On Principal Angles between Subspaces in $\mathbb{R}^n$ *

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Abstract
Let $L, M$ be subspaces in $\mathbb{R}^n$, dim $L = l \leq$ dim $M = m$. Then the principal angles between $L$ and $M$,

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_l \leq \frac{\pi}{2}$$

are given by,

$$\cos \theta_i = \frac{\langle x_i, y_i \rangle}{\|x_i\| \|y_i\|} = \max \left\{ \frac{\langle x, y \rangle}{\|x\| \|y\|} : x \in L, \quad y \in M, \quad x \perp x_k, \quad y \perp y_k, \quad k = 1, \ldots, i - 1 \right\}$$

where

$$(x_i, y_i) \in L \times M, \quad i = 1, \ldots, l,$$

are the corresponding pairs of principal vectors. We also define

$$\sin \{L, M\} := \prod_{i=1}^l \sin \theta_i, \quad \cos \{L, M\} := \prod_{i=1}^l \cos \theta_i.$$

We study relations between the principal angles and the volume of a matrix $A \in \mathbb{R}^{m \times n}$ defined by,

$$\text{vol} A := \sqrt{\sum \det^2 A_{IJ}}, \quad \text{see} \ [2],$$

summing over all $r \times r$ submatrices $A_{IJ}$ of $A$. Sample results are the following generalizations of the Hadamard and Cauchy-Schwarz inequalities.

Theorem 4. Let $A = (A_1, A_2), \ A_1 \in \mathbb{R}^{n_1 \times n_1}, \ A_2 \in \mathbb{R}^{n_2 \times n_2}, \ \text{rank} \ A = l + m$. Then

$$\text{vol} A = \text{vol} A_1 \text{vol} A_2 \sin \{R(A_1), R(A_2)\}.$$

Theorem 5. Let $B, C \in \mathbb{R}^{n \times r}$. Then

$$|\det (B^T C)| = \text{vol} B \text{vol} C \cos \{R(B), R(C)\}.$$

Key words: Principal angles. Singular values. Volume. Compound matrix.

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1 Introduction

Let $L, M$ be subspaces in $\mathbb{R}^n$, and $\dim L = l \leq \dim M = m$. Then the principal angles between $L$ and $M$,

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_l \leq \frac{\pi}{2}$$

(1.1)

are defined by

$$\cos \theta_i := \frac{\langle x_i, y_i \rangle}{\|x_i\| \|y_i\|} = \max \left\{ \frac{\langle x, y \rangle}{\|x\| \|y\|} : x \in L, y \in M, x \perp x_k, y \perp y_k, \right\},$$

(1.2)

where

$$(x_i, y_i) \in L \times M, \ i = 1, \ldots, l,$$

(1.3)

are the corresponding $l$ pairs of principal vectors. Note that

$$\theta_1 = \cdots = \theta_k = 0 \iff \dim L \cap M = k,$$

(1.4)

and that if $\dim L = \dim M = 1$, $\theta_1$ is the (nonobtuse) angle between the lines $L$ and $M$.

We also denote the product of principal sines, and the product of principal cosines, by

$$\sin \{L, M\} := \sin \theta_1 \cdots \sin \theta_l,$$

(1.5)

$$\cos \{L, M\} := \cos \theta_1 \cdots \cos \theta_l,$$

(1.6)

Note that (1.5) and (1.6) are just notation, and not ordinary trigonometrical functions. In particular, $\sin^2 \{L, M\} + \cos^2 \{L, M\} \leq 1$.

Principal angles were introduced by Afriat in his study [1] of the geometry of subspaces in $\mathbb{R}^n$ in terms of their orthogonal and oblique projectors. An important application of principal angles in Statistics is the canonical correlation theory of Hotelling [11], see also [5].

Principal angles and vectors generalize least squares solutions in the following sense: If $\dim L = 1$, say $L$ is the line spanned by the vector $a$, then the principal angle and vector between $L$ and $M$ are found by minimizing $\|a - y\|_2 : y \in M$.

Björck and Golub [3] used the singular value decomposition to compute the principal angles as follows:

**Lemma 1** Let the columns of $Q_L \in \mathbb{R}^{n \times l}$ and $Q_M \in \mathbb{R}^{n \times m}$ be orthonormal bases for $L$ and $M$ respectively, and let

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l \geq 0$$

(1.7)

be the singular values of $Q_M^T Q_L$, then

$$\cos \theta_i = \sigma_i, \ i = 1, \ldots, l,$$

(1.8)

and

$$\sigma_1 = \cdots = \sigma_k = 1 > \sigma_{k+1} \iff \dim L \cap M = k.$$  \hfill (1.9)

In this paper, we discuss some relations between principal angles and the matrix volume defined in [2]. In §3, we express the principal cosines and principal sines in terms of the volume function. Principal angles allow us to “equalize” the Hadamard and Cauchy-Schwarz inequalities in §4.

2 Preliminary results

Let $\mathbb{R}^{m \times n}_r$ be the set of $m \times n$ matrices of rank $r$. The $k$-dimensional volume of $A \in \mathbb{R}^{m \times n}_r$, $0 < k \leq r$, is defined as

$$\text{vol}_k A := \prod_{i=1}^k \sigma_i,$$

(2.1)

where

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

(2.2)

are the nonzero singular values of $A$. In particular, the $r$-dimensional volume of $A \in \mathbb{R}^{m \times n}_r$ is called its volume, and denoted by $\text{vol} A$, [2].

$$\text{vol} A := \begin{cases} 0 & \text{if } r = 0, \\ \prod_{i=1}^r \sigma_i & \text{if } r > 0, \end{cases}$$

(2.3)

or equivalently

$$\text{vol} A = \sqrt{\sum_{i=1}^r \det^2 A_{i,i}},$$

(2.4)

summing over all $r \times r$ submatrices $A_{i,i}$ of $A$. For consistency, set

$$\text{vol}_0 A := \min \{1, \text{rank} A\}. \quad (2.5)$$

In particular, if $A$ has full column rank, then

$$\text{vol} A = \sqrt{\det(A^T A)}, \quad (2.6)$$

the volume of the parallelepiped spanned by the columns of $A$.

The volume function is closely related to compound matrices. The $k$th compound matrix of $A$, $C_k(A)$, is the $(m \times k) \times (n \times k)$ matrix whose elements are determinants of all $k \times k$ submatrices of $A$ in lexicographical order. The singular values of $C_k(A)$ are all products $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$ of singular values of $A$. Therefore the largest singular value of $C_k(A)$ (its spectral norm) is $\text{vol}_k A$. Some well known properties of compound matrices are collected below, (see e.g. [8]).

**Proposition 1**

(a) If $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$, rank $AB = r$, then $C_k(AB) = C_k(A)C_k(B)$, $k \leq \min\{m, n\}$.

(b) $C_k(A^T) = C_k(A)^T$.

(c) $C_k(I) = I$, (appropriate size identity matrix).

(d) If $A$ has orthonormal columns, so does $C_k(A)$.

(e) $\text{vol}_k A = \|C_k(A)\|_2$. \hfill \Box
3 Principal angles and volume

The following lemma is used in the sequel.

Lemma 2 Let the columns of $Q_L \in \mathbb{R}^{n \times l}$ and $Q_M \in \mathbb{R}^{n \times m}$ be orthonormal bases for $L$ and $M$ respectively, and denote

$$Q := (Q_M, Q_L).$$

Let the singular values of $Q^T_M Q_L$ be

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l \geq 0.$$  \hfill (3.2)

Then the singular values of $Q$, in decreasing order, are

$$\sqrt{1 + \sigma_1}, \ldots, \sqrt{1 + \sigma_l}, 1, \ldots, 1, \sqrt{1 - \sigma_l}, \ldots, \sqrt{1 - \sigma_1}. \hfill (3.3)$$

Proof. Since

$$Q^T Q = \left( \begin{array}{c} I \\ Q^T_M Q_L \end{array} \right) \left( \begin{array}{c} I \\ Q^T_M Q_L \end{array} \right)^T = I + \left( \begin{array}{cc} O & Q^T_M Q_L \\ Q^T_M Q_L & O \end{array} \right),$$

the eigenvalues of

$$\left( \begin{array}{cc} O & Q^T_M Q_L \\ Q^T_M Q_L & O \end{array} \right)$$

are (see [9, p. 418])

$$\pm \sigma_1, \pm \sigma_2, \ldots, \pm \sigma_l, 0, \ldots, 0 \hfill \frac{m - l}{m - l}. \hfill \Box$$

Thus the eigenvalues of $Q^T Q$ are

$$1 \pm \sigma_1, \ldots, 1 \pm \sigma_l, 1, \ldots, 1 \hfill \frac{m - l}{m - l}. \hfill \Box$$

Theorem 1 Let the columns of $Q_L \in \mathbb{R}^{n \times l}$ and $Q_M \in \mathbb{R}^{n \times m}$ be orthonormal bases for $L$ and $M$ respectively, and denote

$$Q := (Q_M, Q_L).$$

Let

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_l \leq \frac{\pi}{2}$$

be the principal angles between $L$ and $M$. Then for $i = 0, 1, \ldots, l - k - 1$,

$$\cos \theta_{l-i} = 1 - \frac{\text{vol}_{m+i+1}(Q)}{\text{vol}_{m+i}(Q)} = \frac{\text{vol}_{l-i}(Q)}{\text{vol}_{l-i-1}(Q)} - 1, \hfill (3.6)$$

$$\sin \theta_{l-i} = \frac{\text{vol}_{m+i+1}(Q)}{\text{vol}_{l-i-1}(Q)} (\sin \theta_{l-i-1} \cdots \sin \theta_l)^{-1}, \hfill (3.7)$$

where

$$k = \dim L \cap M. \hfill (3.8)$$

Proof. By Lemmas 1 and 2,

$$\text{vol}_{m+i+1}(Q) = \text{vol}_{m+i}(Q) \sqrt{1 - \cos \theta_{l-i}}, \ i = 0, 1, \ldots, l - 1,$$

$$\text{vol}_{l-i}(Q) = \text{vol}_{l-i-1}(Q) \sqrt{1 + \cos \theta_{l-i}}, \ i = 0, 1, \ldots, l - 1,$$

$$\text{vol}_{m+i+1}(Q) = \text{vol}_{l-i-1}(Q) \sin \theta_{l-i} \cdots \sin \theta_l, \ i = 0, 1, \ldots, l - 1,$$

which, after some arithmetic calculations, prove (3.6) and (3.7).

The following theorem gives the analogous results in terms of the orthogonal projectors $P_L$ and $P_M$ on $L$ and $M$ respectively.

Theorem 2 Let $P_L$ and $P_M$ be the orthogonal projectors on $L$ and $M$ respectively,

$$P := (P_M, P_L),$$

and let

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_l \leq \frac{\pi}{2}$$

be the principal angles between $L$ and $M$. Then for $i = 0, 1, \ldots, l - k - 1$,

$$\cos \theta_{l-i} = 1 - \frac{\text{vol}_{m+i+1}(P)}{\text{vol}_{m+i}(P)} = \frac{\text{vol}_{l-i}(P)}{\text{vol}_{l-i-1}(P)} - 1, \hfill (3.11)$$

$$\sin \theta_{l-i} = \frac{\text{vol}_{m+i+1}(P)}{\text{vol}_{l-i-1}(P)} (\sin \theta_{l-i-1} \cdots \sin \theta_l)^{-1}, \hfill (3.12)$$

where

$$k = \dim L \cap M. \hfill (3.13)$$

Proof. Let $Q_L$, $Q_M$ be as in Lemma 2. Then

$$P_L = Q_L Q_L^T, \quad P_M = Q_M Q_M^T.$$

Therefore

$$P = (Q_M, Q_L) \left( \begin{array}{cc} Q_M^T & O \\ O & Q_L^T \end{array} \right) = QU^T,$$

where

$$Q = (Q_M, Q_L),$$

and

$$U = \left( \begin{array}{cc} Q_M & O \\ O & Q_L \end{array} \right).$$

Thus $U$ has orthonormal columns, and

$$\text{vol}_i(P) = \|C_i(Q)\|_2, \ \text{by \ Proposition \ 1} \hfill (3.14)$$

$$= \text{vol}_i(Q), \quad i = 0, 1, \ldots, m + l. \hfill (3.14)$$

The result follows from Theorem 1. \hfill \Box

In applications to linear equations the subspaces are the null spaces of the coefficient matrices. Let

$$A_i x = b_i, \ i = 1, 2,$$
be two linear systems, with nonempty solution sets

\[ S_i = \{ x : A_i x = b_i \}, \quad i = 1, 2. \]  

(3.15)

Then the principal angles between \( S_1 \) and \( S_2 \) are the principal angles between \( L = N(A_1) \) and \( M = N(A_2) \). The respective projectors are

\[ P_L = I - A_1^\dagger A_1, \quad P_M = I - A_2^\dagger A_2. \]  

(3.16)

The following theorem suggests that instead of using \( P_L \) and \( P_M \), we can use

\[ P_{L_i} = A_i^\dagger A_1, \quad P_{M_i} = A_i^\dagger A_2, \]  

(3.17)

to compute the principal angles between \( L, M \).

**Theorem 3** The non-zero principal angles between \( L, M \) equal to the non-zero principal angles between \( L_i, M_i \).

**Proof.** Let \( (x_i, y_i) \) be a pair of principal vectors corresponding to the \( i \)th non-zero principal angle \( \theta_i \) between \( L \) and \( M \). Then \( P_L y_i = \alpha x_i \), for some scalar \( \alpha \). Let the two vectors \( x_i^\perp \) and \( y_i^\perp \) be obtained from \( x_i \) and \( y_i \), respectively, by a \( \pi/2 \)-rotation in \( \{x_i, y_i\} \)-plane. Then \( x_i^\perp \in L^\perp, y_i^\perp \in M^\perp \), and the angle between \( x_i^\perp \) and \( y_i^\perp \) is also \( \theta_i \). Therefore the \( i \)th non-zero principal angle between \( L_i, M_i \) is \( \leq \theta_i \). The result follows by interchanging \( L, M \) with \( L_i, M_i \). \( \square \)

### 4 Hadamard and Cauchy-Schwarz equalities

Let \( a_1, \ldots, a_m \) be a basis for \( M \), and let \( L \) be 1-dimensional, say \( L = \text{span}\{a\} \), where \( a = a_M + a_{M^\perp} \), \( a_M \in M \) and \( a_{M^\perp} \in M^\perp \). Then ([8])

\[ \|a_{M^\perp}\|_2 = \frac{\|a_{M^\perp}\|}{\|a\|_2} \sin \theta. \]  

(4.1)

That is

\[ \|a_{M^\perp}\|_2 = \frac{\|a_{M^\perp}\|}{\|a\|_2} \sin \theta, \]  

(4.2)

where \( \theta \) is the principal angle between \( a, M \). For the general case, Afriat [1] gave the following “equalized” Hadamard inequality.

**Lemma 3** Let \( A = (A_1, A_2), A_1 \in \mathbb{R}^{n \times l}, A_2 \in \mathbb{R}^{m \times m} \). Then

\[ \|A_{l+m} A\|_2 = \|A_2\|_2 \sin \{R(A_1), R(A_2)\}, \]  

(4.3)

where \( \sin \{R(A_1), R(A_2)\} \) is the product of principal sines between \( R(A_1) \) and \( R(A_2) \), see (1.5). \( \square \)

In Lemma 3 the matrices \( A_1, A_2 \) are of full column-rank.

A further generalization of the Hadamard inequality is:

**Theorem 4** Let \( A = (A_1, A_2), A_1 \in \mathbb{R}_i^{n \times n_1}, A_2 \in \mathbb{R}_m^{n \times n_2} \), rank \( A = l + m \). Then

\[ \det A = \det A_1 \det A_2 \sin \{R(A_1), R(A_2)\}. \]  

(4.4)

**Proof.**

\[ \det^2 A = \sum_{j} \det^2 A_{m,j}, \]  

where the summation is over all \( n \times (l + m) \) submatrices of rank \( l + m \). Since every \( n \times (l + m) \) submatrix of rank \( l + m \) has \( l \) columns \( A_{s,j} \) from \( A_1 \) and \( m \) columns \( A_{s,j} \) from \( A_2 \), then

\[ \det^2 A = \sum_{j} \sum_{j_2} \det^2 A_{s,j_1} \det^2 A_{s,j_2} \sin^2 \{R(A_1), R(A_2)\}, \]  

by Lemma 3,

\[ = \det A_1 \det A_2 \sin^2 \{R(A_1), R(A_2)\}. \]

\( \square \)

For a square matrix \( A \), the above results imply:

**Corollary 1** Let \( A = (A_1, A_2) \) be a square matrix, and \( A_1 \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{m \times m} \). Then

\[ \det A = \det A_1 \det A_2 \sin \{R(A_1), R(A_2)\}. \]  

(4.5)

Now we “equalize” the Cauchy-Schwarz inequality.

**Theorem 5** Let \( B, C \in \mathbb{R}_r^{n \times r} \). Then

\[ \det(B^T C) = \det B \det C \cos \{R(B), R(C)\}, \]  

(4.6)

where \( \cos \{R(B), R(C)\} \) is the product of principal cosines between \( R(B) \) and \( R(C) \), see (1.6).

**Proof.** Let \( Q_B \) and \( Q_C \) be orthonormal bases for \( R(B) \) and \( R(C) \), respectively, so that,

\[ B = Q_B R_B, \quad C = Q_C R_C, \]  

for some matrices \( R_B, R_C \in \mathbb{R}_r^{r \times r} \). Then

\[ \det(B^T C) = \det(Q_B^T) \det(R_B) \det(Q_C^T) \det(Q_B^T Q_C), \]  

\[ = \det(B) \det(C) \cos \{R(B), R(C)\}, \]  

by Lemma 1.

This completes the proof. \( \square \)

Denote the set of strictly increasing sequences of \( k \) elements from \( 1, \ldots, n \), by

\[ Q_{k,n} = \{ i = (i_1, \ldots, i_k) : 1 \leq i_1 < \cdots < i_k \leq n \}. \]

For any sets \( I \subset \{1, 2, \ldots, m\} \), \( J \subset \{1, 2, \ldots, n\} \), let \( A_I, A_{s,J}, A_{J,J} \) denote the submatrices of \( A \) lying in rows
indexed by $I$, in columns indexed by $J$, and in their intersection, respectively. For $A \in \mathbb{R}^{m \times n}$, let
\[
I(A) := \{ I \in Q_{r,m} : \text{rank } A_{I*} = r \}, \\
J(A) := \{ J \in Q_{r,n} : \text{rank } A_{*J} = r \}.
\]

For $I \subset \{1,2, \ldots, m\}$ define the subspace $\mathbb{R}^m_I$ by
\[
\mathbb{R}^m_I := \{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i = 0 \text{ if } i \not\in I \}.
\]

An interesting interpretation of the determinants of maximal nonsingular submatrices is the following

**Corollary 2** Let $A \in \mathbb{R}^{m \times n}, \ I \in Q_{r,m}$. Then
\[
\cos\{R(A), \mathbb{R}^m_I\} = \frac{|\det A_{IJ}|}{\text{vol } A_{*J}} \tag{4.7}
\]
for any $J \in J(A)$.

**Proof.** Let $I = \{i_1,i_2, \ldots, i_r\}, \ B = (e_{i_1}, \ldots, e_{i_r})$, and for any $J \in J(A)$ let $C = A_{*J}$. Then
\[
R(B) = \mathbb{R}^m_I, \ R(C) = R(A), \ \text{and } B^T C = A_{IJ}.
\]
By Theorem 5
\[
\cos\{R(A), \mathbb{R}^m_I\} = \cos\{R(B), R(C)\} = \frac{|\det A_{IJ}|}{\text{vol } A_{*J}}.
\]

Note that for any $I \in Q_{r,m}$, the ratio $|\det A_{IJ}|/\text{vol } A_{*J}$ is independent of the choice of $J \in J(A)$.

**Example 1** Let
\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \tag{4.8}
\]
$I = \{1, 2\}, \ J = \{1, 2\}$. Then $A$ is of rank 2, and
\[
|\det A_{IJ}| = 3, \ \text{vol } A_{*J} = \sqrt{3^2 + 6^2 + 3^2} = 3\sqrt{6},
\]
and, by (4.7),
\[
\cos\{R(A), \mathbb{R}^3_{\{1,2\}}\} = \frac{1}{\sqrt{6}}.
\]

**Corollary 3** Let $L \subset \mathbb{R}^m$ be a subspace of dimension $r$. Then
\[
\sum_{I \in Q_{r,m}} \cos^2\{L, \mathbb{R}_I^m\} = 1. \tag{4.9}
\]

**Proof.** Follows from (4.7) since
\[
\text{vol}^2 A_{*J} = \sum_{I \in I(A)} \det^2 A_{IJ}. \tag{4.10}
\]

The corresponding result for the complementary orthogonal subspace $L^\perp$ is
\[
\sum_{I \in Q_{r,m}} \cos^2\{L^\perp, \mathbb{R}_I^m\} = 1, \tag{4.11}
\]
where $I^c$ is the complement of $I$ in $\{1,2, \ldots, m\}$. The equivalence of (4.9) and (4.10) is by
\[
\cos\{L, \mathbb{R}^m_I\} = \cos\{L^\perp, \mathbb{R}^m_{I^c}\}, \tag{4.12}
\]
see Theorem 3.

**Example 2** Let $L = R(A)$ for $A$ of (4.8). Then $L^\perp = N(A^T)$ is the line spanned by $(1,-2,1)$. The complement of $I = \{1,2\}$ is $I^c = \{3\}$, and $\mathbb{R}^m_{I^c}$ is the $x_3$-axis. The angle between the lines $L^\perp$ and $\mathbb{R}^m_{I^c}$, $\angle(L^\perp, \mathbb{R}^m_{I^c})$, is given by
\[
\cos\angle(L^\perp, \mathbb{R}^m_{I^c}) = \cos \angle((1,-2,1),(0,0,1)) = \frac{1}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{1}{\sqrt{6}},
\]
in agreement with (4.11) and Example 1.

**Theorem 6** Let $B \in \mathbb{R}^{n \times r}, \ C \in \mathbb{R}^{n \times s}$ and rank $(C) \geq r$. Then
\[
\cos\{R(B), R(C)\} = \cos \angle(C_r(B), C_r(C)), \tag{4.13}
\]
where $\angle(C_r(B), C_r(C))$ is the angle between the vector $C_r(B)$ and the subspace $R(C_r(C))$.

**Proof.** Let the columns of
\[
E = (e_1, \ldots, e_r) \ \text{and } F = (f_1, \ldots, f_t)
\]
be orthonormal bases for $R(B)$ and $R(C)$ respectively, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ be the singular values of $F^T E$. Then by Lemma 1
\[
\cos\{R(B), R(C)\} = \sigma_1 \cdots \sigma_r.
\]

On the other hand, the columns of $C_r(E)$ and $C_r(F)$ form orthonormal bases for $C_r(B)$ and $R(C_r(C))$ respectively. Moreover, the singular value of $C_r(F)^T C_r(E)$ ($= C_r(F^T E)$) is
\[
\sigma_1 \cdots \sigma_r.
\]

By the same lemma,
\[
\cos \angle(C_r(B), C_r(C)) = \sigma_1 \cdots \sigma_r.
\]

We now use the above results (the equalized Hadamard and Cauchy-Schwarz inequalities) to equalize a determinantal inequality of Thompson.
Theorem 7 \footnote{We proved this originally for $k = 2$, and we thank the referee for suggesting extension to $k > 2$.} Let $A \in \mathbb{R}^{kr \times kr}$ be partitioned as
\[ A = (B_1, B_2, \cdots, B_k), \quad \text{where} \quad B_i \in \mathbb{R}^{kr \times kr}. \]
Then
\[
\det \begin{pmatrix}
B_1^T B_1 & \cdots & B_1^T B_k \\
\vdots & \ddots & \vdots \\
B_k^T B_1 & \cdots & B_k^T B_k
\end{pmatrix} = \det \begin{pmatrix}
\det(B_1^T B_1) & \cdots & \det(B_1^T B_k) \\
\vdots & \ddots & \vdots \\
\det(B_k^T B_1) & \cdots & \det(B_k^T B_k)
\end{pmatrix} \times \prod_{i=1}^{k-1} \sin^2 \{R(B_i), R(B_{i+1}, \cdots, B_k)\} \left(1 - \cos^2 \theta_i \right)^{-1} \leq 1.
\]
(4.13)

Proof. We prove the case $k = 2$.
\[
\det \begin{pmatrix}
B_1^T B_1 & B_1^T B_2 \\
B_2^T B_1 & B_2^T B_2
\end{pmatrix} = \det A^T A = \det \tilde{A} = \cos \angle(C_r(B_1), C_r(B_2)) \cdot \prod_{i=1}^{k-1} \sin^2 \{R(B_i), R(B_{i+1}, \cdots, B_k)\} \left(1 - \cos^2 \theta_i \right)^{-1} \leq 1.
\]
(4.14)

The last equality follows from Theorem 6. If $\theta_1, \cdots, \theta_k$ are the principal angles between two subspaces $L$ and $M$, then we always have
\[
\sin^2 \{L, M\} \leq \frac{\prod_{i=1}^{k} \left(1 - \cos^2 \theta_i \right)}{1 - \prod_{i=1}^{k} \cos^2 \theta_i} \leq 1.
\]
(4.15)

Thus from (4.14) and (4.15), we have
\[
\prod_{i=1}^{k-1} \sin^2 \{R(B_i), R(B_{i+1}, \cdots, B_k)\} \leq \prod_{i=1}^{k-1} \sin^2 \{R(B_i), R(B_{i+1}, \cdots, B_k)\} \leq 1.
\]
(4.16)

Therefore Theorem 7 implies the following inequalities of Davis [6], Everitt [7] ($k = 2$) and Thompson [13] ($k \geq 2$).

Corollary 4 Let a p.s.d. matrix $A$ of order $kr$ be partitioned as
\[ A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}, \quad \text{where} \quad A_{ij} \in \mathbb{R}^{r \times r}. \]
(4.17)

Then
\[
\det \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix} \leq \det \begin{pmatrix} \det A_{11} & \cdots & \det A_{1k} \\ \vdots & \ddots & \vdots \\ \det A_{k1} & \cdots & \det A_{kk} \end{pmatrix} \cdot \prod_{i=1}^{k-1} \sin^2 \{R(B_i), R(B_{i+1}, \cdots, B_k)\} \left(1 - \cos^2 \theta_i \right)^{-1} \leq 1.
\]
(4.18)

A matrix $A \in \mathbb{R}^{n \times n}_{+}$ is called range-Hermitian (or EP matrix, EP<sub>r</sub> matrix) if
\[ R(A^T) = R(A). \]
(4.19)

Note that the class of range-Hermitian matrices includes normal matrices and nonsingular matrices. A good reference on range-Hermitian matrices is [12].

A characterization of range-Hermitian matrices follows.

Theorem 8 Let $A \in \mathbb{R}_r^{n \times n}$. Then $A$ is range-Hermitian if and only if $A$ has $r$ nonzero eigenvalues $\lambda_1, \cdots, \lambda_r$, and vol $A = \prod_{i=1}^{r} |\lambda_i|$. Proof. Let $A = CB^T$ be a full-rank factorization of $A$. Since
\[ R(A) = R(C), \quad R(A^T) = R(C), \quad R(A) = R(A) \quad \text{if and only if} \quad \text{vol} \{R(B), R(C)\} = 1.
\]
By Theorem 6
\[ \cos \angle(C_r(B), C_r(C)) = \frac{|C_r(B)^T C_r(C)|}{\|C_r(B)\|_2 \|C_r(C)\|_2} = \]
5
\[ = \cos\{R(B), \ R(C)\} = 1. \]

And using the fact, \[2\],

\[ \text{vol} \ A = \text{vol} \ B \ \text{vol} \ C. \]

Then

\[ \frac{\left| \sum \det A_{ij} \right|}{\text{vol} \ A} = 1, \]

where the summation is over all \( r \times r \) principal submatrices of \( A \). And finally note that the \( r \)th elementary symmetric function of the eigenvalues of \( A \) is the sum of all the \( r \times r \) principal minors of \( A \).

**Example 3** The matrix \( A \) of (4.8) is range-Hermitian. Its nonzero eigenvalues are

\[ \lambda_1 = 15 + \frac{3\sqrt{33}}{2}, \quad \lambda_2 = 15 - \frac{3\sqrt{33}}{2}, \]

and its nonzero singular values are, correct to 6 decimals,

\[ \sigma_1 = 16.848103, \quad \sigma_2 = 1.068369. \]

Therefore \( \text{vol} \ A = \sigma_1 \sigma_2 = |\lambda_1| |\lambda_2| = 18. \)

To put Theorem 8 in perspective recall that for any square matrix \( A \in \mathbb{R}^{n \times n} \) with singular values \( \sigma_1 \geq \cdots \geq \sigma_n \geq 0 \), and eigenvalues \( |\lambda_1| \geq \cdots \geq |\lambda_n| \),

\[ \prod_{i=1}^{k} |\lambda_i| \leq \prod_{i=1}^{k} \sigma_i, \quad k = 1, \ldots, n, \tag{4.20} \]

with equality for \( k = n \), [10, Theorem 3.3.2]. For \( k = n \) the common value in (4.20) is nonzero iff \( A \) is nonsingular, in which case it is range-Hermitian. Therefore, the equality in (4.20) for \( k = r = n \) is a special case of Theorem 8 which states that, for range-Hermitian matrices of rank \( r \),

\[ \prod_{i=1}^{r} |\lambda_i| = \prod_{i=1}^{r} \sigma_i. \tag{4.21} \]

Conversely, Theorem 8 follows from the nonsingular case, since a matrix \( A \) is range-Hermitian iff

\[ A = U \begin{pmatrix} B & O \\ O & O \end{pmatrix} U^T, \quad U \text{ orthogonal}, \ B \text{ nonsingular}, \]

see [4, Theorem 4.3.1]. Written analogously to (4.20), a characterization of normal matrices is, [10, Problem 3.3.14]

\[ \prod_{i=1}^{k} |\lambda_i| = \prod_{i=1}^{k} \sigma_i, \quad k = 1, \ldots, n. \tag{4.22} \]

**References**


