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THE DRAZIN INVERSE OF AN INFINITE MATRIX*

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Abstract. Let $A=[a_{ij}]$, $0\leqq i<\infty$, $0\leqq j<\infty$. Then A is called a denumerably infinite matrix. A way to define a Drazin inverse for A is presented. The application of this definition to denumerable Markov chains, infinite linear systems of differential equations, and linear operators on Banach spaces is discussed.

1. Introduction. It has recently been shown that the Drazin inverse [3] of a square matrix has several important applications. In [8], it is used to study Markov chains. In [1], it is used to give closed forms for solutions of systems of linear differential equations with singular coefficient matrices.

The definition of a Drazin inverse usually given depends on the index of a matrix. Thus the only known definitions of Drazin inverses in infinite-dimensional spaces [2] are restricted to rather special classes of operators.

This paper will discuss how to define a Drazin inverse for infinite matrices. The matrices are *not* viewed as linear transformations on some particular linear space.

We shall first review the basic facts about infinite matrices. Then we shall show how to define the inverses and develop some basic information about them. Then we will show that our theory reduces to the usual theory when the matrices are finite. This is necessary since our definitions do not look like the standard ones. Finally, we discuss applications of our theory to denumerable Markov chains, infinite systems of linear differential equations, and operators without finite index.

2. Infinite matrices. Unless stated otherwise, all matrices are infinite matrices with finite entries from the field of complex numbers. If T is a matrix, we shall write it as $T=[T_{ij}]$, $0\leqq i<\infty$, $0\leqq j<\infty$. If $T_{ij}\geqq 0$ for every i, j (in particular, T_{ij} is real), then we write $T\geqq 0$. $|T|$ is defined by $|T|_{ij}=|T_{ij}|$. If $|T_{ij}|<\infty$ for all i, j, we write $|T|<\infty$. I denotes the identity matrix, E a matrix with every entry equal to one, and e a column matrix of ones. C^* is the conjugate transpose of the matrix C.

If S_n is a sequence of matrices, then

$$\lim_n S_n = C$$

is taken to mean $\lim_{n\to\infty}(S_n)_{ij}=C_{ij}$, $0\leqq i<\infty$, $0\leqq j<\infty$. In particular, this applies if $S_n=\sum_{k=0}^n A_n$ for a sequence of matrices $\{A_n\}$.

Matrix multiplication and addition are defined in the usual way. A product or sum is well-defined if one does not encounter any divergent series (series may conditionally converge to $+\infty$ or $-\infty$) or sums of the form $\infty-\infty$. By convention, $0\cdot\infty=0$. We define the following classes of matrices:

(FR) = {all matrices such that each row has a finite number of nonzero entries},
(FC) = {all matrices such that each column has a finite number of nonzero entries},
(FRC) = (FR) ∩ (FC),

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and note without proof that:

PROPOSITION 1. *If A, B are matrices, then*

(i) $A \in (FR), B \in (FR) \Rightarrow AB \in (FR)$, *and*

(ii) $A \in (FC), B \in (FC) \Rightarrow AB \in (FC)$.

Most of the algebraic manipulations one does with finite matrices also work for infinite matrices with one crucial exception. Matrix multiplication is no longer associative. A standard example [7, p. 5] is $e^*(Ae) = 0$, $(e^*A)e = 1$, where A is defined to be 1 on the diagonal, -1 on the superdiagonal and zero elsewhere. Note that $A \in (FRC)$. We shall give an example of a Markov chain with transition matrix P which has $I - P$ a scalar multiple of this A.

The uniqueness of most types of inverses depends upon associativity. Sufficient conditions for associativity are known. The following are taken from [7].

PROPOSITION 2. *Let A, B, C be matrices which possibly have infinite entries. Then $A(BC) = (AB)C$ if any one of the following conditions is met:*

(i) $A \geqq 0, B \geqq 0$, *and* $C \geqq 0$,

(ii) $|A||B||C| < \infty$,

(iii) $A \in (FR)$ *and* $A(BC)$, $(AB)C$ *are well-defined,*

(iv) $C \in (FC)$ *and* $A(BC)$, $(AB)C$ *are well-defined,*

(v) *B has finitely many nonzero entries and* $A(BC)$, $(AB)C$ *are well-defined.*

A useful corollary [7, p. 7] is:

COROLLARY 1. *If $A \geqq 0$, $B \geqq 0$, $C \geqq 0$, $D \geqq 0$, and either*

(i) $$|ABD| < \infty, \quad |AB| < \infty \quad \text{and} \quad |BD| < \infty,$$

or

(ii) $$|ACD| < \infty, \quad |AC| < \infty \quad \text{and} \quad |CD| < \infty,$$

then $[A(B-C)]D = A[(B-C)D]$.

We shall be most interested in Corollary 1 when $B = I$.

3. The Drazin and group inverses

DEFINITION 1. A matrix X, $|X| < \infty$, is a C-(2) inverse of A, $|A| < \infty$, if

(i) $XA = AX$, and

(ii) $X(AX) = (XA)X = A$.

DEFINITION 2. A matrix X, $|X| < \infty$, is a C-(1, 2) inverse of A, $|A| < \infty$, if X is a C-(2) inverse of A and

(iii) $A(XA) = (AX)A = A$.

DEFINITION 3. If X, Y are C-(2) inverses for A and

$$X(AY) = (XA)Y = X, \tag{1}$$

and

$$Y(AX) = (YA)X = X, \tag{2}$$

then we write $X \subseteq Y$.

It is important to note that conditions (1) and (2) are independent. Intuitively, (2) implies that the "range" of X is in the "range" of Y while (1) says that the "nullspace" of X contains the nullspace of Y.

Example 1. Let

$$A=I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad X=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \text{ and } Y=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}.$$

Then X, Y are C-(2) inverses for A and (2) holds but not (1).

If $X \subseteq Y$ for all C-(2) inverses X, we call Y maximal.

DEFINITION 4. If there exists a unique maximal (in the ordering of Definition 3) C-(2) inverse, we call it the Drazin inverse of A and denote it A^D. If A^D exists and is a C-(1, 2) inverse, it is also called the group inverse of A and denoted $A^{\#}$.

Note that $X=0$ is a C-(2) inverse for any A and $0 \subseteq Y$ for any C-(2) inverse Y of A.

PROPOSITION 3. *If $X \subseteq Y$, $Y \subseteq Z$, where X, Y, Z are C-(2) inverses for A, and $XAYAZ$ and $ZAYAX$ are associative products, then $X \subseteq Z$.*

Proof. Let X, Y, Z be C-(2) inverses for A such that $X \subseteq Y$ and $Y \subseteq Z$. Assume the products $XAYAZ$ and $ZAYAX$ are associative. Then

$$(XA)Z = X(AZ) = [X(AY)](AZ) = X(A(YAZ)) = X(AY) = X.$$

Similarly $Z(AX) = X$.

PROPOSITION 4. *If X, Y are C-(2) inverses for A such that $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.*

Proof. Suppose X, Y are C-(2) inverses for A such that $X \subseteq Y$ and $Y \subseteq X$. Then $X = XAY = Y$.

COROLLARY 2. *If A has a maximal C-(2) inverse, then it is unique so that A^D exists.*

PROPOSITION 5. *Suppose that X is a C-(1, 2) inverse of A and Y is a C-(2) inverse of A. If $X(AY)$ and $Y(AX)$ are associative products, and $A \in$ (FCR), then $Y \subseteq X$.*

Proof. Suppose X, Y are as described. Then

$$\begin{aligned} X(AY) &= (XAX)(AY) = (X^2A)(AY)\\ &= (X^2A)(YA) = X^2[A(YA)] = X^2A = X. \end{aligned}$$

Similarly,

$$\begin{aligned} (YA)X &= (YA)(XAX) = (AY)(XAX) = (AY)(AX^2)\\ &= [(AY)A]X^2 = AX^2 = X \end{aligned}$$

as desired.

Note that without some assumptions on A, Proposition 5 is not valid since inverses are not always unique, but they are always C-(1, 2) inverses.

PROPOSITION 6. *Suppose that X, $A \in$ (FR) and X is a C-(1, 2) inverse of A. Then X is the unique C-(1, 2) inverse of A.*

Since (FR) is associative and every C-(1, 2) inverse is a pseudo inverse in the terminology of [3], the proof of Proposition 6 is almost identical to the first half of the proof of Theorem 1 in [3]. It will be omitted.

A similar result holds if X, $A \in$ (FC).

The next example will be referred to several times in what follows.

Example 2. Let

$$P=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdot \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdot \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot \end{bmatrix}, \qquad A=I-P,$$

and

$$X=\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & \cdot \\ 0 & 2 & 2 & 2 & 2 & \cdot \\ 0 & 0 & 2 & 2 & 2 & \cdot \\ 0 & 0 & 0 & 2 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Then $AX=XA=I$. Since $P\in(\mathrm{FCR})$ and $X\in(\mathrm{FC})$, we have $X=(I-P)^{\#}$ by Proposition 6.

Note that in the above case, A has an inverse and yet $Ae=0$. When working with infinite matrices, the concept of an inverse is not as directly related to that of nullspaces and ranges as is the case with finite matrices. For this reason, our development is more algebraic.

Since $(XY)^*=Y^*X^*$ is valid for infinite matrices, we have that A^D exists if and only if $(A^*)^D$ exists and $(A^D)^*=(A^*)^D$.

Example 3. Let S be defined by $S_{ij}=1$ if $j=i+1$, and $S_{ij}=0$ if $j\neq i+1$. The only C-(2) inverse for S is 0 so that $S^D=0$. Hence $(S^*)^D=0$.

If S is thought of as acting on l^2, then S is the "unilateral" shift. In fact, S is an isometry. Now S, S^* have the property that viewed as operators on l^2,

$$\text{range}\,(S^n)\supseteq\text{range}\,(S^{n+1})\quad\text{for every } n,$$

$$\bigcap_{n=0}^{\infty}\text{range}\,(S^n)=\{0\},$$

while

$$\text{range}\,(S^{*n})=\text{range}\,(S^{*n+1})=l^2\quad\text{for every } n,$$

$$\bigcap_{n=0}^{\infty}\text{range}\,(S^{*n})=l^2.$$

Thus index (S), index (S^*) depending on your definition would probably be zero or infinity.

Notice that if X is a C-(2) or C-(1, 2) inverse for some A, then X^* is a C-(2) or C-(1, 2) inverse for A^*. On the other hand, if X_1, X_2 are C-(2) inverses for A and $X_1\subseteq X_2$, then $X_1^*\subseteq X_2^*$.

PROPOSITION 7. *If $A=A^*$ and A^D exists, then $(A^D)^*=A^D$.*

Proof. If $A=A^*$, then A^D, A^{D^*} are C-(2) inverses of A and $A^{D^*}\subseteq A^D$ by the definition of A^D. Thus $A^D\subseteq A^{D^*}$ and $A^{D^*}=A^D$.

While the preceding discussion is helpful in understanding the basic idea behind our definition of a Drazin inverse, it does not provide a means of showing that a given matrix has a Drazin inverse. We shall now develop one such criteria. First we need the following generalization of [7, Prop. I-63]. Its proof is the same as that of Proposition I-63 and will be omitted.

PROPOSITION 8. *Let $\{A_n\}$ be a sequence of matrices. If there exists a matrix C with finite entries such that $|A_n| < C$ for all n, then there is a convergent subsequence $\{A_k\}$ of $\{A_n\}$.*

Given a matrix A, let

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad A_i \text{ a } i \times i \text{ matrix.} \tag{3}$$

Then it is easy to see that

$$\tilde{A}_i^D = \begin{bmatrix} A_i^D & 0 \\ 0 & 0 \end{bmatrix}.$$

THEOREM 1. *Suppose that $A \in$ (FCR). Let $\tilde{A}_i$ be as in (3). If there exists a subsequence $\tilde{A}_k^D$ which converges, then $\lim_k \tilde{A}_k^D$ is a C-(2) inverse of A. If there exists a subsequence $\tilde{A}_l$ which has index 1 and $\tilde{A}_l^{\#}$ converges, then $\lim_l \tilde{A}_l^{\#}$ is a C-(1, 2) inverse of A.*

Proof. Suppose that $Q = \lim_k \tilde{A}_k^D$ exists. Now $\tilde{A}_k^D A \tilde{A}_k^D = \tilde{A}_k^D$. Since $A \in$ (FCR), we have

$$Q = \lim_k \tilde{A}_k^D = \lim_k \tilde{A}_k^D A \tilde{A}_k^D = QAQ. \tag{4}$$

On the other hand,

$$\begin{aligned} \tilde{A}_k^D A - A \tilde{A}_k^D &= \begin{bmatrix} A_k^D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} - \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} A_k^D & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & A_k^D B_k \\ -C_k A_k^D & 0 \end{bmatrix}. \end{aligned}$$

Thus

$$\lim_k (\tilde{A}_k^D A - A \tilde{A}_k^D) = 0$$

since $(\tilde{A}_k^D A - A \tilde{A}_k^D)_{ij} = 0$ if $i, j \leq k$.

Hence

$$QA - AQ = 0, \tag{5}$$

and Q is a C-(2) inverse as desired. Now suppose that $Q = \lim_k \tilde{A}_k^{\#}$. Again (4) and

(5) hold. Since

$$A\tilde{A}_k^{\#}A = \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\lim_k \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} = A,$$

we have

$$AQA = A$$

and Q is a C-$(1, 2)$ inverse as desired.

Note that it is quite possible to have the index of A_i depend on i.

Example 4. Let $A_{ij} = 1$ if $|i-j| = 1$. Then A_i is invertible if and only if i is even. Let $m = 2i$. Then

$$\tilde{A}_2^D = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \qquad \tilde{A}_m^D = \left[\begin{array}{c|c|c} A_{m-2}^D & R & 0 \\ \hline S & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \qquad R, S \text{ matrices.}$$

Thus $\tilde{A}_m^D$ converges to a matrix Q. In fact Q is an inverse of A and

$$Q = \left[\begin{array}{cc|cc|cc|cccc} 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & & \cdot \\ 1 & 0 & 0 & 0 & 0 & 0 & & & \cdot & \cdot \\ \cline{1-2} 0 & 0 & 0 & 1 & 0 & -1 & & & & \\ -1 & 0 & 1 & 0 & 0 & 0 & & & & \\ \cline{1-4} 0 & 0 & 0 & 0 & 0 & 1 & & & \cdot & \\ 1 & 0 & -1 & 0 & 1 & 0 & & & & \\ \cline{1-6} 0 & \cdot & & & & & & & & \\ -1 & \cdot & & & & & & & & \\ \cdot & & & & & & & & & \end{array}\right].$$

PROPOSITION 9. *Suppose that $A = A^*$ and let δ_i be the modulus of the smallest nonzero eigenvalue of A_i. If $\{\delta_i^{-1}\}$ is bounded by m, then*

$$-mE \leqq \tilde{A}_i^D \leqq mE \quad \textit{for all } i.$$

Proof. Assume that $A = A^*$. Let ρ denote the spectral radius. We may assume that $\rho(A_i)>0$. But $\rho(A_i^D)=1/\delta_i$ and

$$\max_{k,l}|A_i^D|_{k,l}\leqq\rho(A_i^D),\qquad 0\leqq k,\, l\leqq i.$$

Proposition 9 combined with Theorem 1 provides a means of not only proving the existence of a group inverse for a self-adjoint matrix but also calculating a given entry.

4. The finite-dimensional case. Let A be a $n\times n$ matrix of complex numbers. The Drazin inverse of A is usually defined as the unique matrix X such that

$$AX=XA, \tag{6}$$

$$XAX=X, \tag{7}$$

and

$$XA^{k+1}=A^k,\quad \text{where } k=\inf_m\{m\colon \operatorname{rank}(A^{m+1})=\operatorname{rank}(A^m)\}. \tag{8}$$

k is called the index of A. In this section, we shall use A^D to denote the inverse defined by (6)–(8) rather than by Definition 4. Note that, by definition, A^D is a C-(2) inverse of A. If $k=1$ or 0, then A^D is a C-(1, 2) inverse and hence is the group inverse. We shall show that our definition of the Drazin inverse is equivalent to the one given in (6)–(8).

Given an $n\times n$ matrix A, there exists an invertible matrix T such that

$$TAT^{-1}=\begin{bmatrix} J & 0\\ 0 & N\end{bmatrix}, \tag{9}$$

where J is invertible and N is nilpotent of index k. Suppose that B commutes with A. Let

$$TBT^{-1}=\begin{bmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}. \tag{10}$$

Then $AB=BA$ implies that

$$JB_{11}=B_{11}J, \tag{11}$$

$$JB_{12}=B_{12}N, \tag{12}$$

$$NB_{21}=B_{21}J, \tag{13}$$

and

$$NB_{22}=B_{22}N. \tag{14}$$

From (12) we get

$$J^kB_{12}=B_{12}N^k=0\quad\text{and hence}\quad B_{12}=0.$$

Similarly (13) gives that

$$B_{21}J^k=N^kB_{21}=0\quad\text{so that}\quad B_{21}=0.$$

Now if B is a (2)-inverse for A, then we have

(15) $$B_{22}NB_{22}=B_{22}.$$

Using (14) and (15) we get

$$N^{k-1}B_{22}=B_{22}N^kB_{22}=0.$$

Hence

$$B_{22}N^{k-1}B_{22}=N^{k-2}B_{22}=0.$$

Continuing in this manner gives $B_{22}=0$. Hence B in (10) is of the form

(16) $$TBT^{-1}=\begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

THEOREM 2. *The two definitions of the Drazin inverse are equivalent for $n\times n$ matrices.*

Proof. Given A write as in (9). Let B be any C-(2) inverse. Then B can be written as in (16). But

$$TA^DT^{-1}=\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus

$$TAA^DT^{-1}=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

and $BAA^D=A^DAB=B$ so that $B\subseteq A^D$. Thus A^D is a maximal C-(2) inverse in the ordering of Definition 3. From Proposition 4 we have that it is the unique maximal one.

5. Markov chains. Let $P\geqq 0$ be such that $Pe=e$. Then P is the transition matrix of a Markov chain. The following information is from [7] and is included for completeness. Let $N=\sum_{k=0}^{\infty}P^k$. Then N_{ij} is the average number of times in state j, starting in state i. Of course, $N\geqq 0$ but not necessarily $N<\infty$. Following [7] we say that the two states i, j are related $i\sim j$ if it is possible to go from i to j. Thus $i\sim j$ if and only if $(P^n)_{ij}>0$ for some n. States are classified as recurrent or transient. A state is transient if and only if $N_{ii}<\infty$. If j is transient, then $N_{ij}<\infty$ for all i. One has, in fact, that with probability one, one is in a transient state only a finite number of times.

A Markov chain is recurrent if its states compromise a single equivalence class and one (hence all) state in the equivalence class is recurrent. A chain is transient if all of its recurrent states are absorbing. Any Markov chain can be viewed as a transient chain whose states are equivalence classes of recurrent states.

If P is a transient chain, then listing the absorbing states first one may write

(17) $$P=\begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}.$$

If one views P as a fixed matrix, then (17) is unitarily equivalent to the original P by way of a permutation. To avoid unnecessary notation we assume P is in the form (17). The possibility that $P = Q$ is not omitted. If

$$M = \sum_{k=0}^{\infty} Q^k, \tag{18}$$

we have $M < \infty$ and $\lim_k Q^k = 0$ [7, p. 107]. Furthermore, $M(I-Q) = (I-Q)M = I$, $QM = MQ$, [7, p. 108]. Clearly $N = \begin{bmatrix} \infty I & 0 \\ X & M \end{bmatrix}$, X a matrix.

In the finite-dimensional chains, if the chain is to be transitive it must have an absorbing state. This is not true in the infinite-dimensional case. The matrix P of Example 2 defines a transient chain. In this section, A is always $I - P$.

PROPOSITION 10. *If P is a transitive Markov chain of the form* (17), $P \in$ (FCR), *and* $M \in$ (FC), *then* $A^{\#} = \begin{pmatrix} 0 & 0 \\ -M^2R & M \end{pmatrix}$. *In particular, $A^{\#}$ exists.*

In [7, p. 109] it is shown that if P is a transient chain, then $\lim_k P^k = \begin{pmatrix} I & 0 \\ MR & 0 \end{pmatrix}$. Thus as in [8, Thm. 2.2], we have $\lim_n P^n = (I - A^{\#}A)$. If A is invertible, this just says, as observed earlier, that $\lim_k P^k = 0$.

Note that $A^{\#}$ may fail to exist if $M \notin$ (FC). The obvious question then is which type of transient chains have $M \in$ (FC). Using (18) and the fact that $M_{ij} = H_{ij}N_{jj}$ where H is the probability of eventually going from i to j for transitive nonabsorbing states i, j, we see that $M_{ij} = 0$ for $i \geqq k$, if and only if there are only a finite number of states which can lead to i since $N_{jj} > 0$. Note that Example 2 is of that form. Not all transitive chains, of course, look like this. A simple example is given in [7, p. 83]. On p. 159 of [7] is an example of a transient chain such that $P \in$ (FCR), $M \notin$ (FC).

We see then, that for transitive chains with associativity of the matrices involved, $A^{\#}$ gives the results of [8] with the added result that if $I - P$ is invertible, then $N = A^{\#}$. Note that Proposition 5-18 of [7] dealing with the infinite drunkard's walk is a special case of our Theorem 1.

If P is recurrent, then

$$\frac{1}{n} \sum_{k=0}^{n-1} P^k \to L,$$

where L satisfies $LP = L = PL = L^2$ [7, p. 130]. If $L = 0$, the chain is called null. If $L \neq 0$, the chain is called ergodic. For finite-dimensional recurrent chains $L = I - A^{\#}A$ [8] and A is never invertible [8]. Thus all finite recurrent chains are ergodic.

If P is recurrent, then there exists a row vector α such that $\alpha P = \alpha$, $\alpha > 0$. Furthermore, P is ergodic if and only if $\alpha e < \infty$, and α is unique up to a constant multiple [7]. If P is a null chain, then $\lim_n P^n = 0$ [7, p. 154].

If X is an inverse of A, then $X + e\alpha$ is another inverse for A since $Ae\alpha$ is associative ($|A|e\alpha \leqq |e\alpha| + |Pe\alpha| < \infty$) and $e\alpha A$ is associative. Thus $A^{\#}$, as we have defined it will not exist for null chains. However, $L = I - XA$ is still true for any inverse of A.

Suppose, on the other hand, that P is ergodic and that $A^{\#}$ exists. Then $(I-A^{\#}A)A=0$ or $(I-A^{\#}A)=(I-A^{\#}A)P$. Since α is summable, assume $\alpha e=1$. Thus each row of $(I-A^{\#}A)$ must be a multiple of α so that $(I-A^{\#}A)=k\alpha$, k a column matrix. Now $L=e\alpha$ [7, p. 133]. But then $(I-A^{\#}A)L=L$ since the product is associative and $AL=0$. Thus $(k\alpha)(e\alpha)=e\alpha$. But $(k\alpha)(e\alpha)=(\alpha e)k\alpha=k\alpha$. Hence $k=e$ and $I-A^{\#}A=L$ just as in the finite-dimensional case.

6. Differential equations. In extending the results of [1] to infinite systems, there is a major difficulty. Consider the infinite system of differential equations

$$A\dot{x}+Bx=f. \tag{19}$$

For simplicity, we shall assume $A, B \in$ (FCR). Such equations occur, for example, in [9]. In [1], it was shown that in the finite-dimensional case, (19) has unique solutions for consistent initial conditions if and only if $(A+\lambda B)$ is invertible for some scalar λ. A closed form for the solutions was also given in [1]. That the same does not hold for (19) in the infinite-dimensional case is shown by the next example.

Example 5. Let $B=-I$ and A be the S^* of Example 3. Then $(A+B)$ is invertible and the inverse is unique. In fact, $[(A+B)^{-1}]_{ij}=(-1)$ if $i\leqq j$ and zero otherwise. Let $f=0$. Then (19) becomes $S^*\dot{x}=x$. Then define x_n by $nS^*x_n=x_{n-1}$. Since S^* is onto the vector space for all column matrices and is not one-to-one, the choice for x_n is not unique. Let $x(t)=\sum_{n=0}^{\infty} t^n x_n$. By taking each $x_n \in l^{\infty}$ such that $\|x_n\|\leqq M$ for some fixed M, $x(t)$ will be a well-defined column matrix whose entries are differentiable functions of t on $(-M, M)$. The choice of x_n is not unique. Since $(S^*)^D=0$, the only solution produced from the formulas of [1] is $x\equiv 0$.

We shall show that provided A is self-adjoint, one gets a reasonably complete extension of the results of [1]. First we need the following result.

PROPOSITION 11. *If $A^{\#}$ exists, $AB=BA$, and products involving B, $A^{\#}$, A are associative, then $A^{\#}B=BA^{\#}$.*

The proof of Proposition 11 is almost identical to the second half of the proof of Theorem 1 in [3] and will be omitted.

We can prove the following infinite-dimensional version of [1, Thm. 3]. The exponential is to be interpreted as a formal power series.

THEOREM 3. *Suppose that $A^{\#}$ exists, $AB=BA$, and all finite products involving A, $A^{\#}$, B are associative. Then*

$$Y=e^{-A^{\#}Bt}A^{\#}A \tag{20}$$

is formally a solution of $A\dot{x}+Bx=0$.

Proof. Let $x_1=A^{\#}Ax$ and $x_2=(I-A^{\#}A)x$. Then (19) becomes

$$A\dot{x}_1+Bx_1+Bx_2=0. \tag{21}$$

Multiply through first by $A^{\#}A$ and then by $(I-AA^{\#})$. Then (21) is equivalent to the pair of equations:

$$A\dot{x}_1+Bx_1=0 \tag{22}$$

and

$$Bx_2=0. \tag{23}$$

But $e^{-A^\# Bt}A^\# A$ is a solution of (22) while $x_2 = 0$ is a solution of (23).

If we knew that $A^\#$ and $B^\#$ both existed, then we could argue that

$$Y = e^{-A^\# Bt}A^\# A + (I - A^\# A)(I - B^\# B)f$$

is the "general" solution of (19), f an "arbitrary" formal series.

7. Operators. Our results suggest a way to define the Drazin inverse of a linear transformation. For convenience we shall assume that A is a bounded linear operator on the complex Banach space B. If A has a finite index, then the Drazin inverse may be defined pretty much as in the finite-dimensional case (see Carradus [2]).

If one wishes to develop a more general theory, to cover say compact and normal operators, then one would want to include the possibility that A^D be unbounded. We shall not develop such a theory here but rather discuss some of the difficulties involved.

To follow the approach given here, a class of operators $\delta(A)$ should be defined, an ordering $\subseteq$ introduced, and the maximal elements studied. The major difficulty is in deciding which X to put in $\delta(A)$. Let $\mathcal{D}(X)$ be the domain of X. Is it best to have

$$AX = XA,$$

or

$$AX = XA \quad \text{on} \quad \mathcal{D}(X)?$$

Similarly, should X be required to be closed or not? To show how it can all make a difference, let us consider two definitions of $\delta(A)$. The symbols $\mathcal{N}$, $\mathcal{R}$ denote nullspaces and ranges respectively.

DEFINITION 5. Let $\delta_1(A)$ be the set of all X such that

$$\mathcal{D}(X) \quad \text{is dense,} \tag{24}$$

$$AX = XA \quad \text{on} \quad \mathcal{D}(X), \tag{25}$$

$$XAX = X, \tag{26}$$

and

$$\bigcup \mathcal{N}(A^k) \subseteq \mathcal{D}(X), \qquad \bigcap \mathcal{R}(A^k) \subseteq \mathcal{D}(X). \tag{27}$$

DEFINITION 6. Let $\delta_2(A)$ be the set of all X such that X satisfies (24), (25), (26), (27), and X is closed. That is, the graph of X is closed.

Example 6. Take $B = l^2$ and $A = S^*$, where $S^*\{\alpha_0, \alpha_1, \alpha_2, \cdots\} = \{\alpha_1, \alpha_2, \alpha_3, \cdots\}$. Then

$$\mathcal{N}(A^k) = \{\{\alpha_n\} \in l^2 : \alpha_n = 0 \text{ if } n > k\}.$$

Thus $\bigcup_k \mathcal{N}(A^k)$ is dense in l^2. Using (25), (26), and (27) we can show that $\bigcup_k \mathcal{N}(A^k) \subseteq \mathcal{N}(X)$. But any closed operator has a closed nullspace. Thus $\delta_2(A) = \{0\}$. To see that $\delta_1(A) \neq \{0\}$, let $\mathcal{M}$ be a finite-dimensional subspace of l^2, invariant under S^*, such that no vector in $\mathcal{M}$ has only a finite number of nonzero components. Let $S^*|\mathcal{M}$ denote the restriction of S^* to $\mathcal{M}$. Then define X as follows. Let $\mathscr{D}(X)=\mathscr{M}+\bigcup_k \mathscr{N}(A^k)$. Note that $\mathscr{M}\cap\bigcup_k\mathscr{N}(A^k)=\{0\}$. Set $X|\mathscr{M}=(S^*|\mathscr{M})^{-1}$ and $X|\bigcup_k \mathscr{N}(A^k)=0$. Extend linearly to $\mathscr{D}(X)$. Then $X\neq 0$ and $X\in \delta_1(A)$.

For the reader unfamiliar with S^*, the existence of the subspace $\mathscr{M}$ may not be obvious. That it exists can be demonstrated as follows. Identify l^2 with H^2, the functions analytic in the open disc whose Taylor series have square summable coefficients. Let b be a finite Blaschke product such that $b(0)\neq 0$. (b is an analytic function of modulus one on the circle with a finite number of zeros inside.) Let $\mathscr{M}=H^2\cap(bH^2)^{\perp}$. Then $\mathscr{M}$ is invariant under S^*, and $S^*|\mathscr{M}$ is invertible since $b(0)\neq 0$ [4] or [5].

We shall conclude by discussing why we included (27) in our Definition 5. Let $\bigcap_{k=0}^{\infty}\mathscr{R}(A^k)$ be denoted by $\mathscr{R}(A^{\infty})$.

How one defines A^D will depend on how one wishes to use it. Let's briefly examine why the Drazin inverse is useful in the finite-dimensional case. Consider the difference equation:

$$Ax_{n+1}=x_n,\quad A \text{ singular},\quad n\geqq 0. \tag{28}$$

As pointed out in [1], one does not want to just solve (34) by $x_{n+1}=A^-x_n$, A^- some (1)-inverse, since $\{(A^-)^n x_0\}$ may not be a solution of (28). But if $\{x_n\}$ is a solution of (28), then

$$x_n=Ax_{n+1}=A^2x_{n+2}=\cdots=A^kx_{n+k}.$$

Note that A maps $\mathscr{R}(A^{\infty})$ onto itself. If $\{x_n\}$ is a solution of (28), then $x_n\in\mathscr{R}(A^{\infty})$ for every n. In the finite-dimensional case, $\mathscr{R}(A^{\infty})=\mathscr{R}(AA^D)$ and $\bigcup_k\mathscr{N}(A^k)=\mathscr{N}(AA^D)=\mathscr{R}(I-AA^D)$, and the unique solution of (28) is $x_n=(A^D)^n x_0=(A^n)^D x_0$. Thus if A^D is to solve (28) we would need $\mathscr{R}(A^{\infty})\subseteq\mathscr{D}(X)$.

Whether or not $\bigcup_k \mathscr{N}(A^k)\subseteq\mathscr{D}(X)$ is necessary is a more difficult question. Its inclusion depends on whether or not the unicity of A^D or the solving of (28) by using different $X\in\delta(A)$ is the more desirable. If unicity is not important in the problem to be worked on, then $\delta(A)$ could be used for (28) much like the set of all (1)-inverses is used to solve finite-dimensional linear systems of equations.

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