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A NOTE ON PARTITIONED MATRICES AND EQUATIONS*

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1. Introduction. Consider the linear equations

$$Ax = b, \qquad A \text{ nonsingular,} \tag{1}$$

and let $\{L_1, L_2\}$ and $\{M_1, M_2\}$ be two pairs of complementary orthogonal subspaces. To every b with components b_1 and b_2 in L_1 and L_2, respectively, there corresponds a solution $x = A^{-1}b$ with components x_1 and x_2 in M_1 and M_2, respectively. In this note (x_1, x_2) are given in terms of (b_1, b_2), under the assumption that $A: M_1 \to L_1$ is nonsingular. This results in some old and new representations for inverses and generalized inverses of partitioned matrices. Other representations were given in [6], [4], [3], [7] and [1].

2. Notations and preliminaries.

C^n is an *n-dimensional complex vector space*,

$C^{m \times n}$ are $m \times n$ *complex matrices*,

$C_r^{m \times n}$ are the same with rank r.

For any $A \in C^{m \times n}$:

A^* is the *conjugate transpose* of A,

$A^\dagger$ is the *generalized inverse* of A (e.g., [5], [2]),

$R(A) = \{Ax : x \in C^n\}$ is the *range space* of A,

$N(A) = \{x \in C^n : Ax = 0\}$ is the *null space* of A.

For any subspace $L \subset C^n$:

P_L is the *perpendicular projection* on L, i.e., $P_L \in C^{n \times n}$, $P_L = P_L^2 = P_L^*$, $L = R(P_L)$,

$L^\perp$ is the *orthogonal complement* of L.

The restriction $A: L \to M$ of $A \in C^{m \times n}$ to the subspaces $L \subset C^n$, $M \subset C^m$ is said to be *nonsingular* if $\dim L = \dim M$ and $AL = M$. Thus for any $A \in C^{m \times n}$, $A: R(A^*) \to R(A)$ and $A^\dagger : R(A) \to R(A^*)$ are nonsingular.

3. Results. Consider

$$Ax = b, \qquad \text{where} \quad A \in C_n^{n \times n}, \quad b \in C^n. \tag{1a}$$

Let L_1, $L_2 = L_1^\perp$, M_1, $M_2 = M_1^\perp$ be subspaces in C^n with corresponding projections P_{L_i}, P_{M_i}, $i = 1, 2$. Using $A_{ij} = P_{L_i} A P_{M_j}$, $b_i = P_{L_i} b$, $x_j = P_{M_j} x$, $i, j = 1, 2$, we rewrite (1a) as:

$$A_{11}x_1 + A_{12}x_2 = b_1, \tag{2}$$

$$A_{21}x_1 + A_{22}x_2 = b_2 \tag{3}$$

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or

(4) $$A_{L,M}x_M = b_L,$$

where

$$A_{L,M} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad b_L = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad x_M = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

From $A_{ij}^* = (P_{L_i}AP_{M_j})^* = P_{M_j}A^*P_{L_i}$ it follows that

(5) $$(A_{L,M})^* = (A^*)_{M,L}.$$

Regarding $A_{L,M}$ as an operator on $C^n \times C^n = C^{2n}$ into itself we observe that

(6) $$R(A_{L,M}) = L_1 \times L_2,$$

(7) $$R((A_{L,M})^*) = M_1 \times M_2,$$

(8) $$N(A_{L,M}) = M_2 \times M_1,$$

(9) $$N((A_{L,M})^*) = L_2 \times L_1.$$

Indeed (6) follows from the nonsingularity of A, (7) follows from (5), and (8) from (7) by using the facts:

$$N(A_{L,M}) = R((A_{L,M})^*)^\perp = (M_1 \times M_2)^\perp = M_2 \times M_1.$$

Similarly (9) follows from (6).

The generalized inverse of $A_{L,M}$ is now given, for the case $AM_1 = L_1$.

THEOREM. *Let A and $A_{L,M}$ be as above and let $A_{11}: M_1 \to L_1$ be nonsingular.*[1] *Then*

(10) $$(A_{L,M})^\dagger = \left(\begin{array}{c:c} A_{11}^\dagger + A_{11}^\dagger A_{12} B A_{21} A_{11}^\dagger & -A_{11}^\dagger A_{12} B \\ \hdashline -BA_{21}A_{11}^\dagger & B \end{array}\right),$$

where

$$B = (A_{22} - A_{21}A_{11}^\dagger A_{12})^\dagger.$$

Proof. Using (6), (7) we see that $(A_{L,M})^\dagger$ is the inverse of the nonsingular restriction $A_{L,M}: M_1 \times M_2 \to L_1 \times L_2$. Computing $(A_{L,M})^\dagger$ thus amounts to solving (4) for all $b_L \in L_1 \times L_2$. From (2) and the nonsingularity of $A_{11}: M_1 \to L_1$ it follows that

(11) $$x_1 = A_{11}^\dagger b_1 - A_{11}^\dagger A_{12} x_2$$

is uniquely determined by x_2. Substituting (11) in (3) we obtain

(12) $$(A_{22} - A_{21}A_{11}^\dagger A_{12})x_2 = b_2 - A_{21}A_{11}^\dagger b_1.$$

Now $(A_{22} - A_{21}A_{11}^\dagger A_{12}): M_2 \to L_2$ is nonsingular. For suppose

(13) $$(A_{22} - A_{21}A_{11}^\dagger A_{12})x_2 = 0$$

[1] Note that the matrix A_{11} is singular except in the uninteresting case $L_1 = M_1 = C^n$.

for some $0 \neq x_2 \in M_2$. Then we can contradict the nonsingularity of A by producing a vector $x_0 \neq 0$ such that

$$Ax_0 = 0. \tag{14}$$

Indeed such a vector is

$$x_0 = x_2 - A_{11}^{\dagger}A_{12}x_2. \tag{15}$$

First, $x_0 \neq 0$ since $x_2 \in M_2$, $x_2 \neq 0$, $A_{11}^{\dagger}A_{12}x_2 \in M_1$ and $M_2 = M_1^{\perp}$. Second, (14) holds because

$$\begin{aligned} Ax_0 &= (A_{11} + A_{12} + A_{21} + A_{22})(x_2 - A_{11}^{\dagger}A_{12}x_2) \\ &= (I - A_{11}A_{11}^{\dagger})A_{12}x_2 + (A_{22} - A_{21}A_{11}^{\dagger}A_{12})x_2 \\ &\qquad\qquad (\text{since } A_{11}x_2 = 0, \quad A_{21}x_2 = 0, \\ &\qquad\qquad A_{12}A_{11}^{\dagger} = 0, \quad A_{22}A_{11}^{\dagger} = 0) \\ &= 0, \end{aligned} \tag{16}$$

since $I - A_{11}A_{11}^{\dagger} = 0$ on L_1 and by using (13). From (12) it follows then that

$$\begin{aligned} x_2 &= (A_{22} - A_{21}A_{11}^{\dagger}A_{12})^{\dagger}(b_2 - A_{21}A_{11}^{\dagger}b_1) \\ &= B(b_2 - A_{21}A_{11}^{\dagger}b_1) \end{aligned} \tag{17}$$

which, when substituted in (11), gives

$$x_1 = (A_{11}^{\dagger} + A_{11}^{\dagger}A_{12}BA_{21}A_{11}^{\dagger})b_1 - A_{11}^{\dagger}A_{12}Bb_2. \tag{18}$$

Recognizing that (17) and (18) stand for

$$x_M = (A_{L,M})^{\dagger}b_L, \tag{19}$$

we now verify (10) term by term. This completes the proof.

We next obtain A^{-1} in terms of A_{ij}, $i, j = 1, 2$.

COROLLARY 1. *Let* A, A_{ij}, $i, j = 1, 2$, *be as above. Then*

$$A^{-1} = A_{11}^{\dagger} + (I - A_{11}^{\dagger}A_{12})(A_{22} - A_{21}A_{11}^{\dagger}A_{12})^{\dagger}(I - A_{21}A_{11}^{\dagger}). \tag{20}$$

Proof. The proof follows by identifying (17) and (18) as $x = A^{-1}b$ for every $b \in C^n$.

If the subspaces $L_1 = M_1$ are spanned by the first k unit vectors, then we obtain from (10) or (20) (by omitting all zero rows and columns) the following well-known result.

COROLLARY 2. *Let* $X \in C_n^{n \times n}$ *be partitioned by*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \qquad \textit{where} \quad X_{11} \in C_k^{k \times k}, \quad 0 \leqq k \leqq n. \tag{21}$$

Then

$$X^{-1} = \begin{pmatrix} X_{11}^{-1} + X_{11}^{-1}X_{12}YX_{21}X_{11}^{-1} & -X_{11}^{-1}X_{12}Y \\ -YX_{21}X_{11}^{-1} & Y \end{pmatrix}, \tag{22}$$

where $Y = (X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1}$.

4. Example. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and the subspaces $L_1, L_2 = L_1^{\perp}, M_1, M_2 = M_1^{\perp}$ be given by

$$P_{L_1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_{L_2} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad P_{M_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{M_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrices $A_{ij} = P_{L_i} A P_{M_j}$ are

$$A_{11} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_{12} = \frac{1}{2}\begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{22} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$A_{11}^{\dagger} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A_{22} - A_{21}A_{11}^{\dagger}A_{12} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = B^{\dagger},$$

and so

$$B = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

From (20) we obtain

$$\begin{aligned} A^{-1} &= A_{11}^{\dagger} + (I - A_{11}^{\dagger}A_{12})B(I - A_{21}A_{11}^{\dagger}) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

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