such as those involving relaxation oscillations, the functional solutions or parts of them represent the "observable" or physically realizable states [7]. Finally, even if (1.2) can be rewritten as a nonsingular subsystem, the form (1.2) may be used to preserve system structure. In such "descriptor systems" the functional solutions are often of most interest.

The system (1.2) often appears in the form

$$x'(t)=A_{11}(t)x(t)+A_{12}(t)y(t)+f_1(t), \tag{2.3a}$$

$$0=A_{21}(t)x(t)+A_{22}(t)y(t)+f_2(t) \tag{2.3b}$$

especially in singular perturbation and nonoptimal control problems [13], [14], [15]. System (2.3) has index one if and only if $A_{22}(t)$ is invertible [7]. This is the case that has been most studied to date [7], [12]. In this paper we shall solve (1.2) when the index is two, so that A_{22} will be singular. Such systems arise in circuits with operational amplifiers and in singular control problems [7]. Examples will be given to show that our analysis does not extend to the index 3 case.

3. Main results. Our starting point will be

$$\hat{A}(t)x'+x=f(t), \qquad 0\leqq t\leqq T, \tag{3.1}$$

where $\hat{A}, f$ are assumed to be differentiable. If the original system is in the form (1.2) and is solvable, the change of variables $x=\exp(\int_0^t \lambda(s)\,ds)z$ may be used to rewrite (1.2) in the form (3.1), where $\hat{A}(t)=(\lambda(t)A(t)+B(t))^{-1}A(t)$ and the new $f(t)$ is

$(\lambda(t)A(t)+B(t))^{-1}f(t)$. Alternatively, one may just multiply (3.1) by $(\lambda(t)E(t)+F(t))^{-1}$ and slightly modify the arguments that follow.

There are also two ways to proceed with the derivation, directly or through "canonical" forms. We shall proceed directly and then mention the canonical form version.

Our main result is the following.

THEOREM 1. *In the system* (3.1) *assume that* f *and* $\hat{A}(t)$ *are continuously differentiable on* $[0\ T]$, $\text{Index}(\hat{A}(t))\leqq 2$ *on* $[0\ T]$ *and* $\text{Rank}(\hat{A}^2(t))=\text{Core-rank}(\hat{A}(t))$ *is constant on* $[0\ T]$. *Define the projections* P, Q *by* $P=\hat{A}^D\hat{A}$, $Q=I-P$. *Let* $N=\hat{A}(I-\hat{A}^D\hat{A})$. *Assume that*

$$I-N'(t) \text{ is invertible on } [0\ T]. \tag{3.2}$$

Then $x(t)$ *is a smooth functional solution of* (3.1) *if and only if* $x=[Px]+[Qx]$, *where* $[Px]$ *and* $[Qx]$ *are given by*

$$[Px]'=(P'-\hat{A}^D)[Px]+P'[Qx]+\hat{A}^Df, \tag{3.3a}$$

$$[Qx]=[Qf]-[I-N']^{-1}N[Qf]'-[I-N']^{-1}NP'[Px]. \tag{3.3b}$$

Thus the dimension of the manifold of consistent initial conditions is the Core-rank of $\hat{A}(0)$ *and the manifold is*; $P(0)x(0)$ *is arbitrary and* $Q(0)x(0)$ *is* (3.3b) *evaluated at* $t=0$.

Proof. Since $\text{Rank}(\hat{A}^2(t))$ is constant, P, Q and $\hat{A}^D$ are as smooth as $\hat{A}$ on $[0\ T]$. Thus N is differentiable and $I-N'$ is well defined.

Differentiating the expressions $N^2=0$, $P^2=P$, $Q^2=Q$ and using $P+Q=I$, $PQ=QP=0$ the following identities are easily derived:

$$(I-N')N=N(N'+I),\quad PP'=P'Q,\quad PP'P=0,$$

$$QQ'=Q'P,\quad P'=-Q',\quad QQ'Q=0.$$

These facts will be used repeatedly in the following derivation. Note that $x=[Px]+[Qx]$. Substituting this into (3.1) gives the equivalent system:

$$\hat{A}[Px]'+\hat{A}[Qx]'+[Px]+[Qx]=f. \tag{3.4}$$

Multiplying (3.4) by $\hat{A}^D=\hat{A}^DP$ and Q yields

$$P[Px]'+P[Qx]'+\hat{A}^D[Px]=\hat{A}^Df, \tag{3.5a}$$

$$N[Px]'+N[Qx]'+[Qx]=Qf. \tag{3.5b}$$

Now

$$\begin{aligned} P[Qx]'&=PQ'x+PQx'=PQ'x=PQ'([Px]+[Qx])\\ &=-PP'Px+PQ'[Qx]=PQ'[Qx]=-PP'Qx=-P'[Qx]. \end{aligned} \tag{3.6}$$

Similarly,

$$Q[Px]'=-Q'[Px]=P'[Px]. \tag{3.7}$$

Thus (3.5) is

$$P[Px]'-P'[Qx]+\hat{A}^D[Px]=\hat{A}^Df, \tag{3.8a}$$

$$NP'[Px]+N[Qx]'+[Qx]=Qf. \tag{3.8b}$$

Now multiply (3.8b) by N so that $N[Qx]=NQf$ or, upon differentiating, $N'[Qx]+N[Qx]'=[NQf]'$. Thus

$$NQ[x]'=[NQf]'-N'[Qx]. \tag{3.9}$$

Substitute (3.9) into (3.8b) to get $[I-N'][Qx]=Qf-[NQf]'-NP'[Px]=[I-N'][Qf]-N[Qf]'-NP'[Px]$. Since $\mathcal{R}(N)=\mathcal{R}([I-N']N)\subseteq\mathcal{R}([I-N']Q)$, this equation can be solved for $[Qx]$ to yield (3.8b). Substituting (3.7) and (3.8a) into $[Px]'=P[Px]'+Q[Px]'$ gives (3.3a). □

The system (3.3) completely determines $[Px]$, $[Qx]$ since if (3.3b) is substituted into (3.3a) a nonsingular system just in $[Px]$ results. Its solution can then be used in (3.3b).

If $\hat{A}^2(t)$ has constant range and nullspace and N is constant, then P is constant and (3.3) becomes $[Px]'=\hat{A}^D[Px]+\hat{A}^Df$, $[Qx]=[Qf]-N[Qf]'$ as in [6]. Thus Theorem 1 includes the constant coefficient index two case. If $\hat{A}$ has index one and constant rank on $[0\ T]$, then (3.2) holds (since $N\equiv 0$) so that the index one case is also completely included.

Note that Theorem 1 does not require that any ranges or nullspaces are constant nor does it require that either Rank $(\hat{A}(t))$ or Index $(\hat{A}(t))$ be constant.

If (3.2) does not hold, there are several possibilities. If $\mathcal{R}(N)\subseteq\mathcal{R}([I-N']Q)$, then (3.1) is consistent for all f but uniqueness depends on whether $\mathcal{N}(I-N')\cap\mathcal{R}(Q)\neq\{0\}$. If $\mathcal{R}(N)\not\subseteq\mathcal{R}([I-N']Q)$, then (3.1) may not be consistent for all f. Alternatively, it is possible that (3.1) could be transformed to a system with index greater than two and the solution involves higher than first derivatives of f [7, Example 5.2.1, p. 117]. See also [6, Example 6.4.1, p. 147] and [7, Examples 5.4.1, 5.4.2, pp. 124–125].

The usefulness, either conceptually or in practice, of Theorem 1 remains to be determined and is under investigation. In any attempt to utilize (3.3) on test problems, for example, to provide "true" solutions to compare implicit numerical methods with, several observations need to be made. First, even if $A(t)$ is invertible in (1.2), then a linear system $E(t)z(t)=u(t)$ will have to be solved for some multiple of the number of time steps and the more rapidly E changes, the more solutions will be needed. Thus frequent "inversions" are intrinsic to problems in the form (1.2) or (3.1).

Second, there are several ways to compute $\hat{A}^D(t)$ or $\hat{A}^D(t)h(t)$ for a known vector $h(t)$, [9], [16] for a given value of t. One of them, [9, Algorithm 7.55, p. 134] which would not ordinarily be used, consists of a procedure which terminates in at most n steps, where $\hat{A}$ is $n\times n$, and involves only matrix products, taking traces and dividing by a scalar function. In principle it is possible to obtain $(\hat{A}^D)'(t)$ and hence $P'(t)$ for small matrices with simple entries by this approach in a program language that does symbolic manipulation.

Another approach is motivated by the observation that in practice it is usually easier or possibly safer to obtain $(\hat{A}')^D$ rather than $(\hat{A}^D)'$ since explicit computation of $\hat{A}^D$ as a function of t is difficult and taking the Drazin inverse numerically and then numerically differentiating is likely to create greater error. The key fact relating $(\hat{A}')^D$ and $(\hat{A}^D)'$ is the following unpublished result of Carl Meyer.

PROPOSITION 1 (Meyer). *Suppose that $A(t)$ is an $n\times n$ differentiable matrix valued function on $[0\ T]$ with constant core-rank. Then*

$$\begin{bmatrix} A & A' \\ 0 & A \end{bmatrix}^D=\begin{bmatrix} A^D & (A^D)' \\ 0 & A^D \end{bmatrix}. \tag{3.10}$$

Proof. From [9, Thm. 7.7.1, p. 139],

$$\begin{bmatrix} A & A' \\ 0 & A \end{bmatrix}^D = \begin{bmatrix} A^D & H \\ 0 & A^D \end{bmatrix},$$

where

$$H = (A^D)^2\left[\sum_{i=0}^{n} (A^D)^i A' A^i\right](I - AA^D) \tag{3.11}$$
$$+ (I - AA^D)\left[\sum_{i=0}^{n} A^i A'(A^D)^i\right](A^D)^2 - A^D A' A^D.$$

But from [2, Thm. 2], $H = (A^D)'$. □

There is a trade-off here, of course, in that

$$\begin{bmatrix} A & A' \\ 0 & A \end{bmatrix}^D$$

is $2n \times 2n$. On the other hand, its computation gives not only $(A^D)'$ but also A^D. Proposition 1 has several nonobvious consequences such as the facts that

$$\begin{bmatrix} A & A' \\ 0 & A \end{bmatrix} \text{ and } \begin{bmatrix} A^D & (A^D)' \\ 0 & A^D \end{bmatrix}$$

have the same core-rank and that the right-hand side of (3.10) has index one. Even for the index two case where $n = 1$ in (3.11), the computation of A^D, A' and using (3.11) is probably more effort than computing A' and the left-hand side of (3.10) by a procedure such as Wilkinson's [16].

It is also possible to derive Theorem 1 from a coordinate change point of view. We shall omit the details. Under the assumptions of Theorem 1, there exists a differentiable matrix $P(t)$ on $[0 \;\; T]$ such that

$$P^{-1}(t)\hat{A}(t)P(t) = \begin{bmatrix} C(t) & 0 \\ 0 & N(t) \end{bmatrix}, \tag{3.12}$$

where C is invertible and $N^2 = 0$ on $[0 \;\; T]$. Let $x = Py$ and $H = P^{-1}P'$, $\tilde{f} = P^{-1}f$. Then, decomposing H, y as in (3.12), (3.1) becomes

$$Cy_1' + (I + CH_{11})y_1 + CH_{12}y_2 = \tilde{f}_1, \tag{3.13a}$$

$$Ny_2' + NH_{12}y_1 + (I + NH_{22})y_2 = \tilde{f}_2. \tag{3.13b}$$

The analogue of (3.3) is

$$y_1' = -(C^{-1} + Q_{11})y_1 - Q_{12}y_2 + C^{-1}\tilde{f}_1 \tag{3.14a}$$

$$y_2 = [N' - I - NQ_{22}]^{-1}\{-\tilde{f}_2 + (N\tilde{f}_2)' + NQ_{12}y_1\}. \tag{3.14b}$$

4. An example. A slight modification of an example in [6] can be used to point out difficulties in the extension of Theorem 1 to the index three case.

Let

$$J=\begin{bmatrix}0&2&0\\0&0&2\\0&0&0\end{bmatrix},\quad R=\begin{bmatrix}1&0&0\\-t&1&0\\0&-t&1\end{bmatrix},\quad R^{-1}=\begin{bmatrix}1&0&0\\t&1&0\\t^2&t&1\end{bmatrix},$$

$$A(t)=RJR^{-1}=\begin{bmatrix}2t&2&0\\0&0&2\\-2t^3&-2t^2&-2t\end{bmatrix}.$$

Then

$$A(t)x'+x=0 \tag{4.1}$$

has constant core-rank (namely zero), constant index 3 and $I-N'=I-A'$ is invertible. Let $x=Ry$. Then (4.1) becomes

$$\begin{bmatrix}0&-2&0\\0&4t^2&-2\\0&0&0\end{bmatrix}\dot{y}+y=0. \tag{4.2}$$

System (4.2) has index two on $(0\ T]$, index three at zero, core-rank one on $(0\ T]$, core-rank zero at zero. The solution of (4.2) is

$$y=\begin{bmatrix}t^2e^{-4/t}c\\-2t^2c\\0\end{bmatrix},\quad c \text{ an arbitrary constant.} \tag{4.3}$$

Thus (4.1) has a nonzero one-dimensional manifold of solutions even though A is nilpotent, which never occurs in the constant coefficient or index two case as shown by Theorem 1. Also consistent initial conditions at $t=0$ do not uniquely determine solutions.

Example 1 also shows that a system with index greater than two can sometimes be changed into an index two system.

5. Conclusion. A fairly general solution of the index two singular linear system has been derived and discussed. An example has been given to show that similar results for higher index systems will probably be less general and involve additional technical assumptions.

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