

## ORDERED INCIDENCE GEOMETRY AND THE GEOMETRIC FOUNDATIONS OF CONVEXITY THEORY <sup>1</sup>

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An **Ordered Incidence Geometry**, that is a geometry with certain axioms of **incidence** and **order**, is proposed as a minimal setting for the fundamental convexity theorems, which usually appear in the context of a linear vector space, but require only incidence, order (and for separation, **completeness**), and none of the linear structure of a vector space.

### INTRODUCTION

We study the following question: What are the relevant geometric assumptions for convexity, especially for separation theorems? For answers, i.e. convexity theorems under different axioms see the survey by Danzer, Grünbaum and Klee [7], Ky Fan's generalization of the Krein-Milman Theorem [9], the Helly-type theorems of Levi [15] and Grünbaum [10], and the separation theorem of Ellis [8]. Axiom systems for convexity geometries were given by Prenowitz and Jantosciak [20] and Bryant and Webster ([3],[4],[5]) using **joins** (intervals joining pairs of points) as primitives.

Our objective is to develop convexity geometry using **affine sets** as primitives, in analogy with the classical (Hilbert) approach to Euclidean geometry. The notions used in separation theorems: **convex sets**, **hyperplanes** and **sides** of a hyperplane, are described here in terms of: **affine sets**, their **incidence** properties and **order** relations. The geometry is called **Ordered Incidence Geometry**, since for two or three dimensions the axioms resemble Hilbert's incidence and order axioms [11], see also [25].

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An application to sub- $\mathcal{F}$  functions, [1], is given in §8.

The order of development allows using standard arguments (e.g. [14]) in proofs, which are mostly omitted. An infinite dimensional case is discussed in [23].

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## 1. AXIOMS

An **Ordered Incidence Geometry** (abbreviated **OIG**),  $G$ , is a triple

$$G = \{X, \mathcal{A}, \dim\}$$

and an order relation (**betweenness**), endowed with ten axioms given below. Here

$X$  is the **space** of elements (**points**),

$\mathcal{A}$  is a family of subsets of  $X$ , called the **affine sets** of  $G$ ,

“ $\dim$ ” a function :  $\mathcal{A} \rightarrow \{\text{integers}\} \cup \{\infty\}$ , the **dimension**.

An affine set  $A \in \mathcal{A}$  is called a  $k$ -**affine** if  $\dim A = k$ . In particular, we use the terms: **point** for a 0-affine<sup>2</sup>, **line** for a 1-affine, **plane** for a 2-affine. By convention  $\dim \emptyset = -1$ .

**AXIOM 1.**  $\mathcal{A}$  contains  $X$ ,  $\emptyset$ , and all singletons  $\{x\}$ ,  $x \in X$ .

**AXIOM 2** (Intersection Axiom).  $\mathcal{A}$  is closed under arbitrary intersections.

**DEFINITION 1.** For  $S \subset X$ , the **affine hull** of  $S$  is

$$a(S) = \cap \{A : A \in \mathcal{A}, S \subset A\}$$

which by Axiom 2 is an affine set. We use this to define a hyperplane (needed in §7):

**DEFINITION 2.** A set  $H \in \mathcal{A}$  is a **hyperplane** if exists  $x \in X \setminus H$  such that  $a(H \cup \{x\}) = X$ .

The next three axioms express **monotonicity properties** of dimension.

**AXIOM 3.** If  $A, B \in \mathcal{A}$ ,  $A \subset B$  then  $\dim A \leq \dim B$ .

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<sup>2</sup>It should be clear from the context whether a “point” is an element of  $X$  or of  $\mathcal{A}$ .

AXIOM 4. If  $x \in X$ ,  $A \in \mathcal{A}$ ,  $x \notin A$  then  $\dim \mathfrak{a}(A \cup \{x\}) = \dim A + 1$ .<sup>3</sup>

AXIOM 5. For  $A, B, H \in \mathcal{A}$ ,

$$\text{if } \begin{cases} B, H \subset A, \dim H = \dim A - 1 \\ B \cap H \neq \emptyset, B \end{cases} \text{ then } \dim (B \cap H) = \dim B - 1$$

Points lying on the same line are called **collinear**. For the geometry to have more than one line we need:

AXIOM 6. There exist three non-collinear points.

The remaining 4 axioms define the order relation **betweenness**. For distinct collinear points  $a, b, c$  we denote by  $abc$  the fact that  $b$  is between  $a$  and  $c$ . The set of all points between  $a$  and  $b$  is called the **open segment** or **open interval** joining  $a, b$  and is denoted by  $(a, b)$ .

AXIOM 7.  $abc$  is equivalent to  $cba$ .

AXIOM 8. If  $a \neq c$  then there exist points  $b, d$  such that  $abc$  and  $acd$ .

AXIOM 9. If  $a, b, c$  are distinct and collinear then one and only one of them is between the other two.

AXIOM 10 (The Pasch Axiom<sup>4</sup>). If  $a, b, c$  are distinct, and if  $L$  is a line in  $\mathfrak{a}(\{a, b, c\})$  with  $a, b, c$  not in  $L$  and  $L \cap (a, b) \neq \emptyset$ , then either  $L \cap (a, c) \neq \emptyset$  or  $L \cap (b, c) \neq \emptyset$ .<sup>5</sup>

## 2. IMMEDIATE CONSEQUENCES

The results in this section are immediate consequences of the axioms. Some could be taken as alternative axioms for the same geometry.

COROLLARY 1. If  $A \in \mathcal{A}$  and  $S \subset A$  then  $\mathfrak{a}(S) \subset A$ .  $\square$

This reduces to Hilbert's Incidence Axiom I,6 [11] for  $\dim A = 2$  and  $\#S = 2$ , where  $\#$  denotes **number of elements**.

COROLLARY 2. If  $A$  is a  $k$ -affine,  $A \neq \emptyset$ , then there is a subset  $S$  of  $A$  with  $\#(S) = k + 1$  and  $\dim \mathfrak{a}(S) > k - 1$   $\square$

For  $k = 1, 2$  and  $3$ , Corollary 2 reduces to Hilbert's Incidence Axioms I,3

<sup>3</sup>Thus, for  $\dim X < \infty$ , hyperplanes are  $(\dim X - 1)$ -affines.

<sup>4</sup>[18], §2, Kernsatz IV.

<sup>5</sup>If  $a, b, c$  are collinear, Axiom 10 can be stated as follows: Let  $h \in (a, b)$ ,  $h \neq c$ . Then either  $h \in (a, c)$  or  $h \in (b, c)$ . (The point  $h$  is the intersection of the lines  $\mathfrak{a}(\{a, b, c\})$  and  $L$ .)

and I,8 specifying the existence of (i) two distinct points on any line, (ii) three non-collinear points on any plane, and (iii) four non-coplanar points in the 3-dimensional space.

COROLLARY 3. IF  $S \subset X$ ,  $x \in X$  then  $a(S \cup \{x\}) = a(a(S) \cup \{x\})$ .  $\square$

COROLLARY 4. If  $S \subset X$ ,  $\#S = k + 1$ , then  $\dim a(S) \leq k$ .  $\square$

COROLLARY 5. Let  $A, B$  be  $k$ -affines,  $S \subset A \cap B$ . Then

either  $a(S) = A = B$  or  $\dim a(S) < k$ .  $\square$

COROLLARY 6.  $A, B \in \mathcal{A}$ ,  $A \subset B \implies \dim A = \dim B$  iff  $A = B$ .  $\square$

A converse of Corollary 2 is:

COROLLARY 7. If  $\emptyset \neq S \subset X$ ,  $\#S = k + 1$ ,  $\dim a(S) > k - 1$  then there is a unique  $k$ -affine containing  $S$ .  $\square$

For  $k = 1$  and  $2$ , Corollary 7 reduces to Hilbert's Incidence Axioms I,1-2 and I,4-5, respectively [11].

The following corollary states roughly that if  $a(S)$  is "overdetermined" by  $S$ , then certain points of  $S$  are "affine combinations" of others.

COROLLARY 8. If  $S \subset X$ ,  $\#S = k + 1$ ,  $\dim a(S) \leq k - 1$  then there is an  $x \in S$  such that  $x \in a(S \setminus \{x\})$ .  $\square$

In plane geometry it is well known that the Pasch Axiom is equivalent to a Plane Separation Axiom, ([16], Chapter 12). This holds also here.

DEFINITION 3. Let  $A, H$  be affine sets,  $H \subset A$ ,  $\emptyset \neq H \neq A$ . Then  $H$  separates  $A$  if for any two points  $x, y \in A \setminus H$  such that

$$(x, y) \cap H \neq \emptyset \quad (1)$$

there is no point  $z \in A \setminus H$  such that<sup>6</sup>,

$$(x, z) \cap H = \emptyset \text{ and } (y, z) \cap H = \emptyset \quad (2)$$

THEOREM 1. Let  $A, H \in \mathcal{A}$ ,  $\dim A \geq 1$ ,  $H \subset A$ ,  $\dim H = \dim A - 1$ . Then  $H$  separates  $A$ .

PROOF. The case  $\dim A = 1$  follows from Axiom 10 (footnote 5). Let  $\dim A \geq 2$ , and suppose  $H$  does not separate  $A$ , i.e. there are distinct points  $x, y, z \in A \setminus H$  satisfying (1) and (2). It follows from Axiom 10 that

<sup>6</sup>I.e.  $x$  and  $y$  cannot be, at the same time, on "opposite" sides of  $H$  and on the "same" side of  $H$ .

$x, y, z$  are non-collinear. Let  $P$  be the plane through  $x, y, z$ . The intersection  $P \cap H$  is

- (i)  $H$  if  $\dim A = 2$  (i.e.  $P = A$ ),
- (ii) a line if  $\dim A > 2$  (by Axiom 5),

so that, in either case,  $P \cap H$  is a line, say  $L$ . Since  $L$  intersects  $(x, y)$ , it follows from Axiom 10 that  $L$  also intersects  $(x, z)$  or  $(y, z)$ , violating (2).  $\square$  Conversely, Axiom 10 follows from Theorem 1, i.e. the two are equivalent.

### 3. MODELS

Concrete models of Ordered Incidence Geometries include: The **real Euclidean  $n$ -dimensional space** ( $\mathbf{X} = R^n$ , with "affine sets" and "dimension" given their standard vector space meanings), the **Poincare half-plane incidence plane** ( $\mathbf{X}$  is the upper half-plane, and 1-affines are the restrictions to  $\mathbf{X}$  of (i) vertical lines, and (ii) circles with centers on the  $x$ -axis) and the **Moulton incidence plane**, [16]. The last two models are special cases of **Beckenbach geometries**, defined below.

**DEFINITION 4** ([1], see also ([21], §§84-85). Let  $(a, b), (c, d)$  be open intervals (not necessarily bounded) in  $R$ . A family  $\mathcal{F}$  of continuous functions  $F : (a, b) \rightarrow (c, d)$  is a **Beckenbach family** (**B-family** for short) if for any two points  $(x_1, y_1), (x_2, y_2)$  with  $a < x_1 < x_2 < b, y_1, y_2 \in (c, d)$  there exists a unique  $F \in \mathcal{F}$ , denoted by  $F_{12}$ , such that

$$F_{12}(x_i) = y_i, \quad (i = 1, 2) \quad (3)$$

**DEFINITION 5.** Let  $(a, b), (c, d)$  be as above and let  $\mathcal{F}$  be a B-family of functions  $(a, b) \rightarrow (c, d)$ . The **Beckenbach geometry** (**B-geometry**),  $\mathbf{G}_{\mathcal{F}}$ , determined by  $\mathcal{F}$ , is a two dimensional geometry with  $\mathbf{X} = (a, b) \times (c, d)$ , and the 1-affine through any pair of points  $(x_1, y_1), (x_2, y_2)$  in  $\mathbf{X}$  is,

- (i) the vertical line  $x = x_1$  if  $x_1 = x_2$ ,
- (ii) the graph of  $F_{12}$  (defined by (3)) if  $x_1 \neq x_2$ .

For a B-geometry Axioms 1-9 are easily verified, and Axiom 10 follows from: **LEMMA 1** (Beckenbach [1]). Let  $a < x_0 < b$  and let  $F_{\alpha}, F_{\beta}$  be two distinct members of  $\mathcal{F}$  such that  $F_{\alpha}(x_0) = F_{\beta}(x_0)$ . Then

- $F_{\alpha}(x) > F_{\beta}(x)$  for all  $x$  in  $(a, b)$  on one side of  $x_0$ ,
- $F_{\alpha}(x) < F_{\beta}(x)$  for all  $x$  in  $(a, b)$  on the other side of  $x_0$ .  $\square$

The next four examples use B-families in the form  $\mathcal{F} = \{F(x; \alpha, \beta)\}$ , where  $\alpha, \beta$  are real parameters.

EXAMPLE 1.  $\mathcal{F}_1 = \{\sqrt{\beta - (x - \alpha)^2} : \beta > 0, \alpha \in R\}$  with  $(a, b) = R$ ,  $(c, d) = (0, \infty)$ , and  $G_{\mathcal{F}_1}$  gives the Poincare half-plane incidence plane.

EXAMPLE 2.  $\mathcal{F}_2 = \{\alpha\phi_1(x) + \beta\phi_2(x) + \phi_3(x)\}$  where:

- (i)  $\phi_i$  are given continuous functions, and
- (ii)  $\phi_2(x) > 0$  on  $(a, b)$ .

A necessary and sufficient condition for  $\mathcal{F}_2$  to be a B-family is that  $\phi_1/\phi_2$  is strictly monotone. For example, with  $(a, b) = R$ ,  $\mathcal{F} = \{\alpha e^x + \beta e^{-x}\}$  is a B-family, while  $\mathcal{F} = \{\alpha x^2 + \beta\}$  is not.

EXAMPLE 3.  $\mathcal{F}_3(x; \alpha, \beta) = \phi(\alpha, x) - \beta$  with  $\phi$  differentiable in  $\alpha$  for all  $x$ . A necessary and sufficient condition for  $\mathcal{F}_3$  to be a B-family is that  $\partial\phi/\partial\alpha$  is a strictly monotone function of  $x$ . For  $\phi(\alpha, x) = \alpha x$ ,  $G_{\mathcal{F}_3}$  is the Euclidean plane geometry.

EXAMPLE 4.  $\mathcal{F}_4(x; \alpha, \beta) = a(\alpha)u(x) + b(\alpha)v(x) - \beta$ , where  $a$  and  $u$  are strictly increasing,  $b$  and  $v$  strictly decreasing. For example,  $\mathcal{F} = \{\cosh(\alpha + x) - \beta : \alpha, \beta \in R\}$ .

#### 4. TRIANGLES

Three non-collinear points  $\{a, b, c\}$  constitute a triangle  $\Delta abc$ . The basic properties of triangles ([18], §2) are sampled in the following two pairs of lemmas, each pair consisting of a result and a (sort of) converse:

LEMMA 2. For any  $u \in (a, c)$  and  $v \in (u, b)$ , there is a point  $w \in (b, c)$  such that  $v \in (a, w)$ .  $\square$

LEMMA 3. For any  $u \in (a, c)$  and  $w \in (b, c)$  there is a point  $v$  in the intersection  $(a, w) \cap (b, u)$ .  $\square$

LEMMA 4. Let  $u \in (a, c)$ ,  $w \in (b, c)$ . Then for any  $v \in (u, w)$  there is a  $z \in (a, b)$  such that  $v \in (c, z)$ .  $\square$

LEMMA 5. Let  $z \in (a, b)$ . Then for any  $v \in (c, z)$  there exist two points  $u \in (a, c)$ ,  $w \in (b, c)$  such that  $v \in (u, w)$ .  $\square$

## 5. LINEAL HULLS

In the Euclidean geometry  $R^n$  a set  $A$  is **affine** if and only if

$$A = \left\{ \sum \lambda_i x_i : x_i \in A, \sum \lambda_i = 1 \right\} \quad (4)$$

i.e.  $A$  coincides with the set of **affine combinations** of its elements. The analogous representation in an OIG (where algebraic constructions such as (4) are not available) is given in Theorem 2. First we require:

**DEFINITION 6.** For a given subset  $S$  of  $\mathbf{X}$ , the **lineal hull** of  $S$ , is

$$\ell(S) = \cup \{ \overline{xy} : x, y \in S \}$$

the union of lines through pairs of points in  $S$ . By convention,  $\ell(\emptyset) = \emptyset$  and  $\ell(\{x\}) = \{x\}$ ,  $\forall x \in \mathbf{X}$ . We also use the abbreviation  $\ell^{(2)}(S) = \ell(\ell(S))$ .

**LEMMA 6.** If  $A$  is affine,  $x \notin A$ , then  $\mathbf{a}(A \cup \{x\}) = \ell^{(2)}(A \cup \{x\})$ .

**PROOF.** Use Corollary 1 and Axiom 8.  $\square$

**THEOREM 2.**  $S$  is an affine set if and only if  $S = \ell(S)$ .  $\square$

## 6. CONVEX SETS

The basic properties of convex sets are developed in this section.

**DEFINITION 7.** A set  $S \subset \mathbf{X}$  is:

- (i) **star shaped** at  $x$  if for all  $y \in S$ ,  $(x, y) \subset S$ ,
- (ii) **convex** if for any two points  $x, y \in S$ ,  $(x, y) \subset S$ .

**DEFINITION 8.** For any set  $S \subset \mathbf{X}$ , the **convex hull** of  $S$ ,  $\text{conv}(S)$ , is the intersection of all convex sets containing  $S$ .

**DEFINITION 9.** For any set  $S \subset \mathbf{X}$ ,

- (i) the **core** of  $S$  is

$$\text{core } S = \{ x \in S : \forall y \in \mathbf{X}, y \neq x, \exists z \in (x, y) \text{ such that } (x, z) \subset S \} \quad (5)$$

- (ii) the **relative core** of  $S$ ,  $\text{relcore } S$ , is defined by (5) with

$$\text{“}\forall y \in \mathbf{a}(S)\text{” replacing “}\forall y \in \mathbf{X}\text{”}$$

(iii) the set **linearly accessible** from  $S$ , is

$$\text{lina } S = \{y \in X : \exists x \in S \text{ such that } (x, y) \subset S\}$$

(iv) the **closure** of  $S$  is  $\text{cl } S = S \cup \text{lina } S$ .

LEMMA 7. Let a set  $S$  with a nonempty relative core<sup>7</sup> be star shaped at  $p$ . Then  $x \in \text{relcore } S \implies (p, x) \subset \text{relcore } S$  i.e.  $\text{relcore } S$  is also star shaped at  $p$ .  $\square$

THEOREM 3. Let  $S$  be a convex set,  $y \in \text{lina } S$ ,  $x \in \text{relcore } S$ ,  $y \neq x$ . Then  $(x, y) \subset \text{relcore } S$ .

PROOF. Use Axioms 8, 10 and Lemma 7.  $\square$

THEOREM 4. If  $S$  is convex, then  $\text{relcore } S$  and  $\text{cl } S$  are convex.  $\square$

This section ends with a result of a topological nature, Theorem 5, that a (nonempty) finite-dimensional convex set has a nonempty relative core.

DEFINITION 10. For any  $S \subset X$ , the **dimension** of  $S$  is defined as the dimension of its affine hull,  $\dim S = \dim \mathbf{a}(S)$ .

DEFINITION 11. An  $n$ -**simplex** is the convex hull of a set  $S$  with

$$\#S = n + 1, \dim S = n$$

We prove first that a simplex has a nonempty relative core.

LEMMA 8. Let  $\Delta_n = \text{conv } \{x_1, \dots, x_{n+1}\}$  be an  $n$ -simplex. Then

$$\text{relcore } \Delta_n \neq \emptyset$$

PROOF. By induction on  $n$ . For  $n = 1$  the result follows from Axiom 8. In the inductive step use Axioms 4, 5 and 10.  $\square$

LEMMA 9. If  $\emptyset \neq C$  is a convex set,  $\dim C < \infty$ , then  $\dim C = \dim \Delta_{\max}$  where  $\Delta_{\max}$  is a maximal dimensional simplex contained in  $C$ .

PROOF. The existence of a maximal  $\Delta_{\max}$  follows since  $C$  is finite-dimensional. Then use Axiom 4 to prove  $\mathbf{a}(C) = \mathbf{a}(\Delta_{\max})$ .  $\square$

THEOREM 5. If  $C$  is a nonempty convex set,  $\dim C < \infty$ , then

$$\text{relcore } C \neq \emptyset. \quad \square$$

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<sup>7</sup>In the finite-dimensional case,  $\text{relcore } S \neq \emptyset$  for any convex set  $S$ , see Theorem 5.



## 7. SEPARATION

The main result here is Theorem 7 stating conditions under which two disjoint convex sets can be separated by a hyperplane.

From Theorem 1 it follows that any hyperplane separates the space in the sense of Definition 3. We elaborate on this statement in the following:

LEMMA 10. Given a hyperplane  $H$  in  $X$ , there exist unique nonempty convex sets  $H^+$ ,  $H^-$  such that

- (a)  $H$ ,  $H^+$ ,  $H^-$  are disjoint, and
- (b)  $X = H \cup H^+ \cup H^-$ .

PROOF. Select any  $x_0 \in X \setminus H$  and define:

$$H^+ = \{y \notin H : (x_0, y) \cap H = \emptyset\}, \quad (6)$$

$$H^- = \{y \notin H : (x_0, y) \cap H \neq \emptyset\}. \quad (7)$$

Then (a) and (b) are obvious. Nonemptiness, convexity and uniqueness<sup>8</sup> of  $H^+$  and  $H^-$  use standard arguments.  $\square$

One can similarly obtain:

LEMMA 11. Let  $H$ ,  $H^+$ ,  $H^-$  be as in Lemma 10. Then

- (i)  $H^+ = \text{core } H^+$ ,  $H^- = \text{core } H^-$
- (ii)  $H = \text{lina } H^+ \cap \text{lina } H^-$
- (iii)  $H \cup H^+ = \text{cl } H^+$ ,  $H \cup H^- = \text{cl } H^-$ .  $\square$

DEFINITION 12. A convex set  $C$  is

- (i) **open** if  $C = \text{core } C$ ,
- (ii) **relatively open** if  $C = \text{relcore } C$ ,
- (iii) **closed** if  $C = \text{cl } C$ .

The following definitions are suggested by Lemma 11.

DEFINITION 13. Let  $H$ ,  $H^+$ ,  $H^-$  be as in Lemma 10. Then

- (i)  $H^+$ ,  $H^-$  are the **open halfspaces** of (i.e. corresponding to)  $H$ .
- (ii)  $H \cup H^+$ ,  $H \cup H^-$  are the **closed halfspaces** of  $H$ .

DEFINITION 14. Let  $A, B \subset X$ , and let  $H$  be a hyperplane. Then

- (i)  $H$  **separates**  $A$  and  $B$  if  $A$  and  $B$  are contained in opposite closed halfspaces of  $H$ .

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<sup>8</sup>Uniqueness means that the (unordered) pair  $\{H^+, H^-\}$  is independent of the particular  $x_0$  used in (6), (7).

(ii) Moreover, if  $A \cup B$  is not a subset of  $H$  then  $H$  separates  $A$  and  $B$  properly.

The next lemma implies a converse of Theorem 1: The only affine sets with the separation property (Definition 3) are hyperplanes.

LEMMA 12. If  $A, B, C$  are affine sets,  $A \subset B$  and  $A \neq B$ , and if  $B$  separates  $C$ , then  $A$  does not separate  $C$ .

PROOF. Suppose  $A$  separates  $C$  and let  $A^+, A^-$  be the "opposite sides" of  $A$  in  $C$ , i.e.

$$C = A \cup A^+ \cup A^-$$

where  $A, A^+, A^-$  are disjoint,

$$x, y \text{ in } A^+ \text{ or in } A^- \implies (x, y) \cap A = \emptyset,$$

$$x \in A^+, y \in A^- \implies (x, y) \cap A \neq \emptyset.$$

Given that  $B$  separates  $C$ , let

$$C = B \cup B^+ \cup B^-$$

be the analogous decomposition of  $C$  with respect to  $B$ .

Now let  $x \in B^+$ . Then  $x \notin A$ , and without loss of generality let  $x \in A^+$ . Any other point in  $B^+$  must also be in  $A^+$  for if  $x \neq y \in B^+, y \in A^-$  then  $(x, y)$  intersects  $A$  but not  $B$ , a contradiction. Therefore

$$B^+ \subset A^+ \tag{8}$$

and similarly,

$$B^- \subset A^- \tag{9}$$

The inclusions (8) and (9) lead to contradiction.  $\square$

DEFINITION 15. A **convex pair** in  $X$  is an unordered pair  $\{C, D\}$  of nonempty convex sets with

$$X = C \cup D, C \cap D = \emptyset$$

A classical result of Mazur [17] and Kakutani [13], (see also [24], Theorem 2.3 and references therein), can be stated for OIG as follows:

THEOREM 6. If  $A, B$  are disjoint convex sets in  $X$ , then there exists a convex pair  $\{C, D\}$  with

$$A \subset C, B \subset D \tag{10}$$

PROOF. We first prove an auxilliary result:

(a) If  $S$  is a nonempty convex set, and  $x_0 \notin S$ , then the set

$$K(S, x_0) = \{x : x \in [x_0, y], y \in S\}$$

is convex. Here we denote by  $[a, b]$  the closed segment joining  $a, b$ :

$$[a, b] = (a, b) \cup \{a\} \cup \{b\}$$

Given (a), the proof of ([24], Theorem 2.3) can be followed, using Zorn's Lemma to obtain  $\{C, D\}$  as a maximal element of the set (partially ordered by inclusion) of disjoint convex sets  $\{C, D\}$  satisfying (10).  $\square$

Convex pairs are used to prove existence of hyperplanes separating disjoint convex sets. First we require:

DEFINITION 16. An affine set is **openly decomposable** if it is the union of two disjoint, relatively open, nonempty convex sets.

DEFINITION 17. A geometry  $\mathbf{G} = \{\mathbf{X}, \mathcal{A}, \text{dim}\}$  is a **Complete Ordered Incidence Geometry (COIG)** for short if it satisfies, in addition to Axioms 1-10 (of §1), the following:

AXIOM 11 (Completeness Axiom). No line in  $\mathbf{G}$  is openly decomposable.

This usage of completeness is standard, see also [5]. An example of a non-complete OIG is the **rational Euclidean  $n$ -dimensional space** with  $\mathbf{X} = \mathbb{Q}^n$ , the set of rational  $n$ -tuples, and  $\mathcal{A}$  and "dim" given their standard vector space meanings.

Lemma 14 below requires the following property (seemingly stronger than completeness): No affine set (of any dimension) is openly decomposable. We show this to be equivalent to completeness.

LEMMA 13. Let  $\mathbf{G} = \{\mathbf{X}, \mathcal{A}, \text{dim}\}$  be a COIG. Then no affine set is openly decomposable.

PROOF. Let  $A \in \mathcal{A}$  be openly decomposable, i.e.

$$A = C_1 \cup C_2 \tag{11}$$

where  $C_1, C_2$  are disjoint, nonempty, relatively open, convex sets. From (11) follows  $A = \mathbf{a}(C_1) \cup \mathbf{a}(C_2)$  and consequently  $A = \mathbf{a}(C_1) = \mathbf{a}(C_2)$ , showing

that, restricted to  $A$ , the relative cores of  $C_1$  and  $C_2$  can be taken as cores, i.e.

$$C_i = \text{core } C_i, \quad i = 1, 2 \quad (12)$$

Choose any two points  $x_i \in C_i$  and let  $L$  be the line  $\overline{x_1 x_2}$ . From (12) follows the existence of points  $z_i \in C_i$ , ( $i = 1, 2$ ), such that

$$(x_i, z_i) \subset L, \quad i = 1, 2$$

Extending the two (relatively) open segments  $(x_i, z_i)$  beyond  $x_i$ , ( $i = 1, 2$ ), we get the intervals (unbounded on one side):

$$I_i = (x_i, z_i) \cup \{y \in L : x_i \in (z_i, y)\}, \quad i = 1, 2$$

By Zorn's Lemma, the set of such intervals has a maximal element  $\{I_1, I_2\}$ , and consequently  $L = I_1 \cup I_2$ , violating Axiom 11.  $\square$

Given a convex pair  $\{C, D\}$ , the following lemma gives conditions for the existence of a hyperplane with opposite sides  $\{\text{core } C, \text{core } D\}$ .

LEMMA 14. If  $\{C, D\}$  is a convex pair in  $\mathbf{X}$ , then the set  $H$  defined by

$$H = \text{cl } C \cap \text{cl } D$$

satisfies:

$$(a) \quad H \cap \text{core } C = \emptyset = H \cap \text{core } D$$

If the geometry is complete then:

$$(b) \quad H \neq \emptyset \text{ and } X = H \cup \text{core } C \cup \text{core } D$$

$$(c) \quad \text{If either } \text{core } C \neq \emptyset \text{ or } \dim \mathbf{X} < \infty, \text{ then } H \text{ is a hyperplane. } \square$$

Combining the above results, we finally prove:

THEOREM 7. (The Separation Theorem). Let  $\mathbf{G} = \{\mathbf{X}, \mathcal{A}, \dim\}$  be a COIG, and let  $A, B$  be disjoint convex sets in  $\mathbf{X}$ . Then a hyperplane  $H$  properly separating  $A$  and  $B$  exists if:

$$(a) \quad \text{core } A \neq \emptyset, \text{ in which case } H \cap \text{core } A = \emptyset,$$

or if

$$(b) \quad \dim \mathbf{X} < \infty.$$

PROOF. Let  $\{C, D\}$  and  $H$  be given by Theorem 6 and Lemma 14. Then  $H$  separates  $A$  and  $B$  in the sense that

$$A \subset H \cup \text{core } C = H \cup H^+, \quad B \subset H \cup \text{core } D = H \cup H^- \quad (13)$$

To prove proper separation (Definition 14(ii)) we show that

$$A \cup B \not\subset H \quad (14)$$

(a) If core  $A \neq \emptyset$  then, by (10), core  $A \subset \text{core } C \neq \emptyset$  and (14) follows from Lemma 14(a).

(b) Let  $\dim \mathbf{X} < \infty$ , core  $A = \emptyset$  (otherwise it is case (a) again), and

$$A \cup B \subset H$$

We restrict the discussion to  $H$  which we denote by  $H_1$ . In  $H_1$  there is a hyperplane  $H_2$  (i.e.  $\dim H_2 = \dim H_1 - 1$ ) separating  $A$  and  $B$  in the sense of (13). Now there are two cases:

(i)  $H_2$  separates  $A$  and  $B$  properly, (ii)  $A \cup B \subset H_2$ .

In case (ii) we repeat the process: Restrict to  $H_2$ , find a hyperplane  $H_3$  (in  $H_2$ ) separating  $A$  and  $B$ , etc. From  $\dim H_{i+1} = \dim H_i - 1$  it follows that after finitely many repetitions, an affine set  $H_i$  is reached in which one of the sets  $A, B$  has a nonempty core, i.e.

$$\dim H_i = \max \{ \dim A, \dim B \}$$

and, by part (a), it is case (i), (although case (i) may occur sooner.) Suppose then that case (i) is reached after  $k$  successive restrictions, a situation described by

$$\mathbf{X} = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_{k+1}$$

where  $H_{i+1}$  separates  $A$  and  $B$  in  $H_i$  ( $i = 0, \dots, k$ ),  $A \cup B \subset H_k$  and  $A \cup B \not\subset H_{k+1}$ . Reversing our steps we construct a sequence of affine sets  $\{\bar{H}_i\}$

$$H_{k+1} = \bar{H}_{k+1} \subset \bar{H}_k \subset \cdots \subset \bar{H}_1 \subset \bar{H}_0 = \mathbf{X} \quad (15)$$

where  $\bar{H}_1$  separates  $A$  and  $B$  properly in  $\mathbf{X}$ . A sequence (15) is defined recursively as follows:

For  $i = k, \dots, 1$   
 choose any  $x_i \in H_{i-1} \setminus \text{a}(A \cup B)$   
 define  $\bar{H}_i = \text{a}(\bar{H}_{i+1} \cup \{x_i\}) \quad \square$

REMARKS. (a) If core  $A \neq \emptyset$ , the assumption " $A \cap B = \emptyset$ " in Theorem 7 can be replaced by "core  $A \cap B = \emptyset$ ".

(b) To show that completeness is needed in Theorem 7, consider the rational line  $Q$ , in which the sets  $\{x : x < \sqrt{2}\}$  and  $\{x : x > \sqrt{2}\}$  cannot be separated

by a hyperplane (the hyperplanes of  $Q$  are its points).

### 8. APPLICATIONS TO FUNCTIONS ON THE REAL LINE: SUB- $\mathcal{F}$ FUNCTIONS AND FENCHEL DUALITY.

We use the terminology of §3. Let  $\mathcal{F}$  be a given B-family on the interval  $(a, b)$  (we take  $(c, d) = R$ ), and let  $\mathbf{G}_{\mathcal{F}}$  be the associated B-geometry.

DEFINITION 18 (Beckenbach [1]). A function  $f : (a, b) \rightarrow R$  is sub- $\mathcal{F}$  if for any two points  $a < x_1 < x_2 < b$  and  $F_{12} \in \mathcal{F}$  defined by  $F_{12}(x_i) = f(x_i)$ ,  $i = 1, 2$ ,

$$f(x) \leq F_{12}(x) \quad \text{for all } x_1 \leq x \leq x_2 \quad (16)$$

$f$  is **super- $\mathcal{F}$**  if the reverse inequality holds in (16). Sub- $\mathcal{F}$  functions are generalizations of convex functions. Indeed, for the family  $\mathcal{F}$  of affine functions, sub- $\mathcal{F}$  and super- $\mathcal{F}$  become **convex** and **concave** (in the ordinary sense), respectively. Sub- $\mathcal{F}$  functions have been applied to 2nd order differential inequalities, e.g. [6], [12] and [19].

In this section we study the geometric properties of sub- $\mathcal{F}$  functions (in the geometry  $\mathbf{G}_{\mathcal{F}}$ ) and establish a Fenchel duality theorem (Theorem 8).

As usual, denote the **epigraph** and **hypograph** of  $f$  by

$$\text{epi } f = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \geq f(x) \right\} \quad (17)$$

$$\text{hypo } f = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \leq f(x) \right\} \quad (18)$$

Here also, "convexity of a set" and "convexity of a function" are related:

LEMMA 15. A function  $f : (a, b) \rightarrow R$  is:

- (a) sub- $\mathcal{F}$  iff epi  $f$  is convex.
- (b) super- $\mathcal{F}$  iff hypo  $f$  is convex.  $\square$

The following characterization of the core of epi  $f$  is useful.

LEMMA 16. Let  $a < a_1 < b_1 < b$ , let  $f$  be sub- $\mathcal{F}$  on  $(a_1, b_1)$ , and

$$A = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \begin{array}{l} a_1 \leq x \leq b_1 \\ \mu \geq f(x) \end{array} \right\}$$

Then

$$\text{core } A = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \begin{array}{l} a_1 < x < b_1 \\ \mu > f(x) \end{array} \right\} \quad \square$$

REMARK. The separation theorem (Theorem 7), applied to the sets

$$A = \text{core}(\text{epi } f) \quad \text{and} \quad B = \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} \right\}$$

for given  $x \in (a, b)$ , shows that a sub- $\mathcal{F}$  function is supported at each point in  $(a, b)$  by a function  $F \in \mathcal{F}$ , [19]. Indeed, the support property is necessary and sufficient for  $f$  to be sub- $\mathcal{F}$ .

From here on we specialize to the B-families of §3, Example 3,

$$\mathcal{F} = \{F(x) = \phi(\alpha, x) - \beta : \alpha, \beta \in R\} \quad (19)$$

where  $\phi$  is differentiable in  $\alpha, x$  and

$$\frac{\partial \phi}{\partial \alpha} \quad \text{is an increasing function of } x. \quad (20)$$

As in [2] define the **dual family**<sup>9</sup>

$$\mathcal{F}^* = \{F^*(\alpha) = \phi(\alpha, x) - \beta : x, \beta \in R\} \quad (21)$$

which is a B-family if <sup>10</sup>

$$\frac{\partial \phi}{\partial x} \quad \text{is an increasing function of } \alpha. \quad (22)$$

The **effective domain** of a function  $f$ , [22], is denoted by  $\text{dom } f$ .

DEFINITION 19. Given  $f : (a, b) \rightarrow R$ ,

(i) the **(convex) conjugate** of  $f$ ,  $f^*$ , is

$$f^*(\alpha) = \sup_{x \in \text{dom } f} \{\phi(\alpha, x) - f(x)\} \quad (23)$$

(ii) the **(concave) conjugate** of  $f$ ,  $f_*$ , is

$$f_*(\alpha) = \inf_{x \in \text{dom } f} \{\phi(\alpha, x) - f(x)\} \quad (24)$$

<sup>9</sup>Note that in  $\mathcal{F}^*$  the argument is  $\alpha$  (one of the parameters of  $\mathcal{F}$ ) and the parameters are  $x, \beta$ . Thus any pair  $\{x, \beta\}$  determines a unique  $F^* = F^*(\cdot; x, \beta)$  in  $\mathcal{F}^*$ .

<sup>10</sup>The conditions (20) and (22) guarantee that both  $\mathcal{F}$  and  $\mathcal{F}^*$  are B-families.

In convex analysis, the conjugate  $f^*$  is convex, regardless of  $f$ . The analogous result here is:

LEMMA 17. For any function  $f : (a, b) \rightarrow R$ ,

- (a)  $f^*$  is sub- $\mathcal{F}^*$ ,
- (b)  $f_*$  is super- $\mathcal{F}^*$ .  $\square$

A duality theorem of Fenchel type (see also [22], Theorem 31.1) now follows. A (somewhat weaker) Fenchel duality theorem was proved in [2] for  $\mathcal{F}$ -convex functions :  $R^n \rightarrow R$ .

THEOREM 8. Let

$f$  be a sub- $\mathcal{F}$  function:  $(a, b) \rightarrow R$ ,

$g$  be a super- $\mathcal{F}$  function:  $(a, b) \rightarrow R$ ,

and consider the pair of problems<sup>11</sup>

$$\inf \{f(x) - g(x) : x \in \text{dom } f \cap \text{dom } g\} \quad (\text{P})$$

$$\sup \{g_*(\alpha) - f^*(\alpha) : \alpha \in \text{dom } f^* \cap \text{dom } g_*\} \quad (\text{D})$$

If<sup>12</sup>

$$\text{int dom } f \cap \text{int dom } g \neq \emptyset \quad (25)$$

then

$$\inf (\text{P}) = \max (\text{D})$$

PROOF. The proof is similar to the proof of ([22], Theorem 31.1). From (23) and (24),

$$g(x) + g_*(\alpha) \leq \phi(\alpha, x) \leq f(x) + f^*(\alpha), \quad \forall x, \alpha$$

so that

$$f(x) - g(x) \geq g_*(\alpha) - f^*(\alpha), \quad \forall x, \alpha$$

proving

$$\inf (\text{P}) \geq \sup (\text{D}) \quad (26)$$

In particular,  $\inf (\text{P}) = -\infty \Rightarrow \sup (\text{D}) = -\infty$ . Let  $\inf (\text{P}) > -\infty$ , and denote

$$\begin{aligned} \gamma &= \inf (\text{P}) \\ &= \sup \{\beta : f(x) \geq g(x) + \beta, \quad \forall x\} \end{aligned} \quad (27)$$

<sup>11</sup>The difference  $f - g$  was shown in ([2], Theorem 4) to be unimodal

<sup>12</sup> $\text{dom } f$  and  $\text{dom } g$  are intervals in  $(a, b)$ , and "int" denotes the interior of a real interval.



By (26) it suffices to show the existence of  $\alpha \in \text{dom } f^* \cap \text{dom } g_*$  such that

$$g_*(\alpha) - f^*(\alpha) \geq \gamma \quad (28)$$

Define two sets

$$A = \text{epi } f, \quad B = \text{hypo } (g + \gamma) = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \leq g(x) + \gamma \right\}$$

Note that for families  $\mathcal{F}$  of type (19),

$$f \text{ is } \left\{ \begin{array}{l} \text{sub-}\mathcal{F} \\ \text{super-}\mathcal{F} \end{array} \right\} \implies (f + \gamma) \text{ is } \left\{ \begin{array}{l} \text{sub-}\mathcal{F} \\ \text{super-}\mathcal{F} \end{array} \right\}, \quad \forall \gamma \in R$$

Therefore, using Lemma 15,  $A$  and  $B$  are convex in  $\mathbf{G}_{\mathcal{F}}$ . Now  $\{\text{core } A\} \cap B = \emptyset$  by Lemma 16, and by theorem 7 there is a hyperplane  $H$  separating core  $A$  and  $B$ , and therefore separating  $A$  and  $B$ . In the geometry  $\mathbf{G}_{\mathcal{F}}$  hyperplanes are the lines of Definition 5. By (25), the separating line  $H$  cannot be vertical, and is therefore of the form

$$H = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu = \phi(x, \alpha^*) - \beta^* \right\}$$

for some pair of parameters  $\alpha^*, \beta^*$ . Since  $H$  separates  $A$  and  $B$ ,

$$\begin{aligned} f(x) &\geq \phi(\alpha^*, x) - \beta^* \geq g(x) + \gamma, \quad \forall x \\ \therefore \beta^* &\geq \sup_x \{\phi(\alpha^*, x) - f(x)\} = f^*(\alpha^*) \\ \therefore \gamma + \beta^* &\leq \inf_x \{\phi(\alpha^*, x) - g(x)\} = g_*(\alpha^*) \end{aligned}$$

And finally,  $\gamma \leq g_*(\alpha^*) - f^*(\alpha^*)$ , proving (28).  $\square$

The following example illustrates Theorem 8. Here (P) is a convex program, and there are infinitely many possible Fenchel duals, corresponding to the various decompositions of the objective function  $f - g$ , and the choice of the underlying family  $\mathcal{F}$ . One such dual is (D) below.

EXAMPLE 5. Let the primal problem be

$$\inf_{x \geq 0} (e^x + e^{-x}) \quad (P)$$

with optimal solution  $x^* = 0$ . We choose

$$\begin{aligned} f(x) &= e^x + e^{-x}, \quad \text{dom } f = R, \\ g(x) &= -\delta(x | R^+) = \begin{cases} 0 & , x \geq 0 \\ -\infty & , \text{otherwise} \end{cases}, \quad \text{dom } g = R^+ \end{aligned}$$

Consider the family  $\mathcal{F} = \{F(x) = \cosh(\alpha + x) - \beta\}$ , (§3, Example 4). Here  $\mathcal{F} = \mathcal{F}^*$  and (20), (22) are satisfied. Since  $f \in \mathcal{F}$ ,  $f$  is sub- $\mathcal{F}$ . Also (since  $\mathcal{F}$  consists of convex functions), the indicator function  $g$  is (strictly) super- $\mathcal{F}$ . The conjugates can be computed to give:

$$f^*(\alpha) = \begin{cases} -\sqrt{(e^\alpha - 2)(e^{-\alpha} - 2)} & \text{if } |\alpha| \leq \log 2 \\ \infty & \text{otherwise} \end{cases}$$

$$g_*(\alpha) = \cosh(\max\{0, \alpha\})$$

so that  $\text{dom } f^* = [-\log 2, \log 2]$ ,  $\text{dom } g_* = \mathbb{R}$  and the dual program is

$$\sup_{|\alpha| \leq \log 2} \{\cosh(\max\{0, \alpha\}) + \sqrt{(e^\alpha - 2)(e^{-\alpha} - 2)}\} \quad (\text{D})$$

It can be verified that the optimal solution of (D) is  $\alpha^* = 0$ , and  $\text{inf (P)} = \text{sup (D)} = 2$ .

## 9. THE THEOREMS OF RADON AND HELLY.

The (closely related) theorems of Radon, Helly and Caratheodory (see e.g. [7]) hold also for OIG.

LEMMA 18. If  $S \subset \mathbf{X}$ ,  $\#S = n + 2$ ,  $\dim S = n$ , then there is a subset  $T \subset S$ ,  $\#T = n$ , such that  $S = T \cup \{x\} \cup \{y\}$  and  $\mathbf{a}(T) \cap [x, y] \neq \emptyset$ .  $\square$

THEOREM 9 (Radon). Let  $S \subset \mathbf{X}$ ,  $\#S \geq n + 2$ ,  $\dim S = n$ . Then  $S$  can be partitioned into  $S = S_1 \cup S_2$  where  $S_1 \cap S_2 = \emptyset$  and  $\text{conv } S_1 \cap \text{conv } S_2 \neq \emptyset$ .  
PROOF. Enough to consider the case  $\#S = n + 2$ . We prove by induction on  $n$ . For  $n = 1$ , the theorem follows from the order axioms. For  $n = 2$  and the inductive step, use Lemma 18.  $\square$

The proof of [24] can now be used, verbatim:

THEOREM 10 (Helly). Let  $\dim \mathbf{X} = n$  and let  $S$  be any family of convex sets in  $\mathbf{X}$ ,  $\#S = k > n + 1$ . If every  $n + 1$  sets in  $S$  have a nonempty intersection, then  $S$  has a nonempty intersection.  $\square$

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