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GENERALIZED INVERSE FORMULAS USING THE SCHUR COMPLEMENT*

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Abstract. A formula for various generalized inverses of a partitioned complex matrix is established under certain general conditions. The use of this formula in obtaining the Moore–Penrose inverse of an arbitrary complex matrix is discussed.

1. Introduction. We deal throughout with complex matrices. A generalized inverse X of an $l \times n$ matrix M is an $n \times l$ matrix X which satisfies one or more of the equations

$$MXM = M, \tag{1.1}$$

$$XMX = X, \tag{1.2}$$

$$(MX)^* = MX, \tag{1.3}$$

$$(XM)^* = XM. \tag{1.4}$$

We use M^* to denote the conjugate transpose of M. If $\{i, j, k\}$ is a subset of $\{1, 2, 3, 4\}$ than any matrix satisfying (1.i), (1.j), and (1.k) will be called an (i, j, k)-inverse of M, denoted by $M^{(i,j,k)}$. Similarly, we define an (i, j)-inverse of M, denoted by $M^{(i,j)}$, and an (i)-inverse of M, denoted by $M^{(i)}$. For each M, there is a unique matrix satisfying (1.1), $\cdots$, (1.4), which is called the Moore–Penrose inverse of M and is denoted by $M^{\dagger}$ [7].

We shall be concerned with matrices partitioned in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{1.5}$$

With respect to this partitioning, a *Schur complement* of A *in* M is a matrix of the form $(M/A) = D - CaB$, where a is some generalized inverse of A. Similarly, if d is some generalized inverse of D, we define $(M/D) = A - BdC$. In the following it will be clear from the context which inverse is appropriate. Properties of Schur complements (using the Moore–Penrose inverse) are examined in some detail in [5].

We define $N(A)$ to be the null-space of A regarded as an operator on column vectors, and $\rho(A)$ denotes the rank of A. Certain relationships among matrices $A, B, C, D, (M/A)$, and (M/D), stated in terms of null-spaces, will be of particular interest to us:

$$N(A) \subseteq N(C), \tag{1.6}$$

$$N(A^*) \subseteq N(B^*), \tag{1.7}$$

$$N((M/A)^*) \subseteq N(C^*), \tag{1.8}$$

$$N((M/A)) \subseteq N(B). \tag{1.9}$$

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Note that (1.6) is equivalent to

$$C = CA^{(1)}A \quad \text{for every } A^{(1)}, \tag{1.10}$$

that (1.7) is equivalent to

$$B = AA^{(1)}B \quad \text{for every } A^{(1)}, \tag{1.11}$$

and similarly for (1.8) and (1.9). We shall say that M satisfies (N_1) if (1.6) and (1.7) hold, (N_3) if (1.7) and (1.8) hold, (N_4) if (1.6) and (1.9) hold, and (N) if M satisfies (N_3) and (N_4). The numbering of the conditions (N_1), (N_3), (N_4) corresponds with the numbering of the equations for the Moore–Penrose inverse. As will be seen (Theorem 1 (ii)) no condition is needed on M for existence of an $M^{(2)}$ in the form (2.1) so (N_2) is omitted.

2. A formula for generalized inverses. Suppose M is in form (1.5), where A is $p \times q$ and D is $(l - p) \times (n - q)$. Consider a matrix of the form

$$m = \begin{bmatrix} a + aBsCa & -aBs \\ -sCa & s \end{bmatrix}, \tag{2.1}$$

where a is $q \times p$ and s is $(n - q) \times (l - p)$. We shall show that under certain conditions m is a generalized inverse of M.

Let $S = D - CaB$. The following statements are easily verified (see Appendix):

If a_1, a_2 are (1)-inverses of A, then $Ca_1B = Ca_2B$ if (1.6) and (1.7) hold. Thus under certain conditions S is independent of the choice of a. However, in the sequel we shall always assume that S is given in terms of a specific choice of a.

If a, s satisfy (1.1) relative to A, S respectively, then m satisfies (1.1) relative to M if and only if $M - MmM = 0$ or

$$(I - Ss)C(I - aA) = 0, \tag{2.2}$$

$$(I - Aa)B(I - sS) = 0, \tag{2.3}$$

$$(I - Aa)BsC(I - aA) = 0. \tag{2.4}$$

Then m satisfies (1.1) relative to M if M satisfies any one of (N_1) or (N_3) or (N_4).

If a, s satisfy (1.2) relative to A, S respectively, then m always satisfies (1.2) relative to M.

If a, s satisfy (1.3) relative to A, S respectively, then m satisfies (1.3) relative to M if M satisfies (N_3).

If $a = A^{(1,3)}$, $s = S^{(1,3)}$, and $m = M^{(1,3)}$, then $(Mm)^* = Mm$ implies

$$[(I - Ss)Ca]^* = (I - Aa)Bs \tag{2.5}$$

which together with (2.3) implies (1.7) holds. Moreover, the conjugate transpose of (2.5) in view of (2.2) shows that (1.8) must hold.

If a, s satisfy (1.4) relative to A, S respectively, then m satisfies (1.4) relative to M if M satisfies (N_4).

Similarly if $a = A^{(1,4)}$, $s = S^{(1,4)}$, and $m = M^{(1,4)}$, then M satisfies (N_4).

Therefore we have the following theorem.

THEOREM 1.

(i) *If* $a = A^{(1)}$ *and* $s = S^{(1)}$, *then* $m = M^{(1)}$ *if* M *satisfies* (N_1) *or* (N_3) *or* (N_4).

(ii) *If* $a = A^{(2)}$ *and* $s = S^{(2)}$, *then* $m = M^{(2)}$.

(iii) *If* $a = A^{(j)}$ *and* $s = S^{(j)}$ *then* $m = M^{(j)}$ *if* M *satisfies* $(N_j), j = 3, 4$.

(iv) *If* $a = A^{(1,j)}$ *and* $s = S^{(1,j)}$, *then* $m = M^{(1,j)}$ *if and only if* M *satisfies* $(N_j), j = 3, 4$.

(v) *If* $A = A^\dagger$ *and* $s = S^\dagger$, *then* $m = M^\dagger$ *if and only if* M *satisfies* (N).

Remark. The conditions for obtaining any $M^{(i,j)}$ or $M^{(i,j,k)}$ in terms of corresponding a and s can be obtained as immediate corollaries. For example, if $a = A^{(1,3,4)}$ and $s = S^{(1,3,4)}$, then $m = M^{(1,3,4)}$ if and only if M satisfies (N).

P. Bhimasankaram [4] has formulas for $M^{(1)}$ and $M^{(1,2)}$ which are identical with ours under conditions equivalent to (2.2), (2.3), (2.4). He has shown (as we remarked above) that (2.2), (2.3), (2.4) are necessary and sufficient for existence of $M^{(1)}$ in the form (2.1). However, Theorem 1 gives a more complete picture for any $M^{(i)}$, $M^{(i,j)}$, $M^{(i,j,k)}$ inverse. Bhimasankaram has proved some other interesting results for special cases.

Rohde [9] has proved that if a, s are (1) or (1, 2) or (1, 2, 3) or Moore–Penrose-inverses of A, S, then m is the corresponding inverse of M when M is positive semidefinite Hermitian and partitioned symmetrically (and, for the (1, 2, 3) or Moore–Penrose inverse, with the assumption that D is nonsingular and $\rho(M) = \rho(A) + \rho(D)$). As was proved by Albert [1] in the real case (his proofs carry over to the complex case) such positive semidefinite Hermitian M always satisfy (1.6) and (1.7); and positive definite Hermitian M satisfy (1.8) and (1.9), as $(M/A) = D - B^*A^\dagger B$ is also positive definite Hermitian. Ben-Israel [2] has shown that $m = M^\dagger$ for certain $2n \times 2n$ matrices M.

We say that M satisfies (N') if, for some specified (M/D),

$$N(D) \subseteq N(B), \tag{1.6'}$$

$$N(D^*) \subseteq N(C^*), \tag{1.7'}$$

$$N((M/D)^*) \subseteq N(B^*), \tag{1.8'}$$

$$N((M/D)) \subseteq N(C). \tag{1.9'}$$

Given M in form (1.5), where A is $p \times q$ and D is $(l - p) \times (n - q)$, if we consider a matrix of the form

$$m' = \begin{bmatrix} t & -tBd \\ -dCt & d + dCtBd \end{bmatrix}, \tag{2.6}$$

where t is $q \times p$ and d is $(n - q) \times (l - p)$, and let $T = A - BdC$, we obtain a result analogous to Theorem 1. The following corollaries apply specifically to the Moore–Penrose inverse $M^\dagger$.

COROLLARY 1. *If* M *satisfies both* (N) *and* (N'), *then*

$$M^\dagger = \begin{bmatrix} (M/D)^\dagger & -A^\dagger B(M/A)^\dagger \\ -D^\dagger C(M/D)^\dagger & (M/A)^\dagger \end{bmatrix}. \tag{2.7}$$

Proof. The proof follows from Theorem 1 and the analogous result mentioned above, by the uniqueness of $M^\dagger$.

COROLLARY 2. *If M satisfies (N), then*

$$(M^\dagger/(M/A)^\dagger)^\dagger = A, \tag{2.8}$$

and if M satisfies (N'), then

$$(M^\dagger/(M/D)^\dagger)^\dagger = D. \tag{2.9}$$

Proof. To prove (2.8) apply the definition of the Schur complement (M/D) to the matrix $M^\dagger$ as represented in (2.1), with $a = A^\dagger$, $s = (D - CA^\dagger B)^\dagger$, to obtain

$$\begin{aligned}(M^\dagger/(M/A)^\dagger) &= A^\dagger + A^\dagger B(M/A)^\dagger CA^\dagger - (-A^\dagger B(M/A)^\dagger)(M/A)(-(M/A)^\dagger CA^\dagger)\\ &= A^\dagger;\end{aligned}$$

(2.8) follows as $(A^\dagger)^\dagger = A$. The proof of (2.9) is similar.

COROLLARY 3. *If M satisfies (N), so that $M^\dagger$ has the form* (2.1), *with $a = A^\dagger$, $s = (D - CA^\dagger B)^\dagger$, then $M^\dagger$ satisfies (N'). Similarly, if M satisfies (N'), then $M^\dagger$ has the form* (2.6), *with $d = D^\dagger$, $t = (A - BD^\dagger C)^\dagger$, and $M^\dagger$ satisfies (N).*

Proof. If M satisfies (N), and $M^\dagger$ has the stated form (2.1), clearly (1.6′) and (1.7′) hold for $M^\dagger$. Since, by Corollary 2, we have $(M^\dagger/(M(S))^\dagger) = A^\dagger$, (1.8′) and (1.9′) also hold for $M^\dagger$; hence $M^\dagger$ satisfies (N'). The proof if M satisfies (N') is similar.

3. The Moore–Penrose inverse of an arbitrary matrix. Mitra [11] and, independently, Zlobec [10] have proved that for any $l \times n$ matrix M,

$$M^\dagger = M^*QM^* \quad \text{for any } Q = (M^*MM^*)^{(1)}. \tag{3.1}$$

Thus the computation of the unique matrix $M^\dagger$ may be reduced to the computation of a (1)-inverse Q (which is a solution of the system of linear equations $(M^*MM^*)Q(M^*MM^*) = M^*MM^*$; cf. [3]).

An alternative representation of $M^\dagger$ was given by Decell [6]:

$$M^\dagger = M^*P_1MP_2M^* \quad \text{for any } P_1 = (MM^*)^{(1)}, \quad P_2 = (M^*M)^{(1)}. \tag{3.2}$$

In this case, MM^* and M^*M are positive semidefinite; and the calculation of P_1 and P_2 may, by our Theorem 1 (in the special case proved by Rohde), be reduced to the calculation of (1)-inverses of four matrices of lower order. In the next section, we give another result which is related to the computation of $M^\dagger$.

4. The Moore–Penrose inverse when $\rho(M) = \rho(A)$. Suppose M is in form (1.5), and that $\rho(M) = \rho(A)$. This implies that (1.6) and (1.7) hold; that for any $a = A^{(1)}$, $(M/A) = D - CaB = 0$; and that

$$M = \begin{bmatrix} A \\ C \end{bmatrix} a[A \quad B].$$

Define

$$R = \begin{bmatrix} A \\ C \end{bmatrix}^* M[A \quad B]^* = A^*AA^* + C^*CA^* + A^*BB^* + C^*CaBB^*.$$

One then can compute that

$$\begin{aligned} M^*MM^* = [A \quad B]^*a^*Ra^*\begin{bmatrix} A \\ C \end{bmatrix}^* &= \begin{bmatrix} A^*a^*Ra^*A^* & A^*a^*Ra^*C^* \\ B^*a^*Ra^*A^* & B^*a^*Ra^*C^* \end{bmatrix}\\ &= \begin{bmatrix} R & Ra^*C^* \\ B^*a^*R & B^*a^*Ra^*C^* \end{bmatrix}. \end{aligned}$$

If $r = R^{(1)}$, we have $M^*MM^*\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}M^*MM^* = M^*MM^*$. By the Zlobec result (3.1), $M^\dagger = M^*\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}M^* = [A \quad B]^* r \begin{bmatrix} A \\ C \end{bmatrix}^*$.

We have proved the following theorem.

THEOREM 2. *Suppose M is in form* (1.5), *and that $\rho(M) = \rho(A)$. Then M satisfies* (1.6) *and* (1.7), *and* $M^\dagger = [A \quad B]^* r \begin{bmatrix} A \\ C \end{bmatrix}^*$, *where* $r = \left(\begin{bmatrix} A \\ C \end{bmatrix}^* M[A \quad B]^*\right)^{(1)}$.

Remark. This generalizes a result of Penrose [8] (rediscovered by Zlobec [10]), in which it was assumed that A was nonsingular. Using permutations of rows and columns, any singular matrix M can be put in form (1.5) with $\rho(A) = \rho(M)$. As suggested by Penrose in [8], this gives another method for the calculation of $M^\dagger$ for any singular M, in terms of the calculation of a (1)-inverse of a matrix smaller than M.

5. Examples. To illustrate the formulas, here, we compute the Moore–Penrose inverse of three matrices. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

1. Suppose $M = \begin{pmatrix} I & J \\ J & 0 \end{pmatrix}$. The matrix M satisfies conditions (N) of Theorem 1 since I is nonsingular and $(M/I) = -2J$. Due to the fact that $(M/I)^\dagger = -\frac{1}{8}J$ one obtains

$$M^\dagger = \begin{pmatrix} I - \frac{1}{2}J & \frac{1}{4}J \\ \frac{1}{4}J & -\frac{1}{8}J \end{pmatrix}.$$

The conditions of Theorem 2 are not satisfied since the rank of M is three.

2. Suppose $M = \begin{pmatrix} I & J \\ 0 & 0 \end{pmatrix}$. Here, neither the conditions (N) nor (N') are satisfied. However, Theorem 2 is applicable and one can compute

$$M^\dagger = \begin{bmatrix} I - \frac{2}{5}J & 0 \\ \frac{1}{5}J & 0 \end{bmatrix}.$$

3. Suppose $M = \begin{pmatrix} J & 0 \\ I & J \end{pmatrix}$. The conditions ($N$) or ($N'$) are not satisfied and Theorem 2 is not applicable. However, in any case, we can use (3.2) with Theorem 1 to obtain

$$M^\dagger = \begin{bmatrix} \frac{1}{4}J & I - \frac{1}{2}J \\ -\frac{1}{8}J & \frac{1}{4}J \end{bmatrix}.$$

Appendix. Computations of Theorem 1. Using M as in (1.5) and m in (2.1) with $S = D - CaB$ we find the following computations are useful in verifying Theorem 1:

$$Mm = \begin{bmatrix} Aa - (I - Aa)BsCa & (I - Aa)Bs \\ (I - Ss)Ca & Ss \end{bmatrix},$$

$$mM = \begin{bmatrix} aA - aBsC(I - aA) & aB(I - sS) \\ sC(I - aA) & sS \end{bmatrix},$$

$$MmM = \begin{bmatrix} AaA + (I - Aa)BsC(I - aA) & AaB + (I - Aa)BsS \\ (I - Ss)CaA + SsC & (I - Ss)CaB + SsD \end{bmatrix}$$

$$= \begin{bmatrix} AaA + (I - Aa)BsC(I - aA) & BsS + AaB(I - sS) \\ CaA + SsC(I - aA) & CaB + SsS \end{bmatrix}$$

$$mMm = \begin{bmatrix} [aAa - aBsC(a - aAa)](I + BsCa) - aB(s - sSs)Ca & -aAaBs + aBsC(a - aAa)Bs + aB(s - sSs) \\ sC(a - aAa)(I + BsCa) - sSsCa & sSs - sC(a - aAa)Bs \end{bmatrix}.$$

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