Contemporary Mathematics

Projectors on intersections of subspaces

Adi Ben–Israel

Abstract. Let $P_{\mathbf{L}}$ denote the orthogonal projector on a subspace $\mathbf{L}$. Two constructions of projectors on intersections of subspaces are given in finite–dimensional spaces. One uses the singular value decomposition of $P_{\mathbf{L}}P_{\mathbf{M}}$ to give an explicit formula for $P_{\mathbf{L}\cap\mathbf{M}}$. The other construction uses the result that the intersection of $m \geq 2$ subspaces, $\mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_m$, is the null–space of the matrix $Q := \sum_{i=1}^{m} \lambda_i\,(I - P_{\mathbf{L}_i})$, for any positive coefficients $\{\lambda_i\}$. The projector $P_{\mathbf{L}_1\cap\mathbf{L}_2\cap\cdots\cap\mathbf{L}_m}$ can then be given in terms of the Moore–Penrose inverse of Q, or as the limit, as $t \to \infty$, of the exponential function $\exp\{-Qt\}$.

Notation

For a linear transformation $A : \mathbb{C}^n \to \mathbb{C}^m$, $\mathbf{R}(A)$ denotes the *range*, $\mathbf{N}(A)$ the *null–space*, A^* the *adjoint*, and $A^\dagger$ the *Moore–Penrose inverse*, [**24**], of A. The same letter is used for the matrix representing A, and A^* is its conjugate transpose, or just transpose if A is real.

For integers $i < j$, the *index set* $\{i, i+1, \ldots, j\}$ is denoted by $\overline{i,j}$.

The (standard) *inner product* of vectors $\mathbf{x}, \mathbf{y}$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$. The Euclidean norm $\|\mathbf{x}\| := \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$, and the corresponding matrix norm,

$$\|A\| := \text{the largest singular value of } A, \ \text{(e.g. [\textbf{15}, Theorem 2.3.1])}, \tag{0.1}$$

are used throughout.

The orthogonal projector P on a subspace $\mathbf{L} \subset \mathbb{C}^n$ is characterized by $P = P^2 = P^*$ and $\mathbf{L} = \mathbf{R}(P)$. It is called here the *projector* on $\mathbf{L}$, and denoted by $P_{\mathbf{L}}$; the projector on the orthogonal complement $\mathbf{L}^\perp$ of $\mathbf{L}$ is denoted by $P_{\mathbf{L}}^\perp$,

$$P_{\mathbf{L}}^\perp = I - P_{\mathbf{L}}. \tag{0.2}$$

SVD is an abbreviation for the *singular value decomposition*, e.g. [**9**, p. 14].

.

1991 *Mathematics Subject Classification.* Primary 15A03, 15A18, 65F10; Secondary 15A09, 65F60, 62H20.

Acknowledgement. Helpful comments by Oskar Baksalary, Yair Censor, Götz Trenkler and the referee are gratefully acknowledged.

1. Introduction

J. von Neumann gave the projector on the intersection of subspaces $\mathbf{L}$, $\mathbf{M}$ of a Hilbert space $\mathbb{H}$ as the limit,

$$P_{\mathbf{L}\cap\mathbf{M}} = \lim_{n\to\infty} (P_{\mathbf{L}} P_{\mathbf{M}})^n, \quad [\mathbf{31}, \text{p. } 55], \tag{1.1}$$

extended by Halperin [**18**] to projectors on the intersection of m subspaces $\{\mathbf{L}_i\}$,

$$P_{\mathbf{L}_1\cap\cdots\cap\mathbf{L}_m} = \lim_{n\to\infty} (P_{\mathbf{L}_1} \cdots P_{\mathbf{L}_m})^n, \tag{1.2}$$

see the history in [**11**, pp. 233–235], and recent proofs by Kopecká and Reich [**21**], Bauschke, Matoušková and Reich [**7**], and Netyanun and Solmon, [**22**].

These ideas are used in the Kaczmarz method [**20**] and other alternating projection methods, e.g. [**32**]. The rate of convergence of (1.1) was established by Aronszajn [**4**, p. 379], Deutsch [**11**, eq. (9.8.1)] and others as

$$\|(P_{\mathbf{L}} P_{\mathbf{M}})^n \mathbf{x} - P_{\mathbf{L}\cap\mathbf{M}}\mathbf{x}\| \le c^{2n-1}\|\mathbf{x}\|, \tag{1.3}$$

where c is the cosine of the minimal angle between $\mathbf{L}\cap(\mathbf{L}\cap\mathbf{M})^\perp$ and $\mathbf{M}\cap(\mathbf{L}\cap\mathbf{M})^\perp$,

$$c = c(\mathbf{L},\mathbf{M}) = \sup\left\{ \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|} :\ \mathbf{x}\in\mathbf{L}\cap(\mathbf{L}\cap\mathbf{M})^\perp,\ \mathbf{y}\in\mathbf{M}\cap(\mathbf{L}\cap\mathbf{M})^\perp \right\}. \tag{1.4}$$

A similar bound for the rate of convergence of (1.2) is given in [**11**, Theorem 9.33].

Anderson and Duffin [**3**] gave an explicit formula for $P_{\mathbf{L}\cap\mathbf{M}}$,

$$P_{\mathbf{L}\cap\mathbf{M}} = 2\,P_{\mathbf{L}}(P_{\mathbf{L}} + P_{\mathbf{M}})^\dagger P_{\mathbf{M}}, \tag{1.5}$$

see also [**12**] and [**25**, Theorem 4].

Results. Specializing to finite–dimensional spaces, three formulas for the projector on the intersection of m subspaces are given.

(a) Theorem 3.2(b) ($m = 2$): a constructive formula (3.6) for $P_{\mathbf{L}\cap\mathbf{M}}$ that uses the SVD of $P_{\mathbf{L}}P_{\mathbf{M}}$.

(b) Corollary 4.2 ($m \ge 2$): an explicit formula (4.7) that uses the Moore–Penrose inverse.

(c) Corollary 5.3 ($m \ge 2$): the projector as the limit (5.8) of an exponential.

Plan. Section 2 is a review of principal angles between subspaces as needed in the sequel.

Section 3 uses the SVD of $P_{\mathbf{L}}P_{\mathbf{M}}$ to get Result (a) above, and the precise error $\|(P_{\mathbf{L}}P_{\mathbf{M}})^n - P_{\mathbf{L}\cap\mathbf{M}}\|$ for all n.

Section 4 represents the intersection of $m \ge 2$ subspaces as the null–space of a matrix given by their projectors, see Lemma 4.1. The projector on the intersection is then given in Corollary 4.2.

Section 5 gives projectors on intersections of subspaces as limits of exponentials, Corollary 5.3.

2. Principal angles

Here and in Section 3, $\mathbf{L}$ and $\mathbf{M}$ are subspaces of $\mathbb{R}^n$ and it is assumed that $P_{\mathbf{L}}P_{\mathbf{M}} \ne O$ (otherwise either $\mathbf{M}\subset\mathbf{L}^\perp$ or $\mathbf{L}\subset\mathbf{M}^\perp$, and $\mathbf{L}\cap\mathbf{M} = \{\mathbf{0}\}$).
(a) A pair of vectors $(\mathbf{x},\mathbf{y}) \in \mathbf{L}\times\mathbf{M}$ is called *reciprocal* if

$$\lambda\,\mathbf{x} = P_{\mathbf{L}}\,\mathbf{y},\ \ \mu\,\mathbf{y} = P_{\mathbf{M}}\,\mathbf{x}\ , \tag{2.1}$$

for some $\lambda, \mu > 0$. It follows that $\langle \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{x} \rangle = \mu \langle \mathbf{y}, \mathbf{y} \rangle$ and the angle between $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\cos^2 \angle\{\mathbf{x}, \mathbf{y}\} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle} = \lambda \, \mu.$$

(b) Any pair of reciprocal vectors $\mathbf{x}$ and $\mathbf{y}$ with $\cos^2 \angle\{\mathbf{x}, \mathbf{y}\} = \sigma^2$ are eigenvectors of $P_{\mathbf{L}} P_{\mathbf{M}}$ and $P_{\mathbf{M}} P_{\mathbf{L}}$, respectively, both with the eigenvalue σ^2,

$$P_{\mathbf{L}} P_{\mathbf{M}} \mathbf{x} = \sigma^2 \, \mathbf{x}, \tag{2.2a}$$

$$P_{\mathbf{M}} P_{\mathbf{L}} \, \mathbf{y} = \sigma^2 \, \mathbf{y}. \tag{2.2b}$$

Conversely, if $\mathbf{x}$ satisfies (2.2a) and $\mathbf{y} := P_{\mathbf{M}} \mathbf{x}$ then $\mathbf{x}$ and $\mathbf{y}$ are reciprocal, [**2**, Theorem 4.4].

(c) The *principal angles* between $\mathbf{L}$ and $\mathbf{M}$,

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_r \leq \tfrac{\pi}{2}, \; r = \operatorname{rank}(P_{\mathbf{L}} P_{\mathbf{M}}), \tag{2.3}$$

are defined recursively by the extremum problems

$$\cos \theta_1 = \frac{\langle \mathbf{x}_1, \mathbf{y}_1 \rangle}{\|\mathbf{x}_1\| \|\mathbf{y}_1\|} = \sup \left\{ \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|} : \; \mathbf{x} \in \mathbf{L}, \, \mathbf{y} \in \mathbf{M}, \right\}, \tag{2.4a}$$

(2.4b)

$$\cos \theta_i = \frac{\langle \mathbf{x}_i, \mathbf{y}_i \rangle}{\|\mathbf{x}_i\| \|\mathbf{y}_i\|} = \sup \left\{ \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \; : \; \begin{array}{ll} \mathbf{x} \in \mathbf{L}, & \mathbf{x} \perp \mathbf{x}_k, \\ \mathbf{y} \in \mathbf{M}, & \mathbf{y} \perp \mathbf{y}_k, \end{array} \quad k \in \overline{1, i-1} \right\}, \; i \in \overline{2, r}.$$

(d) Vectors $(\mathbf{x}_i, \mathbf{y}_i)$ corresponding to a principal angle $\theta_i = \angle \{\mathbf{x}_i, \, \mathbf{y}_i\}$, are reciprocal.

(e) If $i \neq j$ then, $\mathbf{x}_i \perp \mathbf{x}_j$, $\mathbf{y}_i \perp \mathbf{y}_j$, and $\mathbf{x}_i \perp \mathbf{y}_j$.

(f) If $\theta_i = 0$ then $\mathbf{x}_i = \mathbf{y}_i$, a vector in the intersection $\mathbf{L} \cap \mathbf{M}$.

(g) The reciprocal vectors $\{(\mathbf{x}_i, \mathbf{y}_i) : i \in \overline{1, r}\}$ span the space $P_{\mathbf{L}} \mathbf{M} + P_{\mathbf{M}} \mathbf{L}$.

(h) The intersection $\mathbf{L} \cap \mathbf{M}$ is spanned by the vectors $\mathbf{x}_i$ corresponding to $\theta_i = 0$; in particular, $\mathbf{L} \cap \mathbf{M} = \{\mathbf{0}\}$ if all $\theta_i > 0$.

Remark 2.1.

(i) Principal angles between subspaces were introduced by Jordan and studied by Hotelling [**19**], Afriat [**1**]–[**2**], Seidel [**28**], Zassenhaus [**33**] and others [**8**, Theorem 4], see the history in [**30**, p. 45] and [**13**, Section 1.7].

(ii) The main methods for computing principal angles employ the SVD (Björck and Golub [**10**], Golub and Zha [**16**], see also [**15**, Algorithm 12.4.3]) or the CS decomposition (Stewart [**29**]).

(iii) For angles between subspaces of complex vector spaces (where there is no "natural" definition of angle), see [**14**].

3. $P_{\mathbf{L} \cap \mathbf{M}}$ and the singular value decomposition of $P_{\mathbf{L}} P_{\mathbf{M}}$

The SVD of the product $P_{\mathbf{L}} P_{\mathbf{M}}$ is used here to study the von Neumann iteration (1.1), and to obtain a constructive formula for $P_{\mathbf{L} \cap \mathbf{M}}$.

Lemma 3.1. *Let* $(\mathbf{x}, \mathbf{y})$ *be reciprocal vectors satisfying (2.2). Then* $\mathbf{x}$ *and* $\mathbf{y}$ *are eigenvectors of* $(P_{\mathbf{L}} P_{\mathbf{M}})(P_{\mathbf{L}} P_{\mathbf{M}})^*$ *and* $(P_{\mathbf{M}} P_{\mathbf{L}})^* (P_{\mathbf{M}} P_{\mathbf{L}})$*, respectively, corresponding to the eigenvalue* σ^2*,*

$$(P_{\mathbf{L}} P_{\mathbf{M}})(P_{\mathbf{L}} P_{\mathbf{M}})^* \, \mathbf{x} = \sigma^2 \, \mathbf{x}, \tag{3.1a}$$

$$(P_{\mathbf{M}} P_{\mathbf{L}})^* (P_{\mathbf{M}} P_{\mathbf{L}}) \, \mathbf{y} = \sigma^2 \, \mathbf{y}. \tag{3.1b}$$

PROOF.

$$\begin{aligned}(P_{\mathbf{L}}P_{\mathbf{M}})(P_{\mathbf{L}}P_{\mathbf{M}})^*\,\mathbf{x} &= P_{\mathbf{L}}P_{\mathbf{M}}P_{\mathbf{M}}^*P_{\mathbf{L}}^*\,\mathbf{x}\\ &= P_{\mathbf{L}}P_{\mathbf{M}}P_{\mathbf{L}}\,\mathbf{x}\\ &= P_{\mathbf{L}}P_{\mathbf{M}}\,\mathbf{x},\ \text{since } \mathbf{x}\in\mathbf{L}.\end{aligned}$$

Therefore (3.1a) is equivalent to (2.2a). (3.1b) is similarly proved. □

This shows that the σ's in (2.2) are singular values of $P_{\mathbf{L}}P_{\mathbf{M}}$, which allows writing the SVD of $(P_{\mathbf{L}}P_{\mathbf{M}})^n$ for all n.

THEOREM 3.2. *Let $\mathbf{L},\mathbf{M}$ be subspaces of $\mathbb{R}^n$, let $r = rank(P_{\mathbf{L}}P_{\mathbf{M}})$, and let the principal angles $\{\theta_i : i \in \overline{1,r}\}$ and corresponding reciprocal pairs $\{(\mathbf{x}_i,\mathbf{y}_i) : i\in\overline{1,r}\}$ be given. The vectors $\{\mathbf{x}_i,\mathbf{y}_i\}$ are assumed normalized, $\|\mathbf{x}_i\| = 1 = \|\mathbf{y}_i\|$ for all i.*
(a) The SVD of $P_{\mathbf{L}}P_{\mathbf{M}}$ is

$$P_{\mathbf{L}}P_{\mathbf{M}} = X\,\Sigma\,Y^* \tag{3.2}$$

where
(i) X is an $n\times r$ matrix with the vectors $\{\mathbf{x}_i : i\in\overline{1,r}\}$ as columns,
(ii) Y is an $n\times r$ matrix with the vectors $\{\mathbf{y}_i : i\in\overline{1,r}\}$ as columns,
(iii) Σ is the $r\times r$ diagonal matrix with the singular values

$$\sigma_i = \cos\theta_i = \langle \mathbf{x}_i,\mathbf{y}_i\rangle \tag{3.3}$$

on the diagonal, in decreasing order.
(b) Let
(3.4)
$s :=$ the number of singular values $\sigma_i = 1$ (corresponding to angles $\theta_i = 0$), $0\le s\le r$.
Then

$$\mathbf{x}_i = \mathbf{y}_i,\ i\in\overline{1,s}, \tag{3.5}$$

and

$$P_{\mathbf{L}\cap\mathbf{M}} = \begin{cases} O, & \textit{if } s=0;\\ \sum\limits_{i=1}^{s}\mathbf{x}_i\mathbf{x}_i^*, & \textit{otherwise.}\end{cases} \tag{3.6}$$

(c) The SVD of the n^{th} iterate of (1.1) is

$$(P_{\mathbf{L}}P_{\mathbf{M}})^n = X\,\Sigma^{2n-1}\,Y^*. \tag{3.7}$$

(d) The error of the n^{th} iterate

$$(P_{\mathbf{L}}P_{\mathbf{M}})^n - P_{\mathbf{L}\cap\mathbf{M}}, \tag{3.8}$$

has the norm

$$\|(P_{\mathbf{L}}P_{\mathbf{M}})^n - P_{\mathbf{L}\cap\mathbf{M}}\| = \cos^{2n-1}\theta_{s+1}, \tag{3.9}$$

where θ_{s+1} is the smallest positive principal angle.

PROOF. (a) follows from (3.1a)–(3.1b).
(b) If $\mathbf{L}\cap\mathbf{M}\neq\{\mathbf{0}\}$ it is spanned by the orthonormal set $\{\mathbf{x}_i : i\in\overline{1,s}\}$.
(c) $(P_{\mathbf{L}}P_{\mathbf{M}})^n$ is, by (3.2),

$$(P_{\mathbf{L}}P_{\mathbf{M}})^n = (X\,\Sigma\,Y^*)(X\,\Sigma\,Y^*)\cdots(X\,\Sigma\,Y^*),$$

where Σ appears n times, and Y^*X appears $n-1$ times. But Y^*X also $= \Sigma$, by (3.3) and the orthogonality $\mathbf{y}_i \perp \mathbf{x}_j$ if $i \neq j$.
(d) From (3.7) and (3.6) it follows that the error (3.8) has the SVD

$$(P_{\mathbf{L}}P_{\mathbf{M}})^n - P_{\mathbf{L}\cap\mathbf{M}} = X_1\,\Sigma_1^{2n-1}\,Y_1^* \tag{3.10}$$

where the matrices X_1 and Y_1 have as columns the last $r-s$ columns of X and Y respectively, and Σ_1 is the diagonal matrix obtained from Σ by deleting the first s rows and columns. Because of the orthonormality of the columns of X_1 and Y_1, the norm (0.1) of the error (3.8) is the norm of Σ_1^{2n-1}, that is σ_{s+1}^{2n-1}. □

Remark 3.3.
(i) The explicit formula (3.6) for $P_{\mathbf{L}\cap\mathbf{M}}$ follows also from [**15**, Theorem 12.4.2], that uses the SVD of $Q_{\mathbf{L}}^*Q_{\mathbf{M}}$ where the columns of $Q_{\mathbf{L}}$ and $Q_{\mathbf{M}}$ are orthonormal bases of $\mathbf{L}$ and $\mathbf{M}$, respectively. This approach does not yield the SVD of $(P_{\mathbf{L}}P_{\mathbf{M}})^n$ in an obvious way.
(ii) (3.9) is due to Deutsch [**11**, Theorem 9.31] and confirms that the bound (1.3) is the best possible.
(iii) The product $P_{\mathbf{L}}P_{\mathbf{M}}$ was also studied in [**5**], [**8**], [**17**] and elsewhere.
(iv) Baksalary and Trenkler, [**6**], used the spectral factorization

$$P_{\mathbf{L}} = U\begin{pmatrix} I & O \\ O & O\end{pmatrix}U^*, \quad U \text{ unitary,} \tag{3.11}$$

to write

$$P_{\mathbf{M}} = U\begin{pmatrix} A & B \\ B^* & D\end{pmatrix}U^*, \quad \text{for appropriate matrices } A, B, D, \tag{3.12}$$

and showed that

$$(P_{\mathbf{L}}P_{\mathbf{M}})^n = U\begin{pmatrix} A^n & A^{n-1}B \\ O & O\end{pmatrix}U^*, \tag{3.13}$$

from which (3.6) follows in the limit.

Example 3.4. We illustrate (3.9) for the iterations $(P_{\mathbf{L}}P_{\mathbf{M}})^n\,\mathbf{v}_0$, with an arbitrary initial vector

$$\mathbf{v}_0 = \sum_{i=1}^{s}\xi_i\,\mathbf{x}_i + \sum_{i=s+1}^{r}\xi_i\,\mathbf{x}_i + \sum_{i=s+1}^{r}\nu_i\,\mathbf{y}_i + \mathbf{z}, \tag{3.14}$$

where s is as in (3.4), $\sum_{i=1}^{s}\xi_i\,\mathbf{x}_i = P_{\mathbf{L}\cap\mathbf{M}}\mathbf{v}_0$, and the vector $\mathbf{z} \in (P_{\mathbf{L}}\mathbf{M} + P_{\mathbf{M}}\mathbf{L})^{\perp}$. Then the n^{th} iterate

$$\begin{aligned}\mathbf{v}_n := (P_{\mathbf{L}}P_{\mathbf{M}})^n\,\mathbf{v}_0 &= \sum_{i=1}^{s}\xi_i\,\mathbf{x}_i + \sum_{i=s+1}^{r}(\xi_i\,\cos^{2n}\theta_i + \nu_i\,\cos^{2n-1}\theta_i)\,\mathbf{x}_i \\ &\longrightarrow\; P_{\mathbf{L}\cap\mathbf{M}}\,\mathbf{v}_0,\ \text{as } n \to \infty,\end{aligned} \tag{3.15}$$

where (3.15) follows from $P_{\mathbf{L}}P_{\mathbf{M}}\,\mathbf{x}_i = (\cos^2\theta_i)\,\mathbf{x}_i$, $P_{\mathbf{L}}\,\mathbf{y}_i = (\cos\theta_i)\,\mathbf{x}_i$, and $P_{\mathbf{L}}P_{\mathbf{M}}\,\mathbf{z} = \mathbf{0}$. The error

$$\mathbf{v}_n - P_{\mathbf{L}\cap\mathbf{M}}\,\mathbf{v}_0 = \sum_{i=s+1}^{r}(\xi_i\,\cos^{2n}\theta_i + \nu_i\,\cos^{2n-1}\theta_i)\,\mathbf{x}_i$$

is in agreement with (3.10), the "extra" power of $\cos\theta_i$ follows from (3.3).

Remark 3.5. The convergence of the von Neumann iterations is slow if the smallest positive angle θ_{s+1} is small, see (3.9). This cannot be helped, but can be avoided by the direct computation (3.6) that uses only the SVD of $P_{\mathbf{L}}P_{\mathbf{M}}$, an alternative to the Anderson–Duffin formula (1.5).

4. Dual representations

A subspace $\mathbf{L}$ can be represented dually as the vectors orthogonal to a set of vectors (its normals), i.e. as a null space of a matrix with the normals as rows,

$$\mathbf{L} = \mathbf{N}(A) \tag{4.1}$$

in which case the projector on $\mathbf{L}$ is

$$P_{\mathbf{L}} = I - A^{\dagger}A \tag{4.2}$$

which is unique even though A is not. Dual representations allow computing the projectors on intersections of more than 2 subspaces: If m subspaces have dual representations, say $\mathbf{L}_i = \mathbf{N}(A_i)$, then their intersection

$$\mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_m$$

is the null space of the matrix formed from the rows of the m matrices A_i, and the projector on the intersection can be found by (4.2). This approach avoids the computation of the projectors on the subspaces $\mathbf{L}_i$, but requires the matrices A_i.

Given two subspace $\mathbf{L}, \mathbf{M} \subset \mathbb{C}^n$, Afriat gave a dual representation of their intersection

$$\mathbf{L} \cap \mathbf{M} = \mathbf{N}\left(I - P_{\mathbf{L}}P_{\mathbf{M}}\right) \tag{4.3}$$

see [**2**, Theorem 4.5]. The projector $P_{\mathbf{L}\cap\mathbf{M}}$ can then be computed by (4.2) with $A = I - P_{\mathbf{L}}P_{\mathbf{M}}$, but the result does not offer any advantage over (1.5), see [**5**, eq. (2.21)].

Next comes a dual representation of the intersection of m subspaces, $m \geq 2$.

Lemma 4.1. *For $i = 1, \cdots, m$, let*
$\mathbf{L}_i$ be subspaces of $\mathbb{C}^n$,
P_i the corresponding projectors,
$P_i^{\perp} := I - P_i$, and
$\lambda_i > 0$. Then

$$\mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_m = \mathbf{N}\left(\sum_{i=1}^{m} \lambda_i\, P_i^{\perp}\right). \tag{4.4}$$

Proof. Let **LS** and **RS** denote left side and right side, respectively.
LS(4.4) $\subset$ **RS**(4.4): Obvious.

$\mathbf{LS}(4.4) \supset \mathbf{RS}(4.4)$: For any $\mathbf{x} \in \mathbf{N}\left(\sum_{i=1}^{m} \lambda_i\, P_i^{\perp}\right)$, it follows from (0.2) that

$$\left(\sum_{i=1}^{m} \lambda_i\right) \mathbf{x} = \sum_{i=1}^{m} \lambda_i\, P_i\, \mathbf{x}.$$

$$\begin{aligned} \therefore \quad \left(\sum_{i=1}^{m} \lambda_i\right) \|\mathbf{x}\| &= \|\sum_{i=1}^{m} \lambda_i\, P_i\, \mathbf{x}\| \\ &\le \sum_{i=1}^{m} \lambda_i\, \|P_i\, \mathbf{x}\| \\ &\le \sum_{i=1}^{m} \lambda_i\, \|\mathbf{x}\| \end{aligned}$$

with equality iff $\|\mathbf{x}\| = \|P_i\, \mathbf{x}\|$ for all i, i.e. iff $\mathbf{x} \in \mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_m$. □

Equation (4.4) also follows from a result by S. Reich, [**26**, Lemma 1.4, p. 283].

Lemma 4.1 gives a new closed form for the projection on the intersection of m subspaces:

Corollary 4.2. *Let $\mathbf{L}_i$, $P_i^{\perp}, \lambda_i$ be as in Lemma 4.1, and define*

$$Q := \sum_{i=1}^{m} \lambda_i\, P_i^{\perp}, \tag{4.5}$$

in particular, if all $\lambda_i = \frac{1}{m}$,

$$Q := I - \tfrac{1}{m} \sum_{i=1}^{m} P_i. \tag{4.6}$$

Then

$$P_{\mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_m} = I - Q^{\dagger}\, Q. \tag{4.7}$$

Proof. Follows from (4.4) and (4.2). □

Remark 4.3.
(a) The formula (4.7) is independent of (1.5), and does not reduce to it for $m = 2$.
(b) (4.7) gives the projection on the orthogonal complement $(\mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_m)^{\perp}$ as

$$P^{\perp}_{\mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_m} = Q^{\dagger}\, Q. \tag{4.8}$$

5. Projectors as limits of exponentials

For a matrix $A \in \mathbb{C}^{n \times n}$ and a scalar t, recall the formula of the *exponential* function

$$\exp\{At\} := I + At + \tfrac{1}{2!}\, A^2 t^2 + \cdots \tag{5.1}$$

Next come some consequences of the definition (5.1).

Lemma 5.1.
(a) If $A \in \mathbb{C}^{n \times n}$ then

$$\exp\{At\} = P_{\mathbf{N}(A^*)} + P_{\mathbf{R}(A)}\, \exp\{At\}. \tag{5.2}$$

(b) If H is positive semi–definite then

$$\exp\{-Ht\} \longrightarrow P_{\mathbf{N}(H)} \text{ as } t \to \infty. \tag{5.3}$$

(c) If P is a projector and $P^{\perp} := I - P$ then

$$\exp\{-Pt\} \longrightarrow P^{\perp} \text{ as } t \to \infty. \tag{5.4}$$

PROOF.
(a) Writing the matrix I in (5.1) as $I = P_{\mathbf{N}(A^*)} + P_{\mathbf{R}(A)}$ we get

$$\begin{aligned}\exp\{At\} &= P_{\mathbf{N}(A^*)} + P_{\mathbf{R}(A)}\left[I + At + \tfrac{1}{2!}A^2t^2 + \cdots\right]\\ &= P_{\mathbf{N}(A^*)} + P_{\mathbf{R}(A)}\exp\{At\}\end{aligned}$$

(b) If H is positive semi–definite then by (5.2),

$$\begin{aligned}\exp\{-Ht\} &= P_{\mathbf{N}(H)} + P_{\mathbf{R}(H)}\exp\{-Ht\}\\ &\longrightarrow P_{\mathbf{N}(H)} \text{ as } t \to \infty.\end{aligned} \tag{5.5}$$

(c) If P is a projector then by (5.5),

$$\begin{aligned}\exp\{-Pt\} &= P^{\perp} + P\exp\{-t\}\\ &\longrightarrow P^{\perp} \text{ as } t \to \infty.\end{aligned}$$

□

EXAMPLE 5.2. Let P be a projector, $\mathbf{x}_0$ a given vector, and consider the problem of minimizing $\|P^{\perp}(\mathbf{x}-\mathbf{x}_0)\|^2$,

$$\inf_{\mathbf{x}} \langle \mathbf{x}-\mathbf{x}_0, P^{\perp}(\mathbf{x}-\mathbf{x}_0)\rangle, \quad \text{which is equivalent to} \quad \inf_{\mathbf{x}} \{\langle \mathbf{x}, P^{\perp}\mathbf{x}\rangle : P\,\mathbf{x} = P\,\mathbf{x}_0\}.$$

Solution by a gradient method

$$\mathbf{x}_t := \mathbf{x} - t\,P^{\perp}\mathbf{x}, \tag{5.6}$$

or

$$\frac{\mathbf{x}_t - \mathbf{x}}{t} = -P^{\perp}\mathbf{x},$$

gives a trajectory approximated by the differential equation

$$\dot{\mathbf{x}} = -P^{\perp}\,\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0, \tag{5.7}$$

with solution

$$\begin{aligned}\mathbf{x}(t) &= \exp\{-P^{\perp}t\}\,\mathbf{x}_0 = \left(P + P^{\perp}\exp\{-t\}\right)\mathbf{x}_0\\ &\longrightarrow\ P\,\mathbf{x}_0 \text{ as } t \to \infty, \text{ by Lemma 5.1(c).}\end{aligned}$$

Discrete steps along (5.6) are orthogonal to $\mathbf{R}(P)$, as is the trajectory of (5.7).

This is also mentioned in [**27**, p. 244].

The projector $P_{\mathbf{L}_1\cap\mathbf{L}_2\cap\cdots\cap\mathbf{L}_m}$ can be represented as a limit of an exponential.

COROLLARY 5.3. *If $\mathbf{L}_i$, $P_i^{\perp}, \lambda_i$ are as in Lemma 4.1, and Q is given by (4.5),*

$$Q := \sum_{i=1}^{m} \lambda_i\,P_i^{\perp},$$

then

$$P_{\mathbf{L}_1\cap\mathbf{L}_2\cap\cdots\cap\mathbf{L}_m} = \lim_{t\to\infty} \exp\{-Q\,t\} \tag{5.8}$$

PROOF. Follows from Lemma 4.1 and Lemma 5.1(b). □

REMARK 5.4.
(a) A possible implementation for the projection of a given vector $\mathbf{v}_0$ on $\mathbf{N}(Q)$ is the iterative method

$$\mathbf{v}_{t+\Delta t} := (I - \Delta t\, Q)\mathbf{v}_t, \tag{5.9}$$

whose steps

$$\mathbf{v}_{t+\Delta t} - \mathbf{v}_t = -\Delta t\, Q\, \mathbf{v}_t,$$

are all orthogonal to $\mathbf{N}(Q)$, since Q is Hermitian.
(b) The limit (5.8) can be extended to Hilbert spaces (of infinite dimensions) by using the results in [**23**, Chapter 3].

References

[1] S. N. Afriat, *On the latent vectors and characteristic values of products of pairs of symmetric idempotents*, Quart. J. Math. Oxford Ser. **7**(1956), 76–78.

[2] S. N. Afriat, *Orthogonal and oblique projectors and the characteristics of pairs of spaces*, Proc. Cambridge Philos. Soc. **53**(1957), 800–816.

[3] W. N. Anderson, Jr. and R. J. Duffin, *Series and parallel addition of matrices*, SIAM J. Appl. Math. **26**(1969), 576–594.

[4] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[5] O. M. Baksalary and G. Trenkler, *Revisitation of the product of two orthogonal projectors*, Linear Algebra Appl. **430**(2009), 2813–2833.

[6] O. M. Baksalary, G. Trenkler, *On angles and distances between subspaces*, Linear Algebra Appl. **431**(2009) 2243–2260.

[7] H. H. Bauschke, E. Matoušková and S. Reich, *Projections and proximal point methods: Convergence results and counterexamples*, Nonlinear Anal. **56**(2004) 715–738.

[8] A. Ben–Israel, *On the geometry of subspaces in Euclidean n–spaces*, SIAM J. Appl. Math. **15**(1967), 1184–1198.

[9] A. Ben–Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications* (2nd edition), Springer, 2003.

[10] Å. Björck and G. H. Golub, *Numerical methods for computing angles between linear subspaces*, Math. of Computation **27**(1973), 579–594.

[11] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer, 2001.

[12] P. A. Filmore and J. P. Williams, *On operator ranges*, Adv. in Math. **7**(1971), 254–281.

[13] A. Galántai, *Projectors and Projection Methods*, Springer, 2003.

[14] A. Galántai and Cs. J. Hegedűs, *Jordans principal angles in complex vector spaces*, Numer. Linear Algebra Appl. **13**(2006), 589–598.

[15] G. H. Golub and C. E. Van Loan, *Matrix Computation* (3rd edition), The Johns Hopkins University Press, 1996.

[16] G. H. Golub and H. Zha, *Perturbation analysis of canonical correlations of linear subspaces*, Lin. Alg. and its Applic. **210**(1994), 3–28.

[17] J. Groß, *On the product of orthogonal projectors*, Linear Algebra Appl. **289**(1999), 141–150.

[18] I. Halperin, *The product of projection operators*, Acta Sei. Math. (Szeged) **23**(1962), 96–99.

[19] H. Hotelling, *Relations between two sets of variates*, Biometrika **28**(1936), 321–377.

[20] S. Kaczmarz, *Angenäherte Auflösung von Systemen linearer Gleichungen*, Bull. Int. Acad. Polon. Sci. Lettres **A35**(1937), 355–357.

[21] E. Kopecká and S. Reich, *A note on the von Neumann alternating projections algorithm*, J. Nonlinear Convex Anal. **5**(2004), 379–386.

[22] A. Netyanun and D. C. Solmon, *Iterated Products of Projections in Hilbert Space*, Amer. Math. Monthly **113**(2006), 644–648.

[23] J. W. Neuberger, *Sobolev Gradients and Differential Equations* (2nd edition), Springer, 2010.

[24] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51**(1955), 406–413.

[25] R. Piziak, P. L. Odell and R. Hahn, *Constructing projections on sums and intersections*, Computers and Mathematics with Applications **37**(1999), 67–74.

[26] S. Reich, *A limit theorem for projections*, Linear and Multilinear Algebra **13**(1983), 281–290.

[27] S. Reich, *Solutions of two problems of H. Brezis*, J. Math. Anal. Appl. **95**(1983), 243–250.

[28] J. Seidel, *Angles and distances in n–dimensional Euclidean and non–Euclidean geometry*, I, II, III, Nederl. Akad. Wetensch. Proc. Ser. A. **58** = *Indag. Math.* **17** (1955), 329–335, 336–340, 535–541

[29] G. W. Stewart, *On the perturbation of pseudo–inverses, projections and linear least squares problems*, SIAM Review **19**(1977),634–662,

[30] G. W. Stewart and J-G Sun, *Matrix Perturbation Theory*, Academic Press, 1990.

[31] J. von Neumann, *Functional Operators. Vol II: The Geometry of Orthogonal Subspaces*, Annals of Math. Studies, Vol 39, Princeton University Press, Princeton, 1950.

[32] J. Xu and L. Zikatanov, *The method of alternating projections and the method of subspace corrections in Hilbert space*, J. Amer. Math. Soc. **15**(2002), 573–597.

[33] H. Zassenhaus, *Angles of inclination in correlation theory*, Amer. Math. Monthly **71**(1964), 218–219.

Rutgers Business School, Rutgers University, 100 Rockafeller Road, Piscataway, NJ 08854, USA

E-mail address: adi.benisrael@gmail.com