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On Principal Angles between Subspaces in $\mathbb{R}^n$ *

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Abstract

Let L, M be subspaces in $\mathbb{R}^n$, $\dim L = l \leq \dim M = m$. Then the **principal angles** between L and M,

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_l \leq \frac{\pi}{2}$$

are given by,

$$\cos\theta_i = \frac{<\mathbf{x}_i,\ \mathbf{y}_i>}{\|\mathbf{x}_i\|\,\|\mathbf{y}_i\|} = \max\left\{\frac{<\mathbf{x},\ \mathbf{y}>}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\ :\ \begin{array}{ll}\mathbf{x}\in L, & \mathbf{x}\perp\mathbf{x}_k,\\ \mathbf{y}\in M, & \mathbf{y}\perp\mathbf{y}_k,\end{array}\quad k=1,\ldots,i-1\right\},\ \text{ see [1]},$$

where

$$(\mathbf{x}_i,\ \mathbf{y}_i)\ \in\ L\times M,\ i=1,\ldots,l\ ,$$

are the corresponding pairs of **principal vectors**. We also define

$$\sin\{L,M\} := \prod_{i=1}^{l}\sin\,\theta_i,\qquad \cos\{L,M\} := \prod_{i=1}^{l}\cos\,\theta_i.$$

We study relations between the principal angles and the **volume** of a matrix $A\in\mathbb{R}_r^{m\times n}$ defined by,

$$\mathrm{vol}\,A := \sqrt{\sum\det^2 A_{IJ}},\ \text{ see [2]},$$

summing over all $r\times r$ submatrices A_{IJ} of A. Sample results are the following generalizations of the Hadamard and Cauchy-Schwarz inequalities.
Theorem 4. Let $A=(A_1,\ A_2)$, $A_1\in\mathbb{R}_l^{n\times n_1}$, $A_2\in\mathbb{R}_m^{n\times n_2}$, rank $A = l+m$. Then

$$\mathrm{vol}\,A = \mathrm{vol}\,A_1\,\mathrm{vol}\,A_2\ \sin\{R(A_1),R(A_2)\}.$$

Theorem 5. Let $B,\ C\in\mathbb{R}_r^{n\times r}$. Then

$$|\det(B^TC)| = \mathrm{vol}\,B\,\mathrm{vol}\,C\ \cos\{R(B),R(C)\}.$$

Key words: Principal angles. Singular values. Volume. Compound matrix.

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1 Introduction

Let L, M be subspaces in $\mathbb{R}^n$, and $\dim L = l \le \dim M = m$. Then the **principal angles** between L and M,

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_l \le \frac{\pi}{2} \tag{1.1}$$

are defined by

$$\cos\theta_i := \frac{<\mathbf{x}_i,\ \mathbf{y}_i>}{\|\mathbf{x}_i\|\,\|\mathbf{y}_i\|} =$$

$$= \max\left\{\frac{<\mathbf{x},\ \mathbf{y}>}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\ :\ \begin{array}{ll}\mathbf{x}\in L, & \mathbf{x}\perp\mathbf{x}_k,\\ \mathbf{y}\in M, & \mathbf{y}\perp\mathbf{y}_k,\\ & k=1,\ldots,i-1\end{array}\right\}, \tag{1.2}$$

where

$$(\mathbf{x}_i,\ \mathbf{y}_i)\ \in\ L\times M,\ i=1,\ldots,l\ , \tag{1.3}$$

are the corresponding l pairs of **principal vectors**. Note that

$$\theta_1 = \cdots = \theta_k = 0 < \theta_{k+1} \quad \textit{iff} \quad \dim L\cap M = k, \tag{1.4}$$

and that if $\dim L = \dim M = 1$, θ_1 is the (nonobtuse) angle between the lines L and M.

We also denote the product of principal sines, and the product of principal cosines, by

$$\sin\{L,M\} := \sin\theta_1\cdots\sin\theta_l\ , \tag{1.5}$$

$$\cos\{L,M\} := \cos\theta_1\cdots\cos\theta_l\ , \tag{1.6}$$

Note that (1.5) and (1.6) are just notation, and not ordinary trigonometrical functions. In particular, $\sin^2\{L,M\} + \cos^2\{L,M\} \le 1$.

Principal angles were introduced by Afriat in his study [1] of the geometry of subspaces in $\mathbb{R}^n$ in terms of their orthogonal and oblique projectors. An important application of principal angles in Statistics is the canonical correlation theory of Hotelling [11], see also [5].

Principal angles and vectors generalize least squares solutions in the following sense: If $\dim L = 1$, say L is the line spanned by the vector $\mathbf{a}$, then the principal angle and vector between L and M are found by minimizing $\{\ \|\mathbf{a}-\mathbf{y}\|_2 :\ \mathbf{y}\in M\}$.

Björck and Golub [3] used the singular value decomposition to compute the principal angles as follows:

Lemma 1 Let the columns of $Q_L \in \mathbb{R}^{n\times l}$ and $Q_M \in \mathbb{R}^{n\times m}$ be orthonormal bases for L and M respectively, and let

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_l \ge 0 \tag{1.7}$$

be the singular values of $Q_M^T Q_L$, then

$$\cos\theta_i = \sigma_i \quad , \quad i=1,\ldots,l\ , \tag{1.8}$$

and

$$\sigma_1 = \cdots = \sigma_k = 1 > \sigma_{k+1} \quad \textit{iff} \quad \dim L\cap M = k\ .\ \square \tag{1.9}$$

In this paper, we discuss some relations between principal angles and the matrix volume defined in [2] . In § 3, we express the principal cosines and principal sines in terms of the volume function. Principal angles allow us to "equalize" the Hadamard and Cauchy-Schwarz inequalities in § 4.

2 Preliminary results

Let $\mathbb{R}_r^{m\times n}$ be the set of $m\times n$ matrices of rank r. The **k-dimensional volume** of $A\in\mathbb{R}_r^{m\times n}$, $0<k\le r$, is defined as

$$\mathrm{vol}_k A := \prod_{i=1}^{k}\sigma_i, \tag{2.1}$$

where

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0 \tag{2.2}$$

are the nonzero **singular values** of A. In particular, the **r-dimensional volume** of $A\in\mathbb{R}_r^{m\times n}$ is called its **volume**, and denoted by $\mathrm{vol}\,A$, [2],

$$\mathrm{vol}\,A := \begin{cases} 0 & \text{if } r=0\ ,\\ \prod_{i=1}^{r}\sigma_i & \text{if } r>0\ ,\end{cases} \tag{2.3}$$

or equivalently

$$\mathrm{vol}\,A = \sqrt{\sum \det^2 A_{IJ}}\ , \tag{2.4}$$

summing over all $r\times r$ submatrices A_{IJ} of A. For consistency, set

$$\mathrm{vol}_0 A := \min\{1, \mathrm{rank}A\}. \tag{2.5}$$

In particular, if A has full column rank, then

$$\mathrm{vol}\,A = \sqrt{\det(A^T A)}\ , \tag{2.6}$$

the volume of the parallelepiped spanned by the columns of A.

The volume function is closely related to compound matrices. The k^{th}**-compound matrix** of A, $C_k(A)$, is the $\begin{pmatrix} m\\ k\end{pmatrix}\times\begin{pmatrix} n\\ k\end{pmatrix}$ matrix whose elements are determinants of all $k\times k$ submatrices of A in lexicographic order. The singular values of $C_k(A)$ are all products $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ of singular values of A. Therefore the largest singular value of $C_k(A)$ (its **spectral norm**) is $\mathrm{vol}_k A$. Some well known properties of compound matrices are collected below, (see e.g. [8]).

Proposition 1
(a) If $A\in\mathbb{R}^{m\times p}$, $B\in\mathbb{R}^{p\times n}$, $\mathrm{rank}\,AB = r$, then $C_k(AB) = C_k(A)C_k(B)$, $k\le\min\{m,\ r,\ n\}$.
(b) $C_k(A^T) = C_k(A)^T$.
(c) $C_k(I) = I$, (appropriate size identity matrix).
(d) If A has orthonormal columns, so does $C_k(A)$.
(e) $\mathrm{vol}_k A = \|C_k(A)\|_2$. $\square$

3 Principal angles and volume

The following lemma is used in the sequel.

Lemma 2 Let the columns of $Q_L \in \mathbb{R}^{n\times l}$ and $Q_M \in \mathbb{R}^{n\times m}$ be orthonormal bases for L and M respectively, and denote

$$Q := (Q_M,\ Q_L)\ . \tag{3.1}$$

Let the singular values of $Q_M^T Q_L$ be

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l \geq 0\ . \tag{3.2}$$

Then the singular values of Q, in decreasing order, are

$$\sqrt{1+\sigma_1},\ \ldots,\ \sqrt{1+\sigma_l},\ \underbrace{1,\ldots,1}_{m-l}\ ,\ \sqrt{1-\sigma_l},\ \ldots,\ \sqrt{1-\sigma_1}\ . \tag{3.3}$$

Proof. Since

$$Q^TQ = \begin{pmatrix} I & Q_M^TQ_L \\ Q_L^TQ_M & I \end{pmatrix} = I + \begin{pmatrix} \mathrm{O} & Q_M^TQ_L \\ Q_L^TQ_M & \mathrm{O} \end{pmatrix}\ ,$$

the eigenvalues of $\begin{pmatrix} \mathrm{O} & Q_M^TQ_L \\ Q_L^TQ_M & \mathrm{O} \end{pmatrix}$ are (see [9, p. 418])

$$\pm\sigma_1,\ \pm\sigma_2,\ \ldots,\ \pm\sigma_l,\ \underbrace{0,\ldots,0}_{m-l}\ .$$

Thus the eigenvalues of Q^TQ are

$$1\pm\sigma_1,\ \ldots,\ 1\pm\sigma_l,\ \underbrace{1,\ldots,1}_{m-l}\ .$$

□

Theorem 1 Let the columns of $Q_L \in \mathbb{R}^{n\times l}$ and $Q_M \in \mathbb{R}^{n\times m}$ be orthonormal bases for L and M respectively, and denote

$$Q := (Q_M,\ Q_L). \tag{3.4}$$

Let

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_l \leq \frac{\pi}{2} \tag{3.5}$$

be the principal angles between L and M. Then for $i = 0, 1, \ldots, l-k-1$,

$$\cos\theta_{l-i} = 1 - \frac{\mathrm{vol}^2_{m+i+1}(Q)}{\mathrm{vol}^2_{m+i}(Q)} = \frac{\mathrm{vol}^2_{l-i}(Q)}{\mathrm{vol}^2_{l-i-1}(Q)} - 1\ , \tag{3.6}$$

$$\sin\theta_{l-i} = \frac{\mathrm{vol}_{m+l+1}(Q)}{\mathrm{vol}_{l-i-1}(Q)}\,(\sin\theta_{l-i+1}\cdots\sin\theta_l)^{-1}\ , \tag{3.7}$$

where

$$k = \dim\ L\cap M\ . \tag{3.8}$$

Proof. By Lemmas 1 and 2,

$$\mathrm{vol}_{m+i+1}(Q) = \mathrm{vol}_{m+i}(Q)\sqrt{1-\cos\theta_{l-i}}\ ,\ i = 0,1,\ldots,l-1\ ,$$

$$\mathrm{vol}_{l-i}(Q) = \mathrm{vol}_{l-i-1}(Q)\sqrt{1+\cos\theta_{l-i}}\ ,\ i = 0,1,\ldots,l-1\ ,$$

$$\mathrm{vol}_{m+i+1}(Q) = \mathrm{vol}_{l-i-1}(Q)\sin\theta_{l-i}\cdots\sin\theta_l\ ,\ i = 0,1,\ldots,l-1\ ,$$

which, after some arithmetic calculations, prove (3.6) and (3.7). □

The following theorem gives the analogous results in terms of the orthogonal projectors P_L and P_M on L and M respectively.

Theorem 2 Let P_L and P_M be the orthogonal projectors on L and M respectively,

$$P := (P_M,\ P_L), \tag{3.9}$$

and let

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_l \leq \frac{\pi}{2} \tag{3.10}$$

be the principal angles between L and M. Then for $i = 0, 1, \ldots, l-k-1$,

$$\cos\theta_{l-i} = 1 - \frac{\mathrm{vol}^2_{m+i+1}(P)}{\mathrm{vol}^2_{m+i}(P)} = \frac{\mathrm{vol}^2_{l-i}(P)}{\mathrm{vol}^2_{l-i-1}(P)} - 1\ , \tag{3.11}$$

$$\sin\theta_{l-i} = \frac{\mathrm{vol}_{m+l+1}(P)}{\mathrm{vol}_{l-i-1}(P)}\,(\sin\theta_{l-i+1}\cdots\sin\theta_l)^{-1}\ , \tag{3.12}$$

where

$$k = \dim\ L\cap M\ . \tag{3.13}$$

Proof. Let $Q_L,\ Q_M$ be as in Lemma 2. Then

$$P_L = Q_LQ_L^T\quad ,\quad P_M = Q_MQ_M^T\ .$$

Therefore

$$P = (Q_M,\ Q_L)\begin{pmatrix} Q_M^T & \mathrm{O} \\ \mathrm{O} & Q_L^T \end{pmatrix} = QU^T\ ,$$

where

$$Q = (Q_M,\ Q_L)\ ,$$

and

$$U = \begin{pmatrix} Q_M & \mathrm{O} \\ \mathrm{O} & Q_L \end{pmatrix}\ .$$

Thus U has orthonormal columns, and

$$\begin{aligned} \mathrm{vol}_i(P) &= \|C_i(Q)\|_2\ , && \text{by Proposition 1} \\ &= \mathrm{vol}_i(Q)\ , && i = 0,1,\ldots,m+l\ . \end{aligned}$$

The result follows from Theorem 1. □

In applications to linear equations the subspaces are the null spaces of the coefficient matrices. Let

$$A_i\mathbf{x} = \mathbf{b}_i,\ i = 1,2\ , \tag{3.14}$$

be two linear systems, with nonempty solution sets

$$S_i = \{\mathbf{x} : A_i\mathbf{x} = \mathbf{b}_i\},\ i = 1, 2\ . \tag{3.15}$$

Then the principal angles between S_1 and S_2 are the principal angles between $L = N(A_1)$ and $M = N(A_2)$. The respective projectors are

$$P_L = I - A_1^\dagger A_1 \quad , \quad P_M = I - A_2^\dagger A_2\ . \tag{3.16}$$

The following theorem suggests that instead of using P_L and P_M, we can use

$$P_{L^\perp} = A_1^\dagger A_1 \quad , \quad P_{M^\perp} = A_2^\dagger A_2\ , \tag{3.17}$$

to compute the principal angles between L, M.

Theorem 3 The non-zero principal angles between L, M equal to the non-zero principal angles between $L^\perp$, $M^\perp$.

Proof. Let $(\mathbf{x}_i,\ \mathbf{y}_i)$ be a pair of principal vectors corresponding to the ith non-zero principal angle θ_i between L and M. Then $P_L\mathbf{y}_i = \alpha\mathbf{x}_i$, for some scalar α. Let the two vectors $\mathbf{x}_i^\perp$ and $\mathbf{y}_i^\perp$ be obtained from $\mathbf{x}_i$ and $\mathbf{y}_i$ respectively, by a $\pi/2$-rotation in $\{\mathbf{x}_i,\ \mathbf{y}_i\}$–plane. Then $\mathbf{x}_i^\perp \in L^\perp$, $\mathbf{y}_i^\perp \in M^\perp$, and the angle between $\mathbf{x}_i^\perp$ and $\mathbf{y}_i^\perp$ is also θ_i. Therefore the ith non-zero principal angle between $L^\perp$, $M^\perp$ is $\leq \theta_i$. The result follows by interchanging L, M with $L^\perp$, $M^\perp$. □

4 Hadamard and Cauchy-Schwarz equalities

Let $\mathbf{a}_1, \cdots, \mathbf{a}_m$ be a basis for M, and let L be 1-dimensional, say $L = \text{span}\{\mathbf{a}\}$, where $\mathbf{a} = \mathbf{a}_M + \mathbf{a}_{M^\perp}$, $\mathbf{a}_M \in M$ and $\mathbf{a}_{M^\perp} \in M^\perp$. Then ([8])

$$\|\mathbf{a}_{M^\perp}\|_2 = \frac{\text{vol}_{\,m+1}(\mathbf{a}_1, \cdots \mathbf{a}_m, \mathbf{a})}{\text{vol}\,(\mathbf{a}_1, \cdots, \mathbf{a}_m)}\ . \tag{4.1}$$

That is

$$\text{vol}_{\,m+1}(\mathbf{a}_1, \cdots \mathbf{a}_m, \mathbf{a}) = \text{vol}\,(\mathbf{a_1}, \cdots, \mathbf{a}_m)\, \|\mathbf{a}\|_2 \sin\theta\ , \tag{4.2}$$

where θ is the principal angle between $\mathbf{a}$, M. For the general case, Afriat [1] gave the following "equalized" Hadamard inequality.

Lemma 3 Let $A = (A_1,\ A_2)$, $A_1 \in \mathbb{R}_l^{n\times l}$, $A_2 \in \mathbb{R}_m^{n\times m}$. Then

$$\text{vol}_{\,l+m} A = \text{vol}\, A_1\, \text{vol}\, A_2\, \sin\{R(A_1), R(A_2)\}\ , \tag{4.3}$$

where $\sin\{R(A_1), R(A_2)\}$ is the product of principal sines between $R(A_1)$ and $R(A_2)$, see (1.5). □

In Lemma 3 the matrices $A_1,\ A_2$ are of full column-rank. A further generalization of the Hadamard inequality is:

Theorem 4 Let $A = (A_1,\ A_2)$, $A_1 \in \mathbb{R}_l^{n\times n_1}$, $A_2 \in \mathbb{R}_m^{n\times n_2}$, rank $A = l + m$. Then

$$\text{vol}\, A = \text{vol}\, A_1\, \text{vol}\, A_2\, \sin\{R(A_1), R(A_2)\}\ . \tag{4.4}$$

Proof.

$$\text{vol}^{\,2} A = \sum_J \text{vol}^{\,2} A_{*J}\ ,$$

where the summation is over all $n \times (l + m)$ submatrices of rank $l + m$. Since every $n \times (l + m)$ submatrix of rank $l + m$ has l columns A_{*J_1} from A_1 and m columns A_{*J_2} from A_2, then

$$\begin{aligned}
\text{vol}^{\,2} A &= \sum_{J_1}\sum_{J_2} \text{vol}^{\,2}(A_{*J_1},\ A_{*J_2})\ , \\
&= \sum_{J_1}\sum_{J_2} \text{vol}^{\,2} A_{*J_1}\, \text{vol}^{\,2} A_{*J_2}\, \sin^2\{R(A_1), R(A_2)\}\ , \\
&\qquad \text{by Lemma 3}\ , \\
&= \text{vol}^{\,2} A_1\, \text{vol}^{\,2} A_2\, \sin^2\{R(A_1), R(A_2)\}\ . \qquad \square
\end{aligned}$$

For a square matrix A, the above results imply:

Corollary 1 Let $A = (A_1,\ A_2)$ be a square matrix, and $A_1 \in \mathbb{R}^{n\times l}$, $A_2 \in \mathbb{R}^{n\times m}$. Then

$$|\det(A)| = \text{vol}_{\,l}(A_1)\, \text{vol}_{\,m}(A_2)\, \sin\{R(A_1), R(A_2)\}\ . \qquad \square \tag{4.5}$$

Now we "equalize" the Cauchy-Schwarz inequality.

Theorem 5 Let $B,\ C \in \mathbb{R}_r^{n\times r}$. Then

$$|\det(B^TC)| = \text{vol}\, B\, \text{vol}\, C\, \cos\{R(B), R(C)\}\ , \tag{4.6}$$

where $\cos\{R(B), R(C)\}$ is the product of principal cosines between $R(B)$ and $R(C)$, see (1.6).

Proof. Let Q_B and Q_C be orthonormal bases for $R(B)$ and $R(C)$, respectively, so that,

$$B = Q_B R_B \quad , \quad C = Q_C R_C\ ,$$

for some matrices $R_B,\ R_C \in \mathbb{R}_r^{r\times r}$. Then

$$\begin{aligned}
|\det(B^TC)| &= |\det(R_B^T)|\,|\det(R_C)|\,|\det(Q_B^T Q_C)|, \\
&= \text{vol}\ B\, \text{vol}\ C\, \cos\{R(B),\ R(C)\}\ , \quad \text{by Lemma 1.}
\end{aligned}$$

This completes the proof. □

Denote the set of strictly increasing sequences of k elements from $1, \ldots, n$, by

$$Q_{k,n} := \{I = (i_1, \cdots, i_k)\ :\ 1 \leq i_1 < \cdots < i_k \leq n\}.$$

For any sets $I \subset \{1, 2, \ldots, m\}$, $J \subset \{1, 2, \ldots, n\}$, let $A_{I*},\ A_{*J},\ A_{IJ}$ denote the submatrices of A lying in rows indexed by I, in columns indexed by J, and in their intersection, respectively. For $A \in \mathbb{R}_r^{m\times n}$, let

$$\mathcal{I}(A) := \{I \in Q_{r,m} \ : \ \text{rank}\ A_{I*} = r\} ,$$
$$\mathcal{J}(A) := \{J \in Q_{r,n} \ : \ \text{rank}\ A_{*J} = r\} .$$

For $I \subset \{1, 2, \ldots, m\}$ define the subspace $\mathbb{R}_I^m$ by

$$\mathbb{R}_I^m := \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m \ : \ x_i = 0 \text{ if } i \notin I\}.$$

An interesting interpretation of the determinants of maximal nonsingular submatrices is the following

Corollary 2 Let $A \in \mathbb{R}_r^{m\times n}$, $I \in Q_{r,m}$. Then

$$\cos\{R(A),\ \mathbb{R}_I^m\} = \frac{|\det A_{IJ}|}{\text{vol}\, A_{*J}} \tag{4.7}$$

for any $J \in \mathcal{J}(A)$.

Proof. Let $I = \{i_1, i_2, \cdots i_r\}$, $B = (e_{i_1}, \cdots, e_{i_r})$, and for any $J \in \mathcal{J}(A)$ let $C = A_{*J}$. Then

$$R(B) = \mathbb{R}_I^m \ , \quad R(C) = R(A) \ , \ \text{and} \ \ B^T C = A_{IJ} \ .$$

By Theorem 5

$$\cos\{R(A),\ \mathbb{R}_I^m\} = \cos\{R(B),\ R(C)\} = \frac{|\det A_{IJ}|}{\text{vol}\, A_{*J}} . \qquad \square$$

Note that for any $I \in Q_{r,m}$, the ratio $\frac{|\det A_{IJ}|}{\text{vol}\, A_{*J}}$ is independent of the choice of $J \in \mathcal{J}(A)$.

Example 1 Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \ , \tag{4.8}$$

$I = \{1,\ 2\}$, $J = \{1,\ 2\}$. Then A is of rank 2, and

$$|\det A_{IJ}| = 3 \ , \quad \text{vol}\, A_{*J} = \sqrt{3^2 + 6^2 + 3^2} = 3\sqrt{6} \ ,$$

and, by (4.7),

$$\cos\{R(A),\ \mathbb{R}^3_{\{1,2\}}\} = \frac{1}{\sqrt{6}} \ .$$

Corollary 3 Let $L \subset \mathbb{R}^m$ be a subspace of dimension r. Then

$$\sum_{I \in Q_{r,m}} \cos^2\{L, \mathbb{R}_I^m\} = 1 \ . \tag{4.9}$$

Proof. Follows from (4.7) since

$$\text{vol}^2 A_{*J} = \sum_{I \in \mathcal{I}(A)} \det{}^2 A_{IJ} \ . \quad \square$$

The corresponding result for the complementary orthogonal subspace $L^\perp$ is

$$\sum_{I \in Q_{r,m}} \cos^2\{L^\perp, \mathbb{R}_{I^c}^m\} = 1 \ , \tag{4.10}$$

where I^c is the **complement** of I in $\{1, 2, \ldots, m\}$. The equivalence of (4.9) and (4.10) is by

$$\cos\{L, \mathbb{R}_I^m\} = \cos\{L^\perp, \mathbb{R}_{I^c}^m\} \ , \tag{4.11}$$

see Theorem 3.

Example 2 Let $L = R(A)$ for A of (4.8). Then $L^\perp = N(A^T)$ is the line spanned by $(1, -2, 1)$. The complement of $I = \{1, 2\}$ is $I^c = \{3\}$, and $\mathbb{R}^3_{I^c}$ is the x_3-axis. The angle between the lines $L^\perp$ and $\mathbb{R}^3_{I^c}$, $\angle(L^\perp, \mathbb{R}^3_{I^c})$, is given by

$$\cos\angle(L^\perp, \mathbb{R}^3_{I^c}) = \cos\angle(\,(1, -2, 1), (0, 0, 1)\,) = \frac{1}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{1}{\sqrt{6}} \ ,$$

in agreement with (4.11) and Example 1.

Theorem 6 Let $B \in \mathbb{R}_r^{n\times r}$, $C \in \mathbb{R}^{n\times s}$ and rank $(C) \geq r$. Then

$$\cos\{R(B),\ R(C)\} = \cos\angle\{C_r(B),\ R(C_r(C))\} \ , \tag{4.12}$$

where $\angle\{C_r(B),\ R(C_r(C))\}$ is the angle between the vector $C_r(B)$ and the subspace $R(C_r(C))$.

Proof. Let the columns of

$$E = (e_1,\ \cdots,\ e_r) \ \text{ and } \ F = (f_1,\ \cdots,\ f_t)$$

be orthonormal bases for $R(B)$ and $R(C)$ respectively, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ be the singular values of $F^T E$. Then by Lemma 1

$$\cos\{R(B),\ R(C)\} = \sigma_1 \cdots \sigma_r.$$

On the other hand, the columns of $C_r(E)$ and $C_r(F)$ form orthonormal bases for $C_r(B)$ and $R(C_r(C))$ respectively. Moreover, the singular value of $C_r(F)^T C_r(E)$ $\big(= C_r(F^T E)\big)$ is

$$\sigma_1 \cdots \sigma_r.$$

By the same lemma,

$$\cos\angle\{C_r(B),\ R(C_r(C))\} = \sigma_1 \cdots \sigma_r. \qquad \square$$

We now use the above results (the equalized Hadamard and Cauchy-Schwarz inequalities) to equalize a determinantal inequality of Thompson.

Theorem 7 [1] Let $A \in \mathbb{R}^{kr\times kr}$ be partitioned as

$$A = (B_1, B_2, \cdots, B_k), \quad \text{where} \quad B_i \in \mathbb{R}_r^{kr\times r} .$$

Then

$$\det \begin{pmatrix} B_1^T B_1 & \cdots & B_1^T B_k \\ \vdots & \ddots & \vdots \\ B_k^T B_1 & \cdots & B_k^T B_k \end{pmatrix} =$$

$$= \det \begin{pmatrix} \det(B_1^T B_1) & \cdots & \det(B_1^T B_k) \\ \vdots & \ddots & \vdots \\ \det(B_k^T B_1) & \cdots & \det(B_k^T B_k) \end{pmatrix} \times$$

$$\times \prod_{i=1}^{k-1} \frac{\sin^2\{R(B_i),\ R(B_{i+1},\cdots,B_k)\}}{1-\cos^2 \angle\{C_r(B_i),\ R(C_r(B_{i+1}),\cdots,C_r(B_k))\}} . \tag{4.13}$$

Proof. We prove the case $k = 2$.

$$\det \begin{pmatrix} B_1^T B_1 & B_1^T B_2 \\ B_2^T B_1 & B_2^T B_2 \end{pmatrix} = \det A^T A =$$

$$= \operatorname{vol}_r^2 B_1 \operatorname{vol}_r^2 B_2 \sin^2\{R(B_1),\ R(B_2)\} ,$$

by Corollary 1. Now let

$$\widetilde{A} = (C_r(B_1),\ C_r(B_2)) .$$

Then

$$\det \begin{pmatrix} \det(B_1^T B_1) & \det(B_1^T B_2) \\ \det(B_2^T B_1) & \det(B_2^T B_2) \end{pmatrix} = \det(\widetilde{A})^T \widetilde{A} =$$

$$= \|C_r(B_1)\|_2^2 \|C_r(B_2)\|_2^2 \sin^2 \angle\{C_r(B_1),\ C_r(B_2)\} =$$

$$= \operatorname{vol}_r^2 B_1 \operatorname{vol}_r^2 B_2 (1-\cos^2 \angle\{C_r(B_1),\ C_r(B_2)\}) .$$

The general case $k \geq 2$ is proved analogously. □

Remark. If the denominator in (4.13) is zero, then the numerator is also zero. In this case, (4.13) is also valid if we let the ratio to be 1. Note that

$$R(C_r(B_{i+1}),\cdots,C_r(B_k)) \subseteq R(C_r(B_{i+1},\cdots,B_k)) ,$$

and therefore

$$\cos \angle\{C_r(B_i),\ R(C_r(B_{i+1}),\cdots,C_r(B_k))\} \leq$$

$$\leq \cos \angle\{C_r(B_i),\ R(C_r(B_{i+1},\cdots,B_k))\} ,$$

$$= \cos\{R(B_i),\ R(B_{i+1},\cdots,B_k)\} . \tag{4.14}$$

[1] We proved this originally for $k = 2$, and we thank the referee for suggesting extension to $k > 2$.

The last equality follows from Theorem 6. If $\theta_1,\cdots,\theta_r$ are the principal angles between two subspaces L and M, then we always have

$$\frac{\sin^2\{L,\ M\}}{1-\cos^2\{L,\ M\}} = \frac{\prod_{i=1}^{r}(1-\cos^2\theta_i)}{1-\prod_{i=1}^{r}\cos^2\theta_i} \leq 1 . \tag{4.15}$$

Thus from (4.14) and (4.15), we have

$$\prod_{i=1}^{k-1} \frac{\sin^2\{R(B_i),\ R(B_{i+1},\cdots,B_k)\}}{1-\cos^2 \angle\{C_r(B_i),\ R(C_r(B_{i+1}),\cdots,C_r(B_k))\}} \leq$$

$$\leq \prod_{i=1}^{k-1} \frac{\sin^2\{R(B_i),\ R(B_{i+1},\cdots,B_k)\}}{1-\cos^2\{R(B_i),\ R(B_{i+1},\cdots,B_k)\}} \leq 1 . \tag{4.16}$$

Therefore Theorem 7 implies the following inequalities of Davis [6], Everitt [7] ($k = 2$) and Thompson [13] ($k \geq 2$).

Corollary 4 Let a p.s.d. matrix A of order kr be partitioned as

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}, \quad \text{where } A_{ij} \in \mathbb{R}^{r\times r} . \tag{4.17}$$

Then

$$\det \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix} \leq \det \begin{pmatrix} \det A_{11} & \cdots & \det A_{1k} \\ \vdots & \ddots & \vdots \\ \det A_{k1} & \cdots & \det A_{kk} \end{pmatrix} . \ \square \tag{4.18}$$

A matrix $A \in \mathbb{R}_r^{n\times n}$ is called **range-Hermitian** (or *EP* **matrix**, EP_r **matrix**) if

$$R(A^T) = R(A) . \tag{4.19}$$

Note that the class of range-Hermitian matrices includes normal matrices and nonsingular matrices. A good reference on range-Hermitian matrices is [12].

A characterization of range-Hermitian matrices follows.

Theorem 8 Let $A \in \mathbb{R}_r^{n\times n}$. Then A is range-Hermitian if and only if A has r nonzero eigenvalues $\lambda_1,\ldots,\lambda_r$, and $\operatorname{vol} A = \prod_{i=1}^{r} |\lambda_i|$.

Proof. Let $A = CB^T$ be a full-rank factorization of A. Since

$$R(A^T) = R(B) \quad , \quad R(A) = R(C) ,$$

$$R(A^T) = R(A) \quad \text{if and only if} \quad \cos\{R(B),\ R(C)\} = 1 .$$

By Theorem 6

$$\cos \angle(C_r(B), C_r(C)) = \frac{|C_r(B)^T C_r(C)|}{\|C_r(B)\|_2 \|C_r(C)\|_2} =$$

$$= \cos\{R(B),\ R(C)\} \ = \ 1\ .$$

And using the fact , [2],

$$\mathrm{vol}\, A = \mathrm{vol}\, B\, \mathrm{vol}\, C\ .$$

Then

$$\frac{|\sum \det A_{II}|}{\mathrm{vol}\, A} = 1\ ,$$

where the summation is over all $r \times r$ principal submatrices of A. And finally note that the rth elementary symmetric function of the eigenvalues of A is the sum of all the $r \times r$ principal minors of A. □

Example 3 The matrix A of (4.8) is range-Hermitian. Its nonzero eigenvalues are

$$\lambda_1 = \frac{15 + 3\sqrt{33}}{2}\ , \quad \lambda_2 = \frac{15 - 3\sqrt{33}}{2}\ ,$$

and its nonzero singular values are, correct to 6 decimals,

$$\sigma_1 = 16.848103\ , \quad \sigma_2 = 1.068369\ .$$

Therefore $\mathrm{vol}\, A = \sigma_1\sigma_2 = |\lambda_1|\,|\lambda_2| = 18$.

To put Theorem 8 in perspective recall that for any square matrix $A \in \mathbb{R}^{n\times n}$ with singular values $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$, and eigenvalues $|\lambda_1| \geq \cdots \geq |\lambda_n|$,

$$\prod_{i=1}^{k} |\lambda_i| \leq \prod_{i=1}^{k} \sigma_i \quad , \quad k = 1, \ldots, n, \tag{4.20}$$

with equality for $k = n$, [10, Theorem 3.3.2]. For $k = n$ the common value in (4.20) is nonzero iff A is nonsingular, in which case it is range-Hermitian. Therefore, the equality in (4.20) for $k = r = n$ is a special case of Theorem 8 which states that, for range-Hermitian matrices of rank r ,

$$\prod_{i=1}^{r} |\lambda_i| = \prod_{i=1}^{r} \sigma_i\ . \tag{4.21}$$

Conversely, Theorem 8 follows from the nonsingular case, since a matrix A is range-Hermitian iff

$$A = U \begin{pmatrix} B & O \\ O & O \end{pmatrix} U^T\ , \quad U \text{ orthogonal},\ B \text{ nonsingular}\ ,$$

see [4, Theorem 4.3.1]. Written analogously to (4.20), a characterization of normal matrices is, [10, Problem 3.3.14]

$$\prod_{i=1}^{k} |\lambda_i| = \prod_{i=1}^{k} \sigma_i \quad , \quad k = 1, \ldots, n. \tag{4.22}$$

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