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COMPLEMENTARY HALFSPACES AND TRIGONOMETRIC CEVA–BROCARD INEQUALITIES FOR POLYGONS

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Abstract. The product of ratios that equals 1 in Ceva's Theorem is analyzed in the case of non-concurrent Cevians, for triangles as well as arbitrary convex polygons. A general lemma on complementary systems of inequalities is proved, and used to classify the possible cases of non-concurrent Cevians. In the concurrent case, particular consideration is given to the Brocard configuration defined by equal angles between Cevians and polygon sides.

1. Introduction

In his study of triangle geometry, Henri Brocard [1845-1922] focused attention on the points and angle named after him. Given a triangle ΔABC with vertices A, B, C, there is a unique angle ω and a unique point Ω such that

$$\omega = \angle AC\Omega = \angle BA\Omega = \angle CB\Omega ,$$

see Figure 1(a). The angle ω is called the **Brocard angle** and the point Ω is the (**positive**) **Brocard point** of the triangle. The negative Brocard point, Ω', is defined by the same angle

$$\omega = \angle CA\Omega' = \angle AB\Omega' = \angle BC\Omega' ,$$

see Figure 1(b). The Brocard angle is given, in terms of the angles of the triangle, as follows

$$\cot\omega = \cot\alpha + \cot\beta + \cot\gamma \tag{1}$$

The two Brocard points are isogonal conjugates ([12],[13],[19]), and they coincide if the triangle is equilateral, in which case $\omega = \frac{\pi}{6}$.

References on the Brocard points, angle, related constructs and generalizations are contained in [2], [12]–[22] and [24]. See also [10] for a biographical reference on Brocard, the encyclopedia [23] for concise definitions and collections of results, and [5] for a perspective on the role of triangle geometry in classical and contemporary mathematics.

The earliest easily accessible reference to the Brocard point that we are aware of is [1]. According to Honsberger [12] and Mitrinović, Pečarić and Volenec [18], the Brocard point was already known to Crelle [4], Jacobi and others at the beginning of the 19th century. Indeed, the historically more accurate name of **Crelle-Brocard point** is used in [18] (where other references to contemporary work are also given).

The existence of the Brocard points is obvious if we consider a variable angle ω, and three lines AD, BE and CF making an angle ω with the respective sides, see Figure 2. For small values of ω these lines define an inner triangle $\Delta(\omega)$, similar to ΔABC. For $\omega = 0$, $\Delta(0)$ coincides with the original triangle. As ω increases, the triangles $\Delta(\omega)$ shrink, reducing to a point (the Brocard point) when ω is the Brocard angle. The same angle ω gives both the positive and negative Brocard points because these points are isogonal conjugates.

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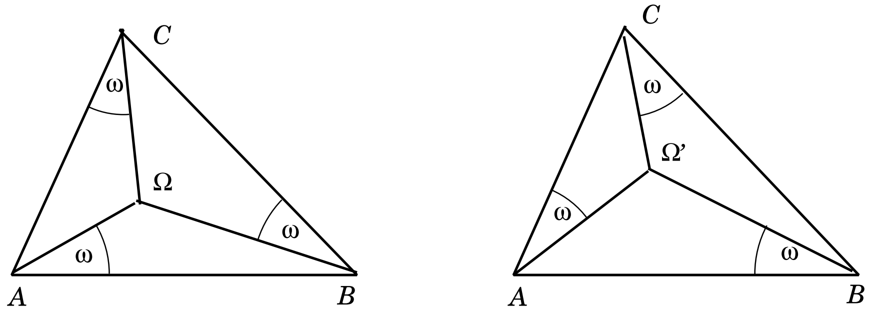

(a) The point Ω and angle ω (b) The point Ω' and angle ω

FIGURE 1. A triangle, its Brocard angle ω, and two Brocard points $\{\Omega, \Omega'\}$.

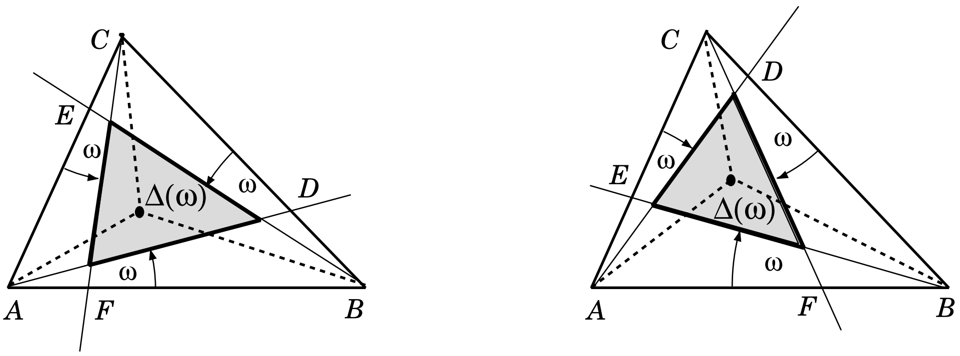

FIGURE 2. Illustration of the triangles $\Delta(\omega)$ that shrink to the Brocard points.

The Brocard points of a triangle are intersections of lines passing through the vertices, and as such are subject to the following theorem generally attributed to GIOVANNI CEVA [1648-1734]. Beutelspacher and Rosenbaum [3], citing Hogendijk [11], indicate that this theorem was stated and proved by Al-Mutaman in the 11th century.

Theorem 1 (Ceva's Theorem)**.** Given a triangle $\Delta\, ABC$ and points D, E, F on the sides, a necessary and sufficient condition for the lines AD, BE and CF to intersect at a point is

$$\frac{|AF|}{|FB|}\,\frac{|BD|}{|DC|}\,\frac{|CE|}{|EA|} \;=\; 1\;, \tag{2}$$

or equivalently,

$$\frac{\sin\angle\, BAD}{\sin\angle\, ABE}\;\frac{\sin\angle\, CBE}{\sin\angle\, BCF}\;\frac{\sin\angle\, ACF}{\sin\angle\, CAD} \;=\; 1\;. \tag{3}$$

The theorems of Ceva and Menelaus were brought to a common denominator and generalized to polygons in dimension two and higher by Grünbaum and Shephard, [6]–[8]. Ceva's Theorem can be used to establish the well-known bound $\pi/6$ on the Brocard angle. (For an account of this idea, due to Abi-Khuzam and pursued by Veldkamp, then by Hoogland and Stroeker, see [18].)

In this paper an idea analogous to the shrinking triangles $\Delta(\omega)$ (of Figure 2) is developed in the context of Ceva's Theorem. Writing the condition (3) as

$$f(\omega_1, \omega_2, \omega_3) \;:=\; \frac{\sin\omega_1}{\sin(\alpha-\omega_1)}\;\frac{\sin\omega_2}{\sin(\beta-\omega_2)}\;\frac{\sin\omega_3}{\sin(\gamma-\omega_3)} \;=\; 1\;, \tag{4}$$

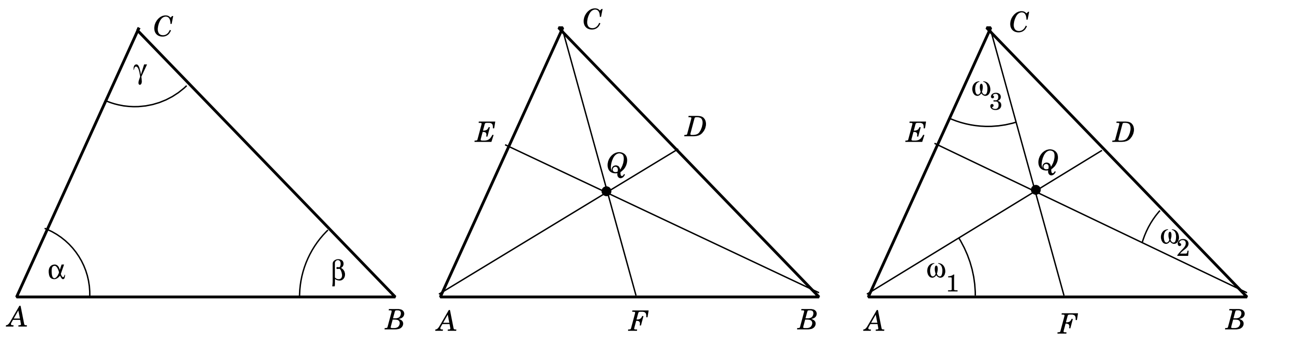

FIGURE 3. Illustration of Ceva's Theorem.

it follows that the inequalities

$$f(\omega_1, \omega_2, \omega_3) \ < \ 1 \quad \text{and} \quad f(\omega_1, \omega_2, \omega_3) \ > \ 1$$

correspond to cases where the lines AD, BE and CF are not concurrent. In this paper we discuss these inequalities for general convex polygons.

2. Complementary Halfspaces

Consider the n–dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$. We make free use of the usual vector space structure and affine geometry on $\mathbb{R}^n$, as well as of the usual notion of convex sets in $\mathbb{R}^n$ and the usual topology. We call a set **concave** when its complement in $\mathbb{R}^n$ is convex.

Recall that for a subset $H \subset \mathbb{R}^n$ the following conditions are equivalent:
(i) H is closed and it is both convex and concave, $H \neq \emptyset$ and $H \neq \mathbb{R}^n$,
(ii) H is the set of solution vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ of a linear inequality, of the form

$$a_1x_1 + \cdots + a_nx_n \leq b \ , \quad \text{or} \quad a_1x_1 + \cdots + a_nx_n \geq b \ , \tag{5}$$

where $(a_1, \cdots, a_n)$ is not the zero vector.
A set satisfying these conditions is called a **closed half-space**. For every closed half-space H there exists a unique closed half-space H^- such that

$$H \cup H^- = \mathbb{R}^n \quad \text{and} \quad H \cap H^- \quad \text{is a hyperplane}$$

The half-spaces H and H^- are said to be **complementary**. Clearly $(H^-)^- = H$. Note that if H is the solution set of (5), then H^- is the solution set of

$$a_1x_1 + \cdots + a_nx_n \geq b$$

and the hyperplane $H \cap H^-$ is the solution set of the equation

$$a_1x_1 + \cdots + a_nx_n = b \ .$$

The intersection of any family of closed half-spaces is always a closed convex set (perhaps empty). It is well known that every closed convex subset of $\mathbb{R}^n$ is the intersection of a (possibly empty) family of closed half-spaces.

Lemma 1. For any family $(H_i : i \in \mathcal{I})$ of closed half-spaces we have one and only one of the following cases:
(a) $\cap\{H_i : i \in \mathcal{I}\} = \cap\{H_i^- : i \in \mathcal{I}\}$ is a singleton,
(b) each one of the intersections $\cap\{H_i : i \in \mathcal{I}\}$ and $\cap\{H_i^- : i \in \mathcal{I}\}$ is either unbounded or empty,
(c) one of the intersections $\cap\{H_i : i \in \mathcal{I}\}$ and $\cap\{H_i^- : i \in \mathcal{I}\}$ is nonempty and bounded, and the other is empty.

Proof. Clearly the three cases are mutually exclusive. We need to show that they cover all possibilities. This is obvious if one of $\cap H_i$ or $\cap H_i^-$ is empty, so we may assume that both intersections are nonempty.

For each $i \in \mathcal{I}$ the closed half-space H_i is the solution set of an inequality

$$a_{i_1} x_1 + \cdots + a_{i_n} x_n \leq b_i \tag{6}$$

We shall use the fact that if $\mathbf{x} \in \cap H_i$ and $\mathbf{y} \in \cap H_i^-$ then the vector $2\mathbf{x} - \mathbf{y}$ also belongs to $\cap H_i$. This is so because for every i,

$$a_{i_1} x_1 + \cdots + a_{i_n} x_n \leq b_i \tag{7}$$

$$a_{i_1} y_1 + \cdots + a_{i_n} y_n \geq b_i \tag{8}$$

imply

$$a_{i_1}(2x_1 - y_1) + \cdots + a_{i_n}(2x_n - y_n) \leq b_i \tag{9}$$

Actually, for any positive k, (7) and (8) imply

$$a_{i_1}((k+1)x_1 - ky_1) + \cdots + a_{i_n}((k+1)x_n - ky_n) \leq b_i \tag{10}$$

It follows from this that if $\cap H_i$ is a singleton $\{\mathbf{x}\}$, then (a) holds, and clearly the same is true if $\cap H_i^-$ is a singleton. It also follows that if $\cap H_i^-$ is unbounded, then $\cap H_i$ is also unbounded, and vise versa, and then we are in case (b).

Suppose now that $\cap H_i$ is a non-singleton and bounded. We have to rule out the possibility that $\cap H_i^-$ is nonempty. Choose vectors $\mathbf{x} = (x_1, \cdots, x_n) \in \cap H_i$ and $\mathbf{y} = (y_1, \cdots, y_n) \in \cap H_i^-$. Since $\cap H_i$ is a non-singleton, we can choose these vectors to be distinct, $\mathbf{x} - \mathbf{y} \neq 0$. According to (10), for all positive k the vectors $(k+1)\mathbf{x} - k\mathbf{y}$ belong to $\cap H_i$. But since k can be arbitrarily large, the set of these vectors is unbounded, contradicting the assumption that $\cap H_i$ is bounded. □

Note that none of the three cases of Lemma 1 is vacuous in any dimension. Examples are easily constructed in dimension 1 or 2 and generalized to higher dimensions. In fact case (b) has three subcases, according to whether both, only one, or none of the intersections is empty. All the three subcases occur in any dimension higher than 1.

Lemma 1 can be expressed in terms of inequalities as follows. Let $((a_{i_1}, \cdots, a_{i_n}) : i \in \mathcal{I})$ be a family of n–vectors, and let $(b_i : i \in \mathcal{I})$ be a corresponding family of scalars. Consider the system of inequalities

$$a_{i_1} x_1 + \cdots + a_{i_n} x_n \leq b_i \ , \quad i \in \mathcal{I} \ , \tag{11}$$

and the complementary system

$$a_{i_1} x_1 + \cdots + a_{i_n} x_n \geq b_i \ , \quad i \in \mathcal{I} \ . \tag{12}$$

If one of the systems (11) or (12) is inconsistent, then the other may well be also inconsistent, or have a unique solution, or have multiple but bounded solutions, or have an unbounded solution set. If both systems are consistent, then we have one and only one of the following two cases:
(i) both systems have a unique solution, and these solutions coincide,
(ii) both systems have infinite unbounded solution sets.

3. Circular Products of Trigonometric Ratios

Lemma 2. Let $0 < \alpha < \pi$. Then the function

$$f(\omega) := \frac{\sin \omega}{\sin(\alpha - \omega)} \tag{13}$$

is monotone increasing for $\omega \in [0, \alpha)$, mapping $[0, \alpha)$ to $[0, \infty)$.

Proof. The derivative

$$f'(\omega) = \frac{\sin \alpha}{\sin^2(\alpha - \omega)}$$

is positive in the given domain. □

Lemma 3. Let $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \cdots, \alpha_n)$ where each $0 < \alpha_i < \pi$, and let

$$f(\boldsymbol{\omega}) := \prod_{i=1}^{n} \frac{\sin \omega_i}{\sin(\alpha_i - \omega_i)} \tag{14}$$

for $\boldsymbol{\omega} := (\omega_1, \omega_2, \cdots, \omega_n)$ with $0 < \omega_i < \alpha_i$. Then

$$\boldsymbol{\omega}^1 \leq \boldsymbol{\omega}^2 \quad \Longrightarrow \quad f(\boldsymbol{\omega}^1) \leq f(\boldsymbol{\omega}^2) \tag{15}$$

where vector inequality is interpreted componentwise. Moreover, if $\boldsymbol{\omega}^1 \leq \boldsymbol{\omega}^2$ and $\boldsymbol{\omega}^1 \neq \boldsymbol{\omega}^2$ then $f(\boldsymbol{\omega}^1) < f(\boldsymbol{\omega}^2)$.

Proof. Apply Lemma 2 to each component of $\boldsymbol{\omega}$. □

4. Convex Polygons

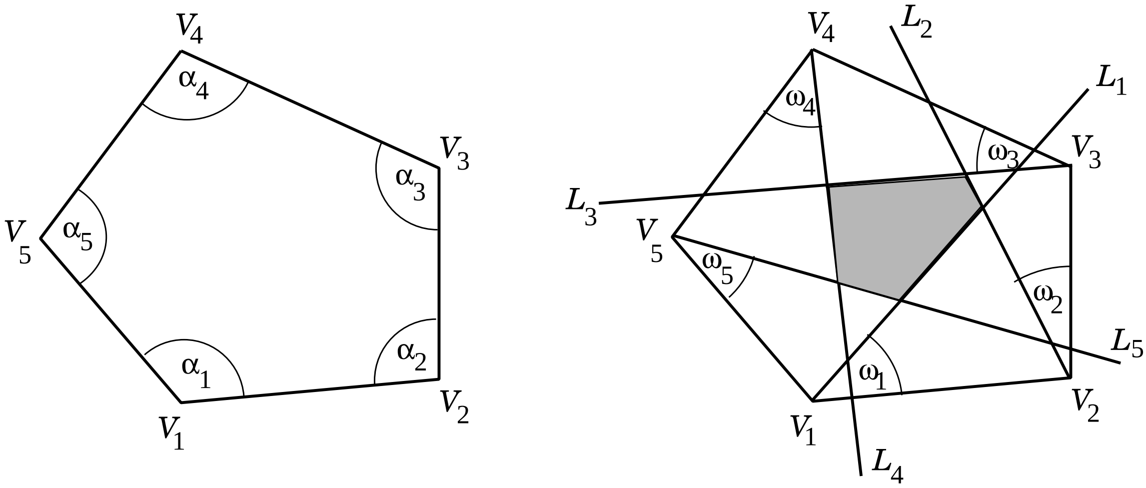

Figure 4. A pentagon $\mathcal{P}$ and the intersection $\mathcal{P}^-(\boldsymbol{\omega})$.

Let $\mathcal{P}$ be a bounded convex n–polygon, number its vertices $V_1, V_2, \cdots, V_n$ counterclockwise, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the corresponding angles of $\mathcal{P}$, where the indexing integers are modulo n (thus $V_n = V_0$, $V_{n+1} = V_1$) and each α_i is less than π. For $i = 1, \cdots, n$ let L_i be a line through the vertex V_i separating V_{i-1} from V_{i+1} i.e. such that none of the two closed complementary half-planes (half-spaces of $\mathbb{R}^2$) containing L_i contains both V_{i-1} and V_{i+1}. Of these two complementary half-planes, there is only one whose interior contains $\{V_{i-1}\} \setminus L_i$ but not $\{V_{i+1}\} \setminus L_i$. Let L_i^- denote this closed half-plane, and let L_i^+ denote the complementary closed half-plane. Note that the line $L_i = L_i^- \cap L_i^+$ makes an angle ω_i, $0 \leq \omega_i \leq \alpha_i$, with the side V_iV_{i+1} of $\mathcal{P}$.

The notation $L_i(\omega_i)$ is used when the angle ω_i varies, causing the line L_i to rotate around V_i. We also denote

$$\mathcal{P}^-(\boldsymbol{\omega}) := \bigcap_{i=1}^{n} L_i^-(\omega_i) \tag{16}$$

$$\mathcal{P}^+(\boldsymbol{\omega}) := \bigcap_{i=1}^{n} L_i^+(\omega_i) \tag{17}$$

for $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_n)$. Clearly

$$\mathcal{P}^-(\mathbf{0}) = \mathcal{P} = \mathcal{P}^+(\boldsymbol{\alpha}) \,,$$

for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, suggesting that the intersection $\mathcal{P}^-(\boldsymbol{\omega})$ "shrinks" from $\mathcal{P}$ to the empty set as $\boldsymbol{\omega}$ increases, componentwise, from $\mathbf{0}$ to $\boldsymbol{\alpha}$.

Therefore let us apply the classification of Lemma 1 to the family $(L_i^- : i = 1, \cdots, n)$ of closed half-spaces and seek corresponding bounds on the value $f(\boldsymbol{\omega})$ defined in (14). Let us assume $0 < \omega_i < \alpha_i$ for every i.

In the case (a), the lines L_i are concurrent at a point Q in the interior of $\mathcal{P}$. In the case where $\mathcal{P}$ is a triangle, $n = 3$, the trigonometric form of the classical Ceva Theorem tells us precisely that $f(\boldsymbol{\omega}) = 1$ (see Shively [19]). Ceva's Theorem has been generalized to polygons and beyond by Grünbaum and Shephard ([6], [7], [8]). From these results, in particular as stated e.g. in [6, Theorem 2] one can derive that $f(\boldsymbol{\omega}) = 1$ for arbitrary n by using an argument similar to the one in Shiveley [19]. A short direct argument goes as follows. Note that the product of the areas of the n triangles ΔQV_iV_{i+1} can be represented, denoting by $\overline{XY}$ the distance between any two points X, Y, as

$$\frac{1}{2^n} \prod_i \overline{V_iV_{i+1}} \sin \omega_i \, \overline{QV_i} \tag{18}$$

but also as

$$\frac{1}{2^n} \prod_i \overline{V_iV_{i-1}} \sin(\alpha_i - \omega_i) \, \overline{QV_i} \tag{19}$$

Thus the quotient of these two expressions, simplifying to $f(\boldsymbol{\omega})$, is 1.

Since both $\mathcal{P}^-(\boldsymbol{\omega})$ and $\mathcal{P}^+(\boldsymbol{\omega})$ are subsets of the polygon $\mathcal{P}$, case (b) of Lemma 1 is possible only when both $\mathcal{P}^-(\boldsymbol{\omega})$ and $\mathcal{P}^+(\boldsymbol{\omega})$ are empty. This is not possible in the case of the triangle, $n = 3$, but possible for any given convex polygon with at least four vertices. Also, by taking sufficiently elongated rectangles, it is easy to show that $f(\boldsymbol{\omega})$ can assume any positive value while $\mathcal{P}^-(\boldsymbol{\omega})$ and $\mathcal{P}^+(\boldsymbol{\omega})$ are both empty.

Finally, in the case (c) two subcases are possible: either $\mathcal{P}^-(\boldsymbol{\omega}) \neq \emptyset$ or $\mathcal{P}^+(\boldsymbol{\omega}) \neq \emptyset$. Note that these sets are contained in the interior of the polygon $\mathcal{P}$.

If $\mathcal{P}^-(\boldsymbol{\omega}) \neq \emptyset$ (and $\mathcal{P}^+(\boldsymbol{\omega}) = \emptyset$), then choose any point Q in $\mathcal{P}^-(\boldsymbol{\omega})$. Replace each line L_i by the line $\overline{L}_i$ through V_i and Q. These new lines (some of which may coincide with the old ones) now make angles $\overline{\boldsymbol{\omega}} = (\overline{\omega}_1, \cdots, \overline{\omega}_n)$ with the sides V_iV_{i+1}. We have $\omega_i \leq \overline{\omega}_i$ for all i, with at least one inequality strict. By Lemma 3, $f(\boldsymbol{\omega}) < f(\overline{\boldsymbol{\omega}})$. But since the new lines are concurrent at Q, we have

$$\mathcal{P}^-(\boldsymbol{\omega}) = \mathcal{P}^+(\boldsymbol{\omega}) = \{Q\} \quad \text{and} \quad f(\overline{\boldsymbol{\omega}}) = 1$$

Therefore $f(\boldsymbol{\omega}) < 1$.

Similarly one can show that if $\mathcal{P}^+(\boldsymbol{\omega}) \neq \emptyset$ (and $\mathcal{P}^-(\boldsymbol{\omega}) = \emptyset$) then $1 < f(\boldsymbol{\omega})$.

We summarize:

Theorem 2. Let $\mathcal{P}$ be a bounded convex n–polygon with angles $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ in a circular enumeration of the vertices. For $\mathbf{0} \leq \boldsymbol{\omega} \leq \boldsymbol{\alpha}$ let

$$f(\boldsymbol{\omega}) := \prod_i \frac{\sin \omega_i}{\sin(\alpha_i - \omega_i)}$$

Then for any $\mathbf{0} \leq \boldsymbol{\omega} \leq \boldsymbol{\alpha}$ there are four possible cases:
(a) $\mathcal{P}^-(\boldsymbol{\omega}) = \mathcal{P}^+(\boldsymbol{\omega})$ is a singleton $\{Q\}$, the lines $\mathcal{L}_i(\omega_i)$ are concurrent at Q, and $f(\boldsymbol{\omega}) = 1$.
(b) $\mathcal{P}^-(\boldsymbol{\omega}) \neq \emptyset\,,\ \mathcal{P}^+(\boldsymbol{\omega}) = \emptyset$ and $0 \leq f(\boldsymbol{\omega}) < 1$.
(c) $\mathcal{P}^+(\boldsymbol{\omega}) \neq \emptyset\,,\ \mathcal{P}^-(\boldsymbol{\omega}) = \emptyset$ and $1 < f(\boldsymbol{\omega}) < \infty$.
(d) Both $\mathcal{P}^-(\boldsymbol{\omega})$ and $\mathcal{P}^+(\boldsymbol{\omega})$ are empty.

5. An Inequality for the Brocard Angle of a Polygon

Given a polygon $\mathcal{P}$, the point of concurrency P of the lines $L_i(\omega_i)$ in case (a) of Theorem 2 is called the **Brocard point of the polygon** if all angles ω_i are the same. It follows from Theorem 2 that a polygon has at most one Brocard point: the corresponding $\omega_1 = \omega_2 = \cdots = \omega_n$ can be called the **Brocard angle**. Not every polygon has a Brocard point: a counter-example is provided by any non-square rectangle. Obviously every regular polygon has a Brocard point. Figure 5 exhibits a non-regular pentagon that has a

Brocard point, with a Brocard angle of $\pi/4$. The polygon has vertices $V_1 = (-1,-1)$, $V_2 = (0,-1)$, $V_3 = (1,0)$, $V_4 = (0,1)$ and the vertex V_5 is the intersection, with negative ordinate, of the line through $(-1,0)$ and $(0,1)$, and the circle through $(0,0)$, $(-1,1)$ and $(-1,-1)$.

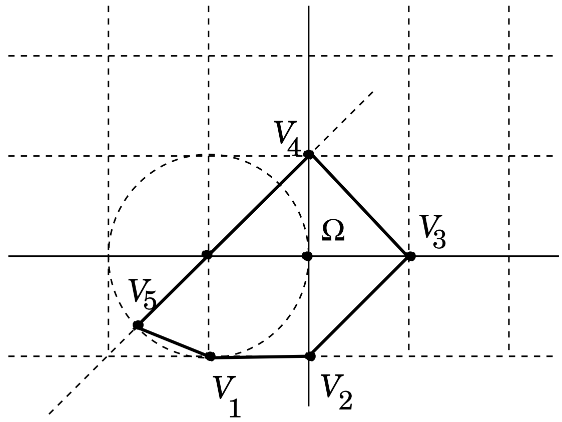

FIGURE 5. A non-regular pentagon with a Brocard point.

Let the n–polygon $\mathcal{P}$ have a Brocard point, and let $\boldsymbol{\omega} = (\omega, \omega, \cdots, \omega)$ be such that $\mathcal{P}^-(\boldsymbol{\omega})$ is non-empty. Then the angle ω is not greater than the Brocard angle, and

$$\sin^n \omega \;\le\; \prod_{i=1}^n \sin(\alpha_i - \omega) \tag{20}$$

with equality if and only if ω is the Brocard angle. Taking the nth root we get

$$\sin \omega \;\le\; \left(\prod_{i=1}^n \sin(\alpha_i - \omega)\right)^{1/n} \;\le\; \frac{1}{n}\sum_{i=1}^n \sin(\alpha_i - \omega) \tag{21}$$

where the second inequality is the arithmetic–geometric inequality, with equality if and only if the angles α_i are equal, in which case

$$\alpha_i = \frac{(n-2)}{n}\pi\ , \quad i = 1, \cdots, n \tag{22}$$

Using the formula $\sin(\alpha_i - \omega) = \sin\alpha_i \cos\omega - \sin\omega \cos\alpha_i$ and simplifying we get from (21)

$$\left(1 + \frac{\sum_{i=1}^n \cos\alpha_i}{n}\right)\sin\omega \;\le\; \left(\frac{\sum_{i=1}^n \sin\alpha_i}{n}\right)\cos\omega$$

or

$$\tan\omega \;\le\; \frac{\sum_{i=1}^n \sin\alpha_i}{n + \sum_{i=1}^n \cos\alpha_i} \tag{23}$$

since $\cos\,\omega$ must be positive. Equality holds in (23) if and only if ω is the Brocard angle and all $\alpha_i = \alpha = \frac{(n-2)}{n}\pi$. In this case the Brocard angle is half the angle α, and (23) reduces to the identity

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

The Brocard point and Brocard angle of course always exist in the case of a triangle. If the three angles of the triangle are α, β, γ, then (23) says that the tangent of the Brocard angle is bounded above by

$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{3 + \cos\alpha + \cos\beta + \cos\gamma}$$

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