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BORDERED MATRICES*

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1. The inspiration for this paper is in the work of M. R. Hestenes, especially in [3, §13, *Eigenvalues by inversion*]. However, the idea of bordering a singular matrix to remove the singularity is an old one and may be found, for example, in Carathéodory's book [1, pp. 169–172]. We explore the usefulness of the technique in a number of different contexts.

Section 2 deals with the problem of computing the eigenvalues of a matrix, and the bordering procedure is shown to be applicable to the separation of close eigenvalues and to the approximate solution of homogenous systems.

In §3 it is demonstrated how bordered matrices may be used to find the projections associated with a square matrix, and in particular how the pseudoinverse of the matrix may be determined in this fashion.

Throughout, capital Roman letters are used to designate matrices over the complex numbers, while small Roman letters following r in the alphabet denote (column) vectors. The conjugate transpose is indicated by *, or by $^{-T}$ (so that $x^* = \bar{x}^T$ means a row vector.) The letters k, m, n, r are employed for certain nonnegative integers, and $\mathcal{V}_n$ is the symbol for complex n-space.

2. In the method of finding eigenvalues by inversion of [3, §13], one proceeds as follows: Let A be an $n \times n$ matrix with simple eigenvalue λ_0, right eigenvector x_0, and left eigenvector y_0^*. If λ_1, x_1, and y_1^* represent initial guesses for these objects, one forms the matrix $\begin{pmatrix} A - \lambda_1 I & x_1 \\ y_1^* & 0 \end{pmatrix}$, inverts it to obtain $\begin{pmatrix} B & x_2 \\ y_2^* & \rho_2 \end{pmatrix}$, then forms $\begin{pmatrix} A - \lambda_1 I & x_2 \\ y_2^* & 0 \end{pmatrix}$, inverts it to obtain $\begin{pmatrix} C & x_3 \\ y_3^* & \rho_3 \end{pmatrix}$, etc. When the quantity $\frac{\rho_k}{y_k^* x_k}$ has become stable, one then puts $\lambda_2 = \lambda_1 - \frac{\rho_k}{y_k^* x_k}$, and repeats with this new value of λ, and x_k, y_k^* as the trial eigenvectors. The process continues until λ_0, x_0, y_0^* have been calculated to the desired accuracy.

It is shown in [3, §13] that the iterations with a fixed trial λ are equivalent

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to the inverse power method, while the correction in λ is computed in a manner equivalent to the use of the generalized Rayleigh quotient (for a thorough discussion of this latter technique, see the series of papers by Ostrowski [4]). Indeed, one can as well employ the inverse power method, compute the correction of the trial λ by means of the Rayleigh quotient, modify the λ accordingly, and continue in this way, except for one thing: when the λ has been sufficiently corrected, the matrix $A - \lambda I$ is difficult to invert accurately. However, the bordered matrix $\begin{pmatrix} A-\lambda I & x \\ y^* & 0 \end{pmatrix}$ is easily inverted, provided λ_0 is a simple eigenvalue and there are no other eigenvalues too close to it. It is this fact that makes the Hestenes' method valuable.

In this section, we prove theorems which extend the usefulness of the Hestenes' method to the cases of multiple characteristic roots and very close eigenvalues. We impose the following conditions:

CONDITIONS. Let A be an $n \times n$ matrix. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be distinct eigenvalues of A such that the number of linearly independent right eigenvectors associated with each λ_i is equal to the multiplicity of that λ_i as a characteristic root (i.e., the elmentary divisors corresponding to these eigenvalues are linear). Let U be an $n \times r$ matrix, where r is the sum of the multiplicities of the λ's, and such that the column vectors of U form a basis for the right eigenspace associated with the totality of the λ's (though the columns of U need not be eigenvectors). Similarly, let V^* be an $r \times n$ matrix whose row vectors are a basis for the left eigenspace of the λ's. Finally, if A is singular, assume that zero is included among the λ's.

Remark 1. The theorems we are about to prove would be valid, with minor changes in their statements, even in the presence of nonlinear elementary divisors corresponding to the λ's, provided the columns of U spanned the complete right invariant subspace associated with the λ's, and similarly for the rows of V^*. The reasons for demanding linear elementary divisors in the above conditions are twofold:

(a) Iterative processes based on our results are slowly convergent in the case of nonlinear elementary divisors. However, Theorem 2.3 demonstrates that this problem could at least be transferred to a lower order matrix, provided one could determine r, the dimension of the right invariant subspace, but

(b) This dimension r is at best difficult to discover in the case of nonlinear elementary divisors.

Remark 2. In the application we have in mind, the λ's of the Conditions will be the eigenvalues of A that are nearest zero. A matrix U and a matrix V^* approximately satisfying the Conditions can then be found by solving approximately the homogeneous systems $Ax = 0$ and $y^*A = 0$. Such approximate solutions can be obtained by a method of this section (see Theorem 2.2 and the Remark following it).

LEMMA 2.1. *Under the above Conditions, we can draw the following conclusions.*

(1) *V^*U is a nonsingular $r \times r$ matrix.*

(2) *If x is any n-vector such that $V^*x = 0$, then $V^*Ax = 0$ also.*

(3) *The column space of U and the right nullspace of V^* are disjoint right invariant subspaces under A, and their (direct) sum is $\mathcal{V}_n$.*

Proof. (1) Because of the assumption of linear elementary divisors corresponding to the λ's, we can choose a set of right eigenvectors $\{x_i\}$ and a set of left eigenvectors $\{y_j^*\}$ for the λ's such that $y_j^*x_i = \delta_{ji}$, the Kronecker δ. Since the column space of U and the row space of V^* are, respectively, the right and left eigenspaces for the λ's, it is therefore possible to find nonsingular $r \times r$ matrices E_1 and E_2 such that $E_1V^*UE_2 = I_r$, the $r \times r$ identity matrix. This implies the nonsingularity of V^*U.

(2) Let $V^* = \begin{pmatrix} v_1^* \\ \vdots \\ v_r^* \end{pmatrix}$. Then $V^*A = \begin{pmatrix} v_1^*A \\ \vdots \\ v_r^*A \end{pmatrix}$. Because of the hypotheses for V^*, we have relations $v_i^*A = \sum_{j=1}^{r} \alpha_{ij}v_j^*$ for each i from 1 to r, where the α_{ij} are complex numbers. If $V^*x = 0$, it is true that $v_i^*x = 0$ for each i, and, hence, that $v_i^*Ax = 0$ for each i from the above relations, so that $V^*Ax = 0$.

(3) Suppose that x is in the column space of U and also in the right nullspace of V^*; i.e., $x = Uy$ for some r-vector y and $V^*x = 0$. Then $V^*Uy = 0$, or $y = 0$, since V^*U is nonsingular by conclusion (1). Therefore $x = 0$, and the two subspaces in question are disjoint.

The column space of U is the right (invariant) eigenspace of the λ's, while the right invariance of the right nullspace of V^* is a consequence of conclusion (2).

Finally, the dimension of the column space of U is r by assumption, and the dimension of the right nullspace of V^* is $n - r$; hence the (direct) sum of these two subspaces is $\mathcal{V}_n$.

THEOREM 2.1. *With the Conditions of this section, the matrix* $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ *(where 0 is the $r \times r$ zero matrix) is nonsingular.*

Proof. Let x be any n-vector and y any r-vector, and consider the $(n + r)$-vector $\begin{pmatrix} x \\ y \end{pmatrix}$. If $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, we must have $Ax + Uy = 0$ and $V^*x = 0$. From $V^*x = 0$ may be inferred that $V^*Ax = 0$ by Lemma 2.1, and this in turn means that $V^* Uy = 0$, as we see by multiplying the equation $Ax + Uy = 0$ on the left by V^*. Since V^*U is nonsingular by Lemma

2.1, y must be the zero vector. Therefore, $Ax = 0$, so that either $x = 0$ or x is a right eigenvector of A for the eigenvalue zero. If the second alternative holds, x must be in the column space of U, by the Conditions. But $V^*x = 0$ says that x is also in the right nullspace of V^*; so, from Lemma 2.1, x is zero anyway. Summing up, we have proved that $\begin{pmatrix} x \\ y \end{pmatrix} = 0$, or that $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ annihilates no nonzero vector and is thus nonsingular.

THEOREM 2.2. *Under the Conditions of this section, and assuming the inverse of* $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ *in the form* $\begin{pmatrix} B & S \\ T^* & R \end{pmatrix}$, *where* B *is* $n \times n$, S *is* $n \times r$, T^* *is* $r \times n$, *and* R *is* $r \times r$, *we have the following:*

(1) (a) $V^*S = I_r$.

(b) $T^*U = I_r$.

(2) *The column space of* S *is identical with the column space of* U.

(3) *The row space of* T^* *is identical with the row space of* V^*.

Proof. Assertion (1a) is established by multiplying the second "row" of $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ by the second "column" of its inverse $\begin{pmatrix} B & S \\ T^* & R \end{pmatrix}$. Assertion (1b) is obtained through multiplication of the second "row" of the latter matrix by the second "column" of the former.

If we multiply the first "row" of $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ by the second "column" of the inverse, we get $AS + UR = 0$. Let $S = (s_1, s_2, \cdots, s_r)$, $U = (u_1, u_2, \cdots, u_r)$, and $R = (\rho_{ij})$. Then, the ith column of the relation $AS = -UR$ is

$$As_i = -\sum_{j=1}^{r} \rho_{ji} u_j .$$

Resolve s_i into two (unique) components, one in the column space of U, the other in the right nullspace of V^* (this resolution being possible by Lemma 2.1). If the second component were not zero, As_i would also have a nonzero component in the right nullspace of V^*, since this subspace as well as the column space of U are right invariant subspaces under A, and the right nullspace of V^* contains no right eigenvector for the eigenvalue zero. But $-\sum_{j=1}^{r} \rho_{ji} u_j$ has no such component, and therefore, s_i is included in the column space of U for each i. Since S has rank r, because $V^*S = I_r$ by conclusion (1a), the column space of S coincides with the column space of U, proving (2). (3) follows similarly.

Remark. If the λ's consist of the eigenvalues of A that are closest to zero, and U and V^* are matrices approximately satisfying the Conditions, then S and T^* will be better approximate matrices for those Conditions. In fact, the undesired components of the column vectors of U and the row vectors of V^* will be decreased relative to the desired components in a way that is analogous to the diminution of unwanted components in the inverse power method. (See also Lemma 3.2 and its Corollary.) On the basis of this observation, one can obtain r linearly independent approximate solutions for $Ax = 0$ and for $y^*A = 0$ by selecting somewhat arbitrary U and V^* and iterating.

LEMMA 2.2. *With the Conditions of this section, the submatrices* U, V^*, S, T^* *of* $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ *and its inverse satisfy*:

(1) (a) $SV^*U = U$,
(b) $UT^*S = S$,

(2) (a) $T^*SV^* = T^*$,
(b) $V^*UT^* = V^*$.

Proof. Since $V^*S = I_r$ by Theorem 2.2, we see that $SV^*S = S$ and $SV^* \cdot SV^* = SV^*$. These two equations say that SV^* (as a left operator) is a projection onto the column space of S, which is the same as the column space of U by Theorem 2.2. Therefore, $SV^*U = U$, which proves (1a). (1b) and (2) follow in like manner.

THEOREM 2.3. *Under the Conditions of this section, the matrix* $-RV^*U$ *has the eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_m$ *with the same multiplicities with which they appear in* A. *Further, if* x_i *(an r-vector) is a right eigenvector of* $-RV^*U$ *for* λ_i, *then* Ux_i *is a right eigenvector of* A *for the same* λ_i. *Similar observations apply to the matrix* $-V^*UR$ *and the left eigenvectors of* A.

Proof. First we have $AS = -UR$ as noticed in the proof of Theorem 2.2. Suppose that $-RV^*Ux_i = \lambda_i x_i$; then, $AUx_i = ASV^*Ux_i$ by Lemma 2.2, and this in turn is $-URV^*Ux_i$, since $AS = -UR$. But $-URV^*Ux_i = \lambda_i Ux_i$; thus, $AUx_i = \lambda_i Ux_i$.

Conversely, assume that $AUx_i = \lambda_i Ux_i$. Then $ASV^*Ux_i = \lambda_i Ux_i$, or $-URV^*Ux_i = \lambda_i Ux_i$. Multiplying the last equation on the left by T^*, and using $T^*U = I_r$ (from Theorem 2.2), we get $-RV^*Ux_i = \lambda_i x_i$.

We have proved that, if x_i is a right eigenvector of $-RV^*U$ for λ_i, then Ux_i is a right eigenvector of A for the same λ_i, and conversely. Therefore, all eigenvalues of $-RV^*U$ are eigenvalues of A. What eigenvalues of A can be eigenvalues of $-RV^*U$? Only those associated with right eigenvectors of A of the form Ux_i—but these are all in the column space of U, so only $\lambda_1, \lambda_2, \cdots, \lambda_m$ can be present. Moreover these must all appear, and with their proper multiplicities, since each eigenvector in the column space of U can be obtained as Ux_i for some r-vector x_i.

The proof for $-V^*UR$ and the left eigenvectors has the same pattern.

COROLLARY 2.1. *If* X *is an* $r \times r$ *matrix whose column vectors are a complete set of right eigenvectors for* $-RV^*U$, *then* UX *us an* $n \times r$ *matrix whose column vectors are a complete set of right eigenvectors of* A *for the values*

$\lambda_1, \lambda_2, \cdots, \lambda_m$, *and conversely. An analogous statement holds for the left eigenvectors and a matrix* Y^*.

Remark. The problem of finding several close (or multiple) eigenvalues of A is therefore reduced to the same problem for $-RV^*U$, a lower order matrix with better relative separation of its eigenvalues. For computational convenience, it is possible to start with matrices U and V^* such that $V^*U = I_r$ (as conclusion (1) of Lemma 2.1 indicates). We then have the following result.

COROLLARY 2.2. *If* $V^*U = I_r$, *it follows that* $S = U$, $T^* = V^*$, *and* $-R$ *has* $\lambda_1, \lambda_2, \cdots, \lambda_m$ *as eigenvalues with the correct multiplicities. Further, the matrices* X *and* Y^* *of Corollary* 2.1 *now consist of complete sets of right and left eigenvectors, respectively, of the one matrix* $-R$.

Proof. $S = U$ and $T^* = V^*$ are here implied by Lemma 2.2. Since $V^*U = I_r$, we have $-RV^*U = -V^*UR = -R$, and the corollary is proved.

Remark. If U and V^* only approximately satisfy the Conditions, then the conclusions of Corollary 2.2 are only approximately true, and S and T^* should be made to satisfy $T^*S = I_r$ before continuing an iterative procedure based on this corollary.

COROLLARY 2.3. *If, in addition to* $V^*U = I_r$, *the columns of* U *are right eigenvectors of* A, *and the rows of* V^* *are left eigenvectors of* A, *then* $-R$ *is a diagonal matrix, and conversely.*

Proof. Use Corollary 2.1, noting that the columns of UI_r are a complete set of right eigenvectors of A for the λ's, so that the columns of I_r are a complete set of right eigenvectors of $-R$, which means that $-R$ is diagonal.

Remark. If U and V^* satisfy the Conditions and $V^*U = I_r$, then, by Corollary 2.1, with X and Y^* as in that corollary, and with the rows of Y^* so ordered and normalized that $Y^*X = I_r$, SX and Y^*T^* will be matrices satisfying the hypotheses of Corollary 2.3. Thus the diagonal form for $-R$ is the indication of when to terminate an iterative procedure based on Theorem 2.3 and its corollaries.

We now summarize a process for computing multiple and close eigenvalues of a matrix, based on the theorems of this section. Let C be the $n \times n$ matrix, and assume that the elementary divisors corresponding to the eigenvalues we seek are linear. Proceed as follows.

(1) Let λ be an initial estimate for the desired eigenvalues. Employ the inverse power method with $C - \lambda I$, and use the generalized Rayleigh quotient, or, equivalently, the scalar product method [2, pp. 212–215] to calculate the correction in λ. Continue until $C - \lambda I$ is singular or nearly so. Call this matrix A.

(2) Discover the value of r such that $Ax = 0$ has r linearly independent approximate solutions. This could be done by row eliminations to reduce

A to triangular form and counting the sufficiently small rows. A multiple characteristic root of zero for A must contribute its full multiplicity to the value of r. If A has a cluster of eigenvalues near zero, with the rest well removed from zero, the value of r will be fairly obvious, and otherwise r is not critical anyway.

(3) Make up trial U and V^*, and iterate as outlined in Theorem 2.2 and the remark following it. If the r found in step (2) is just large enough to encompass all eigenvalues of A near zero, few iterations are required here. However, if one errs by making too few iterations, the succeeding steps will take care of things at the expense of extra labor.

(4) When content with U and V^*, make them fulfill $V^*U = I_r$ and compute the inverse of $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$. Then find the eigenvector matrices X and Y^* of $-R$ and arrange also that $Y^*X = I_r$. In calculating X and Y^*, the inverse power method should suffice for the simple eigenvalues, but a scaled down application of steps (1), (2), and (3) is indicated for the multiple eigenvalues.

(5) Compute SX and Y^*T^* and repeat step (4) with these in place of U and V^*. (In going through step (4) again, one should find that I_r is a good first approximation for the matrices X and Y^*.) Continue this process until $-R$ is diagonal. Then the diagonal elements of $-R$, plus λ, give the eigenvalues of the original C, and the final S and T^* comprise normalized sets of eigenvectors.

3. In this section we pay attention to the submatrix B in the inverse $\begin{pmatrix} B & S \\ T^* & R \end{pmatrix}$ of $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$. In particular, we show how to compute the projections associated with a singular square matrix A, and Theorem 3.2 gives a convenient construction of the pseudoinverse of A.

LEMMA 3.1. *Let A be an $n \times n$ matrix of rank $n - r$. Suppose U is an $n \times r$ matrix whose column vectors are linearly independent and, together with the column vectors of A, span $\mathcal{V}_n$. Assume likewise that V^* is an $r \times n$ matrix whose row vectors are linearly independent and, together with the row vectors of A, span $\mathcal{V}_n$. Then the matrix $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ is nonsingular.*

Proof. The $n \times (n + r)$ matrix (A, U) is of rank n by the assumptions on A and U. Therefore its rows are linearly independent. The rows of $(V^*, 0)$ are also linearly independent; and these, together with the rows of (A, U), span $\mathcal{V}_{n+r}$, for otherwise some row of V^* would be a linear combination of the rows of A and the other rows of V^*, contrary to hypothesis. Therefore $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ has linearly independent rows and is nonsingular.

LEMMA 3.2. *Under the conditions of Lemma 3.1, the inverse of $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$*

has the form $\begin{pmatrix} B & S \\ T^* & 0 \end{pmatrix}$.

Proof. Assume the inverse is $\begin{pmatrix} B & S \\ T^* & R \end{pmatrix}$, where R is $r \times r$. Multiplying the first "row" of $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ by the second "column" of the inverse we get the familiar relation $AS + UR = 0$.

The column space of AS is a subspace of the column space of A, while the column space of UR is a subspace of the column space of U. By hypothesis, these spaces are disjoint, and hence we conclude that $AS = UR = 0$. Since U has rank r, and $UR = 0$, where R is $r \times r$, we must also admit that therefore $R = 0$, as was to be proved.

COROLLARY 3.1. *With the assumptions of Lemma* 3.1, *we get* $AS = T^*A = 0$.

Remark. Corollary 3.1 shows that we have here a method of solving homogeneous systems of equations. If A is not quite singular, the method is fully equivalent to that of Ostrowski in [5]. In the paper referred to, Ostrowski proves that the procedure yields approximate solutions in this case. (Cf. Theorem 2.2 and the Remark following it.) The process of Lemma 3.2 and its corollary may be looked upon as an extension of the inverse power method to the case of a singular matrix A.

THEOREM 3.1. *With the conditions of Lemma* 3.1, *the submatrices* A, U, V^*, *and* B *of* $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ *and its inverse satisfy the relations*:

(1) (a) $ABA = A$,
(b) $BAB = B$,

(2) (a) $BU = 0$,
(b) $V^*B = 0$.

Proof. Since $\begin{pmatrix} B & S \\ T^* & 0 \end{pmatrix}$ is the inverse of $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$, we obtain (among others) these relations:

(i) $$AB + UT^* = I_n,$$

(ii) $$V^*B = 0,$$

(iii) $$BU = 0,$$

(iv) $$T^*A = 0.$$

Equations (iii) and (ii) give conclusions (2a) and (2b), respectively. To get (1a), multiply (i) on the right by A and use (iv). For (1b), multiply (i) on the left by B and use (iii).

Remark. The conclusions of Corollary 3.1 and Theorem 3.1 mean that AB, as a left operator, is the projection on the column space of A along the column space of U, while, as a right operator, AB is the projection on the row space of B along the row space of T^*. Similar remarks apply to the (oblique) projection BA.

THEOREM 3.2. *Let the submatrices of* $\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix}$ *satisfy the conditions of Lemma* 3.1, *and let the inverse of this matrix be* $\begin{pmatrix} B & S \\ T^* & 0 \end{pmatrix}$. *Then the matrix* $\begin{pmatrix} A & T \\ S^* & 0 \end{pmatrix}$ *is nonsingular. Furthermore, its inverse has the form* $\begin{pmatrix} A' & M \\ N^* & 0 \end{pmatrix}$, *where* A' *is the pseudoinverse* (*generalized inverse in the sense of E. H. Moore*) *of* A.

Proof. By Corollary 3.1, $AS = T^*A = 0$, so the columns of $\bar{S}$ are orthogonal to the rows of A. Hence, the rows of S^* are orthogonal to the rows of A, and S must have rank r since $\begin{pmatrix} B & S \\ T^* & 0 \end{pmatrix}$ is nonsingular. Thus S^* and T (by similar reasoning) satisfy the conditions imposed on V^* and U, respectively, in Lemma 3.1; so $\begin{pmatrix} A & T \\ S^* & 0 \end{pmatrix}$ is nonsingular, and by Lemma 3.2 has an inverse of the form $\begin{pmatrix} A' & M \\ N^* & 0 \end{pmatrix}$.

By Theorem 3.1 and the Remark following it, AA' as a left operator is the projection on the column space of A along the column space of T. But, since $T^*A = 0 = A^*T = A^T\bar{T}$, and T has rank r, the column space of T is the orthogonal complement of the row space of A^T, which is the same thing as the orthogonal complement of the column space of A. Therefore AA' as a left operator is the orthogonal projection on the column space of A. Similarly, $A'A$ as a right operator is the orthogonal projection on the row space of A, which makes A' the pseudoinverse of A.

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