The Matrix Volume

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Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and consider the minimum norm least squares solution (MNLSS) $x^*$ of

$$Ax = b$$

Notation:

$$\overline{1,n} := \{1, 2, \ldots, n-1, n\}$$

$$M(A) := \{IJ \in \overline{1,m} \times \overline{1,n} : A_{IJ} \in \mathbb{R}^{r \times r}\}$$

For any $IJ \in M(A)$, the solution $x_{IJ}$ of

$$A_{IJ}x = b_I$$

is called a basic solution. It is $x_{IJ} = A_{IJ}^{-1}b_I$.

The MNLSS, $x^* = A_{IJ}^\dagger b$, is a convex combination of the $x_{IJ}$'s

$$x^* = \sum_{IJ \in M(A)} \lambda_{IJ} A_{IJ}^{-1}b_I$$

summing over all nonsingular $r \times r$ submatrices $A_{IJ}$ of $A$. 
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The MNLSS, $x^* = A^{\dagger}b$, is a convex combination of the $x_{IJ}$'s

$$x^* = \sum_{IJ \in M(A)} \lambda_{IJ} \widehat{A_{IJ}^{-1}b_I}$$

summing over all nonsingular $r \times r$ submatrices $A_{IJ}$ of $A$. 
The convex weights $\lambda_{IJ}$ are proportional to $\det^2 A_{IJ}$

$$
\lambda_{IJ} = \frac{\det^2 A_{IJ}}{\sum_{KL \in M(A)} \det^2 A_{KL}}
$$

**Theorem**

If $A \in \mathbb{R}^{m \times n}$ then

$$
A^\dagger = \sum_{IJ \in M(A)} \lambda_{IJ} \widehat{A_{IJ}}^{-1}
$$

with weights $\lambda_{IJ}$ as above.

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If $A \in \mathbb{R}^{m \times n}$ then

$$A^\dagger = \sum_{IJ \in M(A)} \lambda_{IJ} \hat{A}_{IJ}^{-1}$$

with weights $\lambda_{IJ}$ as above.

Let $A \in \mathbb{R}^{m \times n}$. The volume of $A$, $\text{vol}(A)$, is defined as zero if $r = 0$, and otherwise,

$$\text{vol}(A) := \sqrt{\sum \det^2 A_{IJ}}$$

summing over all $r \times r$ nonsingular submatrices $A_{IJ}$ of $A$.

Equivalently, if $r > 0$,

$$\text{vol}(A) := \prod_{i=1}^{r} \sigma_i$$

the product of the singular values of $A$.

Given $A \in \mathbb{R}^{m \times n}$, every unit cube in $R(A^T)$ is mapped by $A$ into a parallelepiped in $R(A)$ of volume $\text{vol}(A)$. 

Representations
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Given $A \in \mathbb{R}^{m \times n}$, every unit cube in $R(A^T)$ is mapped by $A$ into a parallelepiped in $R(A)$ of volume $\text{vol}(A)$.
Volume as determinant

If \( \{ \mathbf{w}_1, \cdots, \mathbf{w}_k \} \subset \mathbb{R}^n, \ k \leq n, \) the volume of the parallelepiped \( \mathbf{\diamond} \{ \mathbf{w}_1, \cdots, \mathbf{w}_k \} \) is given by the Gram determinant,

\[
\text{vol}^2(\mathbf{\diamond} \{ \mathbf{w}_1, \cdots, \mathbf{w}_k \}) = \det W^T W, \quad W = (\mathbf{w}_1, \cdots, \mathbf{w}_k)
\]

Change of variables formula in integration

\[
\int_V f(\mathbf{v}) \, d\mathbf{v} = \int_U (f \circ \phi)(\mathbf{u}) \left| \det J_\phi(\mathbf{u}) \right| \, d\mathbf{u}
\]

\( U, V \) are sets in \( \mathbb{R}^n, \)
\( \phi : U \rightarrow V \) a sufficiently well-behaved function,
\( f : V \rightarrow \mathbb{R} \) integrable on \( V, \)
\( d\mathbf{x} \) denotes the volume element \( |d\mathbf{x}_1 \wedge d\mathbf{x}_2 \wedge \cdots \wedge d\mathbf{x}_n|, \)
\( J_\phi \) is the Jacobi matrix (or Jacobian)

\[
J_\phi := \left( \frac{\partial \phi_i}{\partial u_j} \right), \quad \text{also denoted} \quad \frac{\partial (v_1, v_2, \cdots, v_n)}{\partial (u_1, u_2, \cdots, u_n)},
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representing the derivative of \( \phi. \)
Volume as determinant

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  representing the derivative of \( \phi \).
Given

\( A \in \mathbb{R}_r^{m \times n}, \ r > 0, \)

\( U \in \mathbb{R}^{m \times (m-r)}, \) columns of \( U = \text{o.n. basis of } N(A^T), \)

\( V \in \mathbb{R}^{n \times (n-r)}, \) columns of \( V = \text{o.n. basis of } N(A), \)

the bordered matrix

\[
B(A) = \begin{pmatrix}
    A & U \\
    V^T & O
\end{pmatrix} \in \mathbb{R}^{(m+n-r) \times (m+n-r)}
\]

is nonsingular, and for any \( b \in \mathbb{R}^m \) the minimum-norm least-squares solution (MNLSS) of \( Ax = b, \)

\[
x = A^\dagger b,
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is the solution \( x \) of

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\end{pmatrix}
\begin{pmatrix}
    x \\
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\end{pmatrix} = \begin{pmatrix}
    b \\
    0
\end{pmatrix},
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with residual \( P_{N(A^T)} b = Uu. \)
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Theorem

If
\[ A \in \mathbb{R}^{m \times n}, \ r > 0, \]
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then

\[
\text{vol}^2(A) = \det^2 \begin{pmatrix} A & U \\ V^T & O \end{pmatrix} = \det \left( A A^T + U U^T \right) = \det \left( A^T A + V V^T \right)
\]

Remark: A signed volume can be defined by

\[
\text{svol} (A) := \det \begin{pmatrix} A & U \\ V^T & O \end{pmatrix}
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the sign determined by the matrices \( U, V \).
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Full rank factorizations

- **Notation**

\[ Q_{r,n} = \{ I = \{ i_1, \cdots, i_r \} \in \overline{1,n} : 1 \leq i_1 < i_2 < \cdots < i_r \leq n \} \]

\[ I(A) = \{ I \in Q_{r,m} : \text{rank } A_{I*} = r \} \]

\[ J(A) = \{ J \in Q_{r,n} : \text{rank } A_{*J} = r \} \]

\[ M(A) = \{ IJ \in Q_{r,m} \times Q_{r,n} : \text{rank } A_{IJ} = r \} \]

- A full rank factorization (FRF) of \( A \in \mathbb{R}_{r}^{m \times n} \) is

\[ A = CR, \ C \in \mathbb{R}_{r}^{m \times r}, \ R \in \mathbb{R}_{r}^{r \times n} \]

- If \( A = CR \) is a FRF,

\[ I(A) = I(C) \]

\[ J(A) = J(R) \]

\[ M(A) = I(A) \times J(A) \]

\[(I \in I(A), J \in J(A) \implies A_{IJ} = C_{I*}R_{*J} \text{ is nonsingular})\]
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Definition

The volume of $A \in \mathbb{R}^{m \times n}$ is

$$\text{vol}(A) := \sqrt{\sum_{IJ \in M(A)} \det^2 A_{IJ}}$$

Theorem

If $A \in \mathbb{R}^{m \times n}$, $r > 0$, and $A = CR$ is any FRF, then

$$\text{vol}(A) = \text{vol}(C) \text{vol}(R)$$

Proof.

$$\text{vol}^2(A) = \sum_{IJ \in M(A)} \det^2 A_{IJ} = \sum_{IJ \in M(A)} \det^2 C_{I*} R_{*J}$$

$$= \sum_{IJ \in M(A)} \det^2 C_{I*} \det^2 R_{*J}$$

$$= \left( \sum_{I \in I(A)} \det^2 C_{I*} \right) \left( \sum_{J \in J(A)} \det^2 R_{*J} \right)$$

$$= \text{vol}^2(C) \text{vol}^2(R).$$
## Definition

The **volume** of $A \in \mathbb{R}^{m \times n}$ is 

$$\text{vol}(A) := \sqrt{\sum_{IJ \in M(A)} \det^2 A_{IJ}}$$

## Theorem

*If* $A \in \mathbb{R}^{m \times n}$, $r > 0$, *and* $A = CR$ *is any FRF, then*

$$\text{vol}(A) = \text{vol}(C) \text{vol}(R)$$

## Proof.

$$\text{vol}^2(A) = \sum_{IJ \in M(A)} \det^2 A_{IJ} = \sum_{IJ \in M(A)} \det^2 C_{I*} R_{*J}$$

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$$= \text{vol}^2(C) \text{vol}^2(R).$$
Theorem

If \( A \in \mathbb{R}^{m \times n} \) with singular values \( \{\sigma_i : i \in \overline{1,r}\} \), then

\[
\text{vol}(A) = \prod_{i=1}^{r} \sigma_i
\]

Proof.

Given a singular value decomposition (SVD) of \( A \),

\[
A = U\Sigma V^T
\]

with \( \Sigma = \text{diag}(\sigma_i) \in \mathbb{R}^{r \times r} \), \( U^T U = V^T V = I_r \)

\[
\because \quad \text{vol}(A) = \text{vol}(U) \text{vol}(\Sigma V^T)
\]

\[
= \text{vol}(U) \text{vol}(\Sigma) \text{vol}(V^T)
\]

\[
= \prod_{i=1}^{r} \sigma_i
\]
Singular values

**Theorem**

If \( A \in \mathbb{R}^{m \times n} \) with singular values \( \{ \sigma_i : i \in 1,r \} \), then

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\text{vol}(A) = \prod_{i=1}^{r} \sigma_i
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Given a singular value decomposition (SVD) of \( A \),

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A = U\Sigma V^T
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with \( \Sigma = \text{diag}(\sigma_i) \in \mathbb{R}^{r \times r} \), \( U^T U = V^T V = I_r \)

\[
\therefore \text{vol}(A) = \text{vol}(U) \text{vol}(\Sigma V^T) = \text{vol}(U) \text{vol}(\Sigma) \text{vol}(V^T)
\]

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\]
Let $A \in \mathbb{R}^{m \times n}$, $r > 0$, with SVD,

$$A = U \Sigma V^T,$$

where $\Sigma = \text{diag}(\sigma_i) \in \mathbb{R}^{r \times r}$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$

$U = (u_1, \ldots, u_r)$, o.n. basis of $R(A)$,

$V = (v_1, \ldots, v_r)$, o.n. basis of $R(A^T)$,

$$A v_i = \sigma_i u_i, \ i \in 1, r.$$ 

$A$ maps the $r$–dimensional unit cube $\square\{v_1, \ldots, v_r\}$ into the cube $\square\{\sigma_1 u_1, \ldots, \sigma_r u_r\}$, with volume

$$\prod_{i=1}^{r} \sigma_i = \text{vol} (A)$$
Let \( A \in \mathbb{R}^{m \times n} \), \( r > 0 \), with SVD,

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where \( \Sigma = \text{diag}(\sigma_i) \in \mathbb{R}^{r \times r} \), \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \)

\[
U = (u_1, \ldots, u_r), \text{ o.n. basis of } R(A),
\]

\[
V = (v_1, \ldots, v_r), \text{ o.n. basis of } R(A^T),
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A v_i = \sigma_i u_i, \quad i \in \overline{1, r}.
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\( A \) maps the \( r \)-dimensional unit cube \( \square \{ v_1, \ldots, v_r \} \) into the cube \( \square \{ \sigma_1 u_1, \ldots, \sigma_r u_r \} \), with volume

\[
\prod_{i=1}^{r} \sigma_i = \text{vol}(A)
\]
**Theorem**

If $A \in \mathbb{R}^{m \times n}$, $r > 0$, then $A$ maps any unit cube in $R(A^T)$ into a parallelepiped in $R(A)$ of volume $\text{vol}(A)$.

**Proof.**

Consider the SVD $A = U \Sigma V^T$, $U = (u_1, \cdots, u_r)$, $V = (v_1, \cdots, v_r)$

$$Av_i = \sigma_i u_i, \; i \in \overline{1,r}, \text{ or } A \square \{V\} = \square \{U \Sigma\}$$

Any unit cube in $R(A^T)$ is $\square \{VQ\}$, $Q$ orthogonal.

$$A \square \{VQ\} = \diamond \{U \Sigma V^T VQ\} = \diamond \{U \Sigma Q\}$$

$$\therefore \text{vol}^2(\diamond \{U \Sigma Q\}) = \det (U \Sigma Q)^T (U \Sigma Q) = \det (Q^T \Sigma^2 Q)$$

$$= \det \Sigma^2$$

$$= \text{vol}^2(A)$$
Theorem
If \( A \in \mathbb{R}^{m \times n} \), \( r > 0 \), then \( A \) maps any unit cube in \( \mathbb{R}(A^T) \) into a parallelepiped in \( \mathbb{R}(A) \) of volume \( \text{vol}(A) \).

Proof.
Consider the SVD \( A = U\Sigma V^T \), \( U = (u_1, \cdots, u_r) \), \( V = (v_1, \cdots, v_r) \)

\[
A v_i = \sigma_i u_i, \quad i \in \overline{1,r}, \quad \text{or} \quad A \square \{V\} = \square \{U \Sigma\}
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Any unit cube in \( \mathbb{R}(A^T) \) is \( \square \{VQ\} \), \( Q \) orthogonal.

\[
A \square \{VQ\} = \diamond \{U\Sigma V^T VQ\} = \diamond \{U\Sigma Q\}
\]

\[
\therefore \quad \text{vol}^2(\diamond \{U\Sigma Q\}) = \det(U\Sigma Q)^T (U\Sigma Q) = \det(Q^T \Sigma^2 Q)
\]

\[
= \det \Sigma^2
\]

\[
= \text{vol}^2(A)
\]
Exterior products

- $V$ = finite–dimensional linear space over field $F$
- An exterior product is an operation $\wedge: V \times V \to V$ that is
  
  (a) anti–commutative, $u \wedge v = -v \wedge u$
  
  (b) $(\lambda \cdot u) \wedge v = \lambda \cdot (u \wedge v)$
  
  (c) distributive in both variables:

  \[
  (u + v) \wedge w = u \wedge w + v \wedge w \quad w \wedge (u + v) = w \wedge u + w \wedge v
  \]

  for all $u, v, w \in V, \lambda \in F$.

- $\wedge^k V$ = the $k_{th}$–exterior space over $V$, spanned by all exterior products $x_1 \wedge \cdots \wedge x_k$ of $k$ elements in $V$

  \[
  \dim \wedge^k \mathbb{R}^n = \binom{n}{k}
  \]
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\]
Compound matrices

- $V, U = \text{finite–dimensional linear spaces}$
- $L(V, U) = \text{the linear transformations: } V \rightarrow U$
- Linear transformations $\longleftrightarrow$ their matrix representations.

For $V = \mathbb{R}^n$, $U = \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, $r > 0$, $k \in \{1, r\}$, the $k$–compound matrix of $A$ is (the matrix representation of) the linear transformation $C_k(A) \in L(\wedge^k V, \wedge^k U)$, defined by

$$C_k(A) (x_1 \wedge \cdots \wedge x_k) := A x_1 \wedge \cdots \wedge A x_k, \quad \forall \{x_i\} \subset V,$$

- The compound matrix $C_k(A)$ is $\binom{n}{k} \times \binom{m}{k}$ of rank $\binom{r}{k}$.
- In particular, $C_r(A)$ is $\binom{n}{r} \times \binom{m}{r}$ of rank 1.
Compound matrices

- \( V, U = \) finite–dimensional linear spaces
  \( L(V, U) = \) the linear transformations: \( V \rightarrow U \)
  Linear transformations \( \leftrightarrow \) their matrix representations.

- For \( V = \mathbb{R}^n, U = \mathbb{R}^m, \)
  \( A \in \mathbb{R}^{m \times n}, r > 0, k \in 1, r, \)
  the \textit{k–compound matrix} of \( A \) is (the matrix representation of) the linear transformation \( C_k(A) \in L(\wedge^k V, \wedge^k U), \) defined by

\[
C_k(A) (x_1 \wedge \cdots \wedge x_k) := Ax_1 \wedge \cdots \wedge Ax_k, \quad \forall \{x_i\} \subset V,
\]

- The compound matrix \( C_k(A) \) is \( \binom{n}{k} \times \binom{m}{k} \) of rank \( \binom{r}{k} \).
  In particular, \( C_r(A) \) is \( \binom{n}{r} \times \binom{m}{r} \) of rank 1.
Compound matrices

- $V, U =$ finite–dimensional linear spaces
  $L(V, U) =$ the linear transformations: $V \rightarrow U$
  Linear transformations $\leftrightarrow$ their matrix representations.

- For $V = \mathbb{R}^n$, $U = \mathbb{R}^m$, $A \in \mathbb{R}_r^{m \times n}$, $r > 0$, $k \in 1, r$,
  the $k$–compound matrix of $A$ is (the matrix representation of) the linear transformation $C_k(A) \in L(\wedge^k V, \wedge^k U)$, defined by
  
  $C_k(A) (x_1 \wedge \cdots \wedge x_k) := A x_1 \wedge \cdots \wedge A x_k$, $\forall \{x_i\} \subset V$.

- The compound matrix $C_k(A)$ is $\binom{n}{k} \times \binom{m}{k}$ of rank $\binom{r}{k}$.
  In particular, $C_r(A)$ is $\binom{n}{r} \times \binom{m}{r}$ of rank 1.
To any subspace $W \subset \mathbb{R}^n$, $\dim W = r$, there corresponds a $1$–dimensional subspace $\wedge W \subset \wedge \mathbb{R}^n$, spanned by
\[ w^\wedge = w_1 \wedge \cdots \wedge w_r \]
where $\{w_1, \cdots, w_r\}$ is any basis of $W$.

The $\binom{n}{r}$ components of $w^\wedge$ (determined up to a multiplicative constant) are the Plücker coordinates of $W$.

Let $A \in \mathbb{R}^{m \times n}$ and $A = U \Sigma V^T$ its SVD, $U = (u_1, \cdots, u_m)$, $V = (v_1, \cdots, v_n)$.

The Plücker coordinates of $R(A)$ and $R(A^T)$ are
\[ u^\wedge = u_1 \wedge \cdots \wedge u_r, \quad v^\wedge = v_1 \wedge \cdots \wedge v_r \]
\[ C_r(A) \ v^\wedge = (\text{vol}(A)) \ u^\wedge, \quad C_r(A^\dagger) \ u^\wedge = \left(1/\text{vol}(A)\right) \ v^\wedge \]
\[ C_r(A^\dagger) = (C_r(A))^\dagger, \quad \text{vol}(A^\dagger) = 1/\text{vol}(A) \]
Plücker coordinates

To any subspace \( W \subset \mathbb{R}^n \), \( \dim W = r \), there corresponds a 1–dimensional subspace \( \bigwedge^r W \subset \bigwedge^r \mathbb{R}^n \), spanned by

\[
\mathbf{w}^\wedge = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r
\]

where \( \{\mathbf{w}_1, \cdots, \mathbf{w}_r\} \) is any basis of \( W \).

The \( \binom{n}{r} \) components of \( \mathbf{w}^\wedge \) (determined up to a multiplicative constant) are the Plücker coordinates of \( W \).

Let \( A \in \mathbb{R}^{m \times n} \) and \( A = U \Sigma V^T \) its SVD,

\[
U = (\mathbf{u}_1, \cdots, \mathbf{u}_m), \quad V = (\mathbf{v}_1, \cdots, \mathbf{v}_n)
\]

The Plücker coordinates of \( R(A) \) and \( R(A^T) \) are

\[
\mathbf{u}^\wedge = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r, \quad \mathbf{v}^\wedge = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r
\]

\[
C_r(A) \mathbf{v}^\wedge = (\text{vol}(A)) \mathbf{u}^\wedge, \quad C_r(A^\dagger) \mathbf{u}^\wedge = (1/\text{vol}(A)) \mathbf{v}^\wedge
\]

\[
C_r(A^\dagger) = (C_r(A))^\dagger, \quad \text{vol}(A^\dagger) = 1/\text{vol}(A)
\]
Plücker coordinates

- To any subspace $W \subset \mathbb{R}^n$, $\dim W = r$, there corresponds a $1$–dimensional subspace $\mathcal{W} \subset \bigwedge^r \mathbb{R}^n$, spanned by
  $$w^\wedge = w_1 \wedge \cdots \wedge w_r$$

  where $\{w_1, \cdots, w_r\}$ is any basis of $W$.

- The $\binom{n}{r}$ components of $w^\wedge$ (determined up to a multiplicative constant) are the Plücker coordinates of $W$.

- Let $A \in \mathbb{R}^{m \times n}$ and $A = U\Sigma V^T$ its SVD,
  $$U = (u_1, \cdots, u_m), \quad V = (v_1, \cdots, v_n)$$

  The Plücker coordinates of $R(A)$ and $R(A^T)$ are
  $$u^\wedge = u_1 \wedge \cdots \wedge u_r, \quad v^\wedge = v_1 \wedge \cdots \wedge v_r$$

  $C_r(A) v^\wedge = (\text{vol}(A)) u^\wedge, \quad C_r(A^\dagger) u^\wedge = (1/\text{vol}(A)) v^\wedge$

  $C_r(A^\dagger) = (C_r(A))^\dagger, \quad \text{vol}(A^\dagger) = 1/\text{vol}(A)$
Plücker coordinates

To any subspace \( W \subset \mathbb{R}^n \), \( \dim W = r \), there corresponds a 1–dimensional subspace \( \wedge^r W \subset \wedge \mathbb{R}^n \), spanned by

\[
\mathbf{w}^\wedge = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r
\]

where \( \{ \mathbf{w}_1, \cdots, \mathbf{w}_r \} \) is any basis of \( W \).

The \( \binom{n}{r} \) components of \( \mathbf{w}^\wedge \) (determined up to a multiplicative constant) are the Plücker coordinates of \( W \).

Let \( A \in \mathbb{R}^{m \times n} \) and \( A = U \Sigma V^T \) its SVD,

\[
U = (\mathbf{u}_1, \cdots, \mathbf{u}_m), \quad V = (\mathbf{v}_1, \cdots, \mathbf{v}_n)
\]

The Plücker coordinates of \( R(A) \) and \( R(A^T) \) are

\[
\mathbf{u}^\wedge = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r, \quad \mathbf{v}^\wedge = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r
\]

\[
C_r(A) \mathbf{v}^\wedge = (\text{vol}(A)) \mathbf{u}^\wedge, \quad C_r(A^\dagger) \mathbf{u}^\wedge = (1/\text{vol}(A)) \mathbf{v}^\wedge
\]

\[
C_r(A^\dagger) = (C_r(A))^\dagger, \quad \text{vol}(A^\dagger) = 1/\text{vol}(A)
\]
Lemma

Let $r > 0$, $A \in \mathbb{R}_r^{m \times n}$, $C \in \mathbb{R}_r^{m \times r}$ have columns $c^i$, and $R \in \mathbb{R}_r^{r \times n}$ have rows $r^i$.

Then:

(a) $C_r(R) \ (r^1 \wedge \cdots \wedge r^r) = \text{vol}^2(R)$

(b) $C_r(C^T) \ (c^1 \wedge \cdots \wedge c^r) = \text{vol}^2(C)$

(c) If $A = CR$ is a FRF, then

$C_r(A) = (c^1 \wedge \cdots \wedge c^r) \ (r^1 \wedge \cdots \wedge r^r)$

and the volume of $A$ is given by,

$\text{vol}^2(A) = (c^1 \wedge \cdots \wedge c^r, Ar^1 \wedge \cdots \wedge Ar^r) = \text{vol}^2(C_r(A)) = \text{vol}^2(C_r(C))\text{vol}^2(C_r(R))$
Lemma

Let $r > 0$, $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{m \times r}$ have columns $c_j$, and $R \in \mathbb{R}^{r \times n}$ have rows $r_i$. Then:

(a) \[ C_r(R) \ (r^1 \wedge \cdots \wedge r^r) = \text{vol}^2(R) \]

(b) \[ C_r(C^T) \ (c^1 \wedge \cdots \wedge c^r) = \text{vol}^2(C) \]

(c) If $A = CR$ is a FRF, then

\[ C_r(A) = (c^1 \wedge \cdots \wedge c^r) \ (r^1 \wedge \cdots \wedge r^r) \]

and the volume of $A$ is given by,

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\[ = \text{vol}^2(C_r(A)) = \text{vol}^2(C_r(C))\text{vol}^2(C_r(R)) \]
Outline

1. Motivation
2. Representations
3. Factorizations
4. A multilinear setting
5. Angles
6. Integrals
7. Concentration of measure
8. Probability
9. Application
10. References
Let $L, M$ be subspaces in $\mathbb{R}^n$, $\dim L = \ell \leq \dim M = m$. The principal angles between $L$ and $M$,

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_\ell \leq \frac{\pi}{2}$$

are computed recursively as follows

$$\cos \theta_i = \frac{\langle x_i, y_i \rangle}{\|x_i\| \|y_i\|} = \max \left\{ \frac{\langle x, y \rangle}{\|x\| \|y\|} : x \in L, \quad x \perp x_k, \quad y \in M, \quad y \perp y_k, \quad k \in \overline{1,i-1} \right\}$$

where

$$(x_i, y_i) \in L \times M, \quad i \in \overline{1,\ell}$$

are the corresponding pairs of principal vectors. We also define

$$\sin\{L, M\} := \prod_{i=1}^{\ell} \sin \theta_i, \quad \cos\{L, M\} := \prod_{i=1}^{\ell} \cos \theta_i.$$
Let $L, M$ be subspaces in $\mathbb{R}^n$, $\dim L = \ell \leq \dim M = m$. The principal angles between $L$ and $M$,

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$$\sin\{L,M\} := \prod_{i=1}^{\ell} \sin \theta_i, \quad \cos\{L,M\} := \prod_{i=1}^{\ell} \cos \theta_i.$$
Hadamard’s inequality

The determinant of $A = (v_1, \cdots, v_n)$ satisfies

$$|\det A| \leq \prod_{i=1}^{n} \|v_i\|$$

with equality if and only if the vectors are orthogonal, or at least one of them is zero.

**Theorem**

Let $A = (A_1, A_2)$, $A_1 \in \mathbb{R}^{n \times n_1}$, $A_2 \in \mathbb{R}^{n \times n_2}$, rank $A = \ell + m$. Then

$$vol_{\ell+m}(A) = vol_{\ell}(A_1) \cdot vol_{m}(A_2) \cdot \sin\{R(A_1), R(A_2)\}.$$ 

**Corollary**

Let $A = (A_1, A_2) \in \mathbb{R}^{n \times n}$, with $A_1 \in \mathbb{R}^{n \times \ell}$, $A_2 \in \mathbb{R}^{n \times m}$. Then

$$|\det(A)| = vol_{\ell}(A_1) \cdot vol_{m}(A_2) \cdot \sin\{R(A_1), R(A_2)\}.$$
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**Theorem**

Let $A = (A_1, A_2), A_1 \in \mathbb{R}_{\ell}^{n \times n_1}, A_2 \in \mathbb{R}_{m}^{n \times n_2}, \text{rank } A = \ell + m$. Then

$$\text{vol}_{\ell+m}(A) = \text{vol}_{\ell}(A_1) \text{vol}_m(A_2) \sin\{R(A_1), R(A_2)\}.$$  

**Corollary**

Let $A = (A_1, A_2) \in \mathbb{R}^{n \times n}$, with $A_1 \in \mathbb{R}^{n \times \ell}, A_2 \in \mathbb{R}^{n \times m}$. Then

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**Theorem**

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Let $A = (A_1, A_2) \in \mathbb{R}^{n \times n}$, with $A_1 \in \mathbb{R}^{n \times \ell}$, $A_2 \in \mathbb{R}^{n \times m}$. Then

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Orthogonal projections

\[ V := \{v_1, \cdots, v_k\} \text{ set of vectors in } \mathbb{R}^n \]

\[ S := \text{span}\{V\}, \text{ the subspace spanned by } V \]

\[ \dim S = r \]

Any \( w \in \mathbb{R}^n \) can be written as \( w = w_S + w_{S\perp} \).

**Theorem**

*Let \( V, S \) be as above. Then, for any \( w \in \mathbb{R}^n \),*

\[ \|w_{S\perp}\| = \frac{\text{vol}_{r+1}(v_1, \cdots, v_k, w)}{\text{vol}_r(v_1, \cdots, v_k)} \]

*where \( (v_1, \cdots, v_k) \) is the matrix with \( v_j \) as columns.*

**Proof.**

If \( w \in S \), \( 0 = 0 \). If \( w \notin S \) then

\[ \text{vol}_{r+1}(v_1, \cdots, v_k, w) = \text{vol}_r(v_1, \cdots, v_k) \text{vol}_1(w) \sin\{S, w\} \]
Orthogonal projections

$V := \{v_1, \cdots, v_k\}$ set of vectors in $\mathbb{R}^n$
$S := \text{span} \{V\}$, the subspace spanned by $V$
$\dim S = r$
Any $w \in \mathbb{R}^n$ can be written as $w = w_S + w_{S^\perp}$.

**Theorem**

Let $V, S$ be as above. Then, for any $w \in \mathbb{R}^n$,

$$\|w_{S^\perp}\| = \frac{\text{vol}_{r+1}(v_1, \cdots, v_k, w)}{\text{vol}_r(v_1, \cdots, v_k)} ,$$

where $(v_1, \cdots, v_k)$ is the matrix with $v_j$ as columns.

**Proof.**

If $w \in S$, $0 = 0$. If $w \not\in S$ then

$$\text{vol}_{r+1}(v_1, \cdots, v_k, w) = \text{vol}_r(v_1, \cdots, v_k) \text{vol}_1(w) \sin\{S, w\}$$
Almost Cramer

\[ V := \{v_1, \cdots, v_k\} \text{ set of vectors in } \mathbb{R}^n \]
\[ S := \text{span } \{V\}, \text{ the subspace spanned by } V \]
\[ \dim S = r \]

Corollary

Let \( V, S \) be as above. Then, for any \( v, w \in \mathbb{R}^n, v \not\in S \),

\[
\frac{vol_{r+1}(v_1, \cdots, v_k, w)}{vol_{r+1}(v_1, \cdots, v_k, v)} = \frac{\|w_S\|}{\|v_S\|}
\]
Cauchy–Schwarz inequality

- For any two vectors \( u, v \in \mathbb{R}^n \),
  \[
  |\langle u, v \rangle| \leq \|u\| \|v\|,
  \]
  with equality if and only if \( u, v \) are collinear.

- For any \( u, v \) as above,
  \[
  \langle u, v \rangle = \|u\| \|v\| \cos \angle \{u, v\}
  \]

**Theorem**

Let \( B, C \in \mathbb{R}^{n \times r} \). Then

\[
|\det(B^T C)| = \text{vol}(B) \text{vol}(C) \cos \{R(B), R(C)\}.
\]
For any two vectors $u, v \in \mathbb{R}^n$,

$$|\langle u, v \rangle| \leq \|u\|\|v\|,$$

with equality if and only if $u, v$ are collinear.

For any $u, v$ as above,

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---

**Theorem**

Let $B, C \in \mathbb{R}^{n \times r}$. Then

$$|\det(B^TC)| = \text{vol}(B) \text{vol}(C) \cos \{R(B), R(C)\}.$$
**Cauchy–Schwarz inequality**

- For any two vectors \( u, v \in \mathbb{R}^n \),
  \[
  |\langle u, v \rangle| \leq \|u\| \|v\|,
  \]
  with equality if and only if \( u, v \) are collinear.
- For any \( u, v \) as above,
  \[
  \langle u, v \rangle = \|u\| \|v\| \cos \angle \{u, v\}
  \]

**Theorem**

Let \( B, C \in \mathbb{R}^{n \times r} \). Then

\[
|\det(B^T C)| = \text{vol}(B) \text{vol}(C) \cos \{R(B), R(C)\}.
\]
Change of variables

- The change of variables formula is

\[ \int_V f(v) \, dv = \int_U (f \circ \phi)(u) \left| \det J_\phi(u) \right| \, du \]  

(A)

\( U, V \subset \mathbb{R}^n, \)
\( \phi : U \rightarrow V \) is a sufficiently well-behaved function,
\( f \) is integrable on \( V, \)
\( dx \) is the volume element \( |dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|, \) and
\( J_\phi \) is the Jacobi matrix (or Jacobian)

\[ J_\phi := \left( \frac{\partial \phi_i}{\partial u_j} \right), \quad \text{also denoted} \quad \frac{\partial (v_1, v_2, \cdots, v_n)}{\partial (u_1, u_2, \cdots, u_n)}, \]

representing the derivative of \( \phi. \)

- An advantage of (A) is that integration on \( V \) is translated to (perhaps simpler) integration on \( U. \)
The change of variables formula is

\[ \int_V f(v) \, dv = \int_U (f \circ \phi)(u) \, |\det J_\phi(u)| \, du \quad (A) \]

where $U, V \subset \mathbb{R}^n$, $\phi : U \rightarrow V$ is a sufficiently well-behaved function, $f$ is integrable on $V$, $dx$ is the volume element $|dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|$, and $J_\phi$ is the Jacobi matrix (or Jacobian)

\[ J_\phi := \left( \frac{\partial \phi_i}{\partial u_j} \right), \quad \text{also denoted} \quad \frac{\partial (v_1, v_2, \cdots, v_n)}{\partial (u_1, u_2, \cdots, u_n)}, \]

representing the derivative of $\phi$.

An advantage of (A) is that integration on $V$ is translated to (perhaps simpler) integration on $U$. 
The change of variables formula is

\[ \int_{V} f(v) \, dv = \int_{U} (f \circ \phi)(u) \left| \det J_{\phi}(u) \right| \, du \]  

(A)

\( U, V \subset \mathbb{R}^n, \)
\( \phi : U \to V \) is a sufficiently well-behaved function,
\( f \) is integrable on \( V, \)
d\( x \) is the volume element \( |dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|, \) and
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\[
J_{\phi} := \left( \frac{\partial \phi_i}{\partial u_j} \right), \quad \text{also denoted} \quad \frac{\partial (v_1, v_2, \cdots, v_n)}{\partial (u_1, u_2, \cdots, u_n)},
\]

representing the derivative of \( \phi. \)

An advantage of (A) is that integration on \( V \) is translated to (perhaps simpler) integration on \( U. \)
\[ \int_V f(v) \, dv = \int_U (f \circ \phi)(u) \left| \det J_\phi(u) \right| \, du \]  

(A)

- If \( U \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \) and \( n \neq m \), (A) cannot be used.
- If \( J_\phi \) is of full column rank throughout \( U \), then \( \left| \det J_\phi \right| \) can be replaced in (A) by \( \text{vol} J_\phi \)

\[ \int_V f(v) \, dv = \int_U (f \circ \phi)(u) \text{vol} J_\phi(u) \, du. \]  

(B)

- The volume of an \( m \times n \) matrix of rank \( r \) is

\[ \text{vol} A := \sqrt{\sum_{(I,J) \in M(A)} \det^2 A_{IJ}} \]

\( A_{IJ} \) is the submatrix of \( A \) with rows \( I \) and columns \( J \), \( M(A) \) is the index set of \( r \times r \) nonsingular submatrices of \( A \).
- If \( A \) is of full column rank, its volume is

\[ \text{vol} A = \sqrt{\det A^T A} \]

- If \( m = n \) then \( \text{vol} J_\phi = \left| \det J_\phi \right| \), and (B) reduces to (A).
\[
\int_V f(v) \, dv = \int_U (f \circ \phi)(u) \left| \det J_\phi(u) \right| \, du \quad (A)
\]

- If \( U \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \) and \( n \neq m \), (A) cannot be used.
- If \( J_\phi \) is of full column rank throughout \( U \), then \( \left| \det J_\phi \right| \) can be replaced in (A) by \( \text{vol} J_\phi \)

\[
\int_V f(v) \, dv = \int_U (f \circ \phi)(u) \text{vol} J_\phi(u) \, du. \quad (B)
\]

- The **volume** of an \( m \times n \) matrix of rank \( r \) is

\[
\text{vol} A := \sqrt{\sum_{(I,J) \in M(A)} \det^2 A_{IJ}}
\]

\( A_{IJ} \) is the submatrix of \( A \) with rows \( I \) and columns \( J \),
\( M(A) \) is the index set of \( r \times r \) nonsingular submatrices of \( A \).
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\[
\text{vol} A = \sqrt{\det A^T A}
\]

- If \( m = n \) then \( \text{vol} J_\phi = \left| \det J_\phi \right| \), and (B) reduces to (A).
Surface integral in $\mathbb{R}^3$

$$S = \{(x, y, z) : z = g(x, y)\}$$

$$\phi \begin{pmatrix} x \\ y \\ g(x, y) \end{pmatrix} = \begin{pmatrix} x \\ y \\ g(x, y) \end{pmatrix}, \quad J_\phi (x, y) = \frac{\partial (x, y, z)}{\partial (x, y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{pmatrix}$$

$$\text{vol} (J_\phi (x, y)) = \sqrt{1 + g_x^2 + g_y^2}$$

$$\int_V f(x, y, z) \, ds = \int_U f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} \, dx \, dy$$
Surface integral in $\mathbb{R}^3$

$S = \{(x, y, z) : z = g(x, y)\}$

$\phi \begin{pmatrix} x \\ y \\ g(x, y) \end{pmatrix} = \begin{pmatrix} x \\ y \\ g(x, y) \end{pmatrix}$, \quad $J_{\phi}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{pmatrix}$

$\text{vol}(J_{\phi}(x, y)) = \sqrt{1 + g_x^2 + g_y^2}$

$\int_{V} f(x, y, z) \, ds = \int_{U} f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} \, dx \, dy$
Surface integral in $\mathbb{R}^3$

$$S = \{(x, y, z) : z = g(x, y)\}$$

$$\phi \left( \begin{array}{c} x \\ y \\ g(x, y) \end{array} \right) = \left( \begin{array}{c} x \\ y \\ g(x, y) \end{array} \right), \quad J_\phi(x, y) = \frac{\partial(x, y, z)}{\partial(x, y)} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

$$\text{vol}(J_\phi(x, y)) = \sqrt{1 + g_x^2 + g_y^2}$$

$$\int_V \int_U f(x, y, z) ds = \int_U \int_V f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} \; dx \; dy$$
Cylindrical coordinates

Let $S$ be a surface in $\mathbb{R}^3$ represented by $z = z(r, \theta)$ where $\{r, \theta\}$ are polar coordinates, or by the mapping

$$
\phi(r, \theta) \rightarrow (x, y, z) = (r \cos \theta, r \sin \theta, z(r, \theta))
$$

$$
\therefore J_\phi(r, \theta, z) = \frac{\partial(x, y, z)}{\partial(r, \theta)} = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta}
\end{pmatrix}
$$

$$
\therefore \text{vol}(J_\phi) = \sqrt{r^2 + r^2 \left( \frac{\partial z}{\partial r} \right)^2 + \left( \frac{\partial z}{\partial \theta} \right)^2} = r \sqrt{1 + \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2},
$$

An integral over a domain $V \subset S$ is therefore

$$
\int_V f(x, y, z) \, dS =
$$

$$
\int_U f(r \cos \theta, r \sin \theta, z(r, \theta)) r \sqrt{1 + \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2} \, dr \, d\theta.
$$
Cylindrical coordinates

Let \( S \) be a surface in \( \mathbb{R}^3 \) represented by \( z = z(r, \theta) \) where \( \{r, \theta\} \) are polar coordinates, or by the mapping

\[
\phi(r, \theta) \rightarrow (x, y, z) = (r \cos \theta, r \sin \theta, z(r, \theta))
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\therefore J_{\phi}(r, \theta, z) = \frac{\partial (x, y, z)}{\partial (r, \theta)} = \begin{pmatrix}
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\[
\int_V f(x, y, z) \, dS =
\int_U f(r \cos \theta, r \sin \theta, z(r, \theta)) r \sqrt{1 + \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2} \, dr \, d\theta.
\]
Cylindrical coordinates

Let \( S \) be a surface in \( \mathbb{R}^3 \) represented by \( z = z(r, \theta) \) where \( \{r, \theta\} \) are polar coordinates, or by the mapping

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\phi(r, \theta) \mapsto (x, y, z) = (r \cos \theta, r \sin \theta, z(r, \theta))
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\therefore \quad J_\phi(r, \theta, z) = \frac{\partial (x, y, z)}{\partial (r, \theta)} = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta \\
r \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta}
\end{pmatrix}
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\therefore \quad \text{vol}(J_\phi) = \sqrt{r^2 + r^2 \left( \frac{\partial z}{\partial r} \right)^2 + \left( \frac{\partial z}{\partial \theta} \right)^2} = r \sqrt{1 + \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2},
\]

An integral over a domain \( V \subset S \) is therefore

\[
\int_V f(x, y, z) \, dS = \int_U f(r \cos \theta, r \sin \theta, z(r, \theta)) \, r \sqrt{1 + \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2} \, dr \, d\theta.
\]
Let $H(v, p)$ be a (“non–vertical”) hyperplane in $\mathbb{R}^n$

$$H(v, p) := \{ x : \langle v, x \rangle = p \} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} v_i x_i = p \right\}, \quad (v_n \neq 0)$$

$$H(v, p) = \phi(\mathbb{R}^{n-1}), \quad x_n := \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i$$

$$\text{vol} J\phi = \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{v_i}{v_n} \right)^2} = \frac{\|v\|}{|v_n|}$$

The Radon transform $(Rf)(v, p)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is,

$$(Rf)(v, p) := \int_{H(v,p)} f(x) \, dx$$

$$= \frac{\|v\|}{|v_n|} \int_{\mathbb{R}^{n-1}} f \left( x_1, \cdots, x_{n-1}, \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \, dx_1 \, dx_2 \cdots \, dx_{n-1}$$
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The Radon transform $(Rf)(v,p)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is,

$$\begin{align*}
(Rf)(v,p) &:= \int_{H(v,p)} f(x) \, dx \\
&= \frac{\| v \|}{|v_n|} \int_{\mathbb{R}^{n-1}} f \left( x_1, \ldots, x_{n-1}, \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \, dx_1 \, dx_2 \cdots dx_{n-1}
\end{align*}$$
Let $H(v, p)$ be a ("non–vertical") hyperplane in $\mathbb{R}^n$

$$H(v, p) := \{x : <v, x> = p\} = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{n} v_i x_i = p \right\}, \quad (v_n \neq 0)$$

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The Radon transform $(Rf)(v, p)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is,

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Radon transform

Let $H(v,p)$ be a (“non–vertical”) hyperplane in $\mathbb{R}^n$

$$H(v,p) := \{ x : \langle v, x \rangle = p \} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} v_i x_i = p \right\}, \quad (v_n \neq 0)$$

$$H(v,p) = \phi(\mathbb{R}^{n-1}), \quad x_n := \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i$$

$$\text{vol} J_{\phi} = \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{v_i}{v_n} \right)^2} = \frac{\| v \|}{|v_n|}$$

The Radon transform $(Rf)(v,p)$ of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is,

$$(Rf)(v,p) := \int_{H(v,p)} f(x) \, dx$$

$$= \frac{\| v \|}{|v_n|} \int_{\mathbb{R}^{n-1}} \left( x_1, \cdots, x_{n-1}, \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \, dx_1 dx_2 \cdots dx_{n-1}$$
\( \mathbb{R}^n \) is a union of (parallel) hyperplanes,

\[
\mathbb{R}^n = \bigcup_{p=-\infty}^{\infty} H(v,p), \quad \text{where } v \neq 0.
\]

Therefore an integral over \( \mathbb{R}^n \),

\[
\int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} f(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_n = \int_{-\infty}^{\infty} \frac{dp}{\|v\|} (\mathbb{R}f)(v,p)
\]

\[
= \frac{1}{|v_n|} \int_{-\infty}^{\infty} \left\{ \int_{\mathbb{R}^{n-1}} f \left( x_1, \ldots, x_{n-1}, \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \, dx_1 \, dx_2 \cdots dx_{n-1} \right\} \, dp
\]

where \( dp/\|v\| \) is the differential of the distance along \( v \).
$\mathbb{R}^n$ is a union of (parallel) hyperplanes, 

$$\mathbb{R}^n = \bigcup_{p=-\infty}^{\infty} H(v,p), \text{ where } v \neq 0.$$ 

Therefore an integral over $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} f(x_1, x_2, \cdots, x_n) \, dx_1 \, dx_2 \cdots \, dx_n = \int_{-\infty}^{\infty} \frac{dp}{\|v\|} (Rf)(v,p)$$

$$= \frac{1}{|v_n|} \int_{-\infty}^{\infty} \left\{ \int_{\mathbb{R}^{n-1}} f \left( x_1, \cdots, x_{n-1}, \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \, dx_1 \, dx_2 \cdots \, dx_{n-1} \right\} \, dp$$

where $dp/\|v\|$ is the differential of the distance along $v$. 
$\mathbb{R}^n$ is a union of (parallel) hyperplanes,

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Therefore an integral over $\mathbb{R}^n$,

$$
\int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} f(x_1, x_2, \cdots, x_n) \, dx_1 \, dx_2 \cdots dx_n = \int_{-\infty}^{\infty} \frac{dp}{\|v\|} (Rf)(v,p)
$$

$$
= \frac{1}{|v_n|} \int_{-\infty}^{\infty} \left\{ \int_{\mathbb{R}^{n-1}} f \left( x_1, \cdots, x_{n-1}, \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \, dx_1 \, dx_2 \cdots dx_{n-1} \right\} \, dp
$$

where $dp/\|v\|$ is the differential of the distance along $v$.
The unit ball & sphere in $\mathbb{R}^n$

* $\| \cdot \|$ the Euclidean norm,
  
  $B_n(r) := \{ x \in \mathbb{R}^n : \| x \| \leq r \}; B_n(1) = B_n$, the unit ball in $\mathbb{R}^n$,
  
  $S_n(r) := \{ x \in \mathbb{R}^n : \| x \| = r \}; S_n(1) = S_n$, the unit sphere,

  $V_n(r)$ and $V_n$ denote the volume of $B_n(r)$ and $B_n$,
  
  $A_n(r)$ and $A_n$ the area of $S_n(r)$ and $S_n$, resp.

For $n = 2, 3, \cdots$,

$$dV_n(r) = V'_n(r) \, dr = A_n(r) \, dr,$$

$$V_n(r) = V_n r^n, \quad V_n = \frac{A_n}{n},$$

$$A_n(r) = A_n r^{n-1}, \quad A_n = \frac{2 \pi^{n/2}}{\Gamma(n/2)},$$

where $\Gamma(\cdot)$ is the Gamma function

$$\Gamma(p) = \int_0^\infty x^{p-1} \, e^{-x} \, dx.$$
The unit ball & sphere in $\mathbb{R}^n$

- $\| \cdot \|$ the Euclidean norm,
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- For $n = 2, 3, \cdots$ ,
  
  \[
dV_n(r) = V'_n(r) \, dr = A_n(r) \, dr ,
  \]
  
  \[
  V_n(r) = V_n r^n , \quad V_n = \frac{A_n}{n} ,
  \]
  
  \[
  A_n(r) = A_n r^{n-1} , \quad A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} ,
  \]

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\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} \, dx .
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The unit ball & sphere in $\mathbb{R}^n$

- $\| \cdot \|$ the Euclidean norm,
  
  $\mathcal{B}_n(r) := \{ x \in \mathbb{R}^n : \| x \| \leq r \}$; $\mathcal{B}_n(1) = \mathcal{B}_n$, the unit ball in $\mathbb{R}^n$,

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- For $n = 2, 3, \cdots$,

  
  $$dV_n(r) = V'_n(r) \, dr = A_n(r) \, dr,$$

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$$\Gamma(p) = \int_0^\infty x^{p-1} \, e^{-x} \, dx.$$
The unit ball & sphere in $\mathbb{R}^n$

- $\| \cdot \|$ the Euclidean norm,
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  - $V_n(r)$ and $V_n$ denote the volume of $B_n(r)$ and $B_n$,
  - $A_n(r)$ and $A_n$ the area of $S_n(r)$ and $S_n$, resp.

For $n = 2, 3, \ldots$,

$\begin{align*}
dV_n(r) &= V_n'(r) \, dr = A_n(r) \, dr, \\
V_n(r) &= V_n \, r^n, \quad V_n = \frac{A_n}{n}, \\
A_n(r) &= A_n \, r^{n-1}, \quad A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},
\end{align*}$

where $\Gamma(\cdot)$ is the Gamma function

$\begin{align*}
\Gamma(p) &= \int_0^\infty x^{p-1} \, e^{-x} \, dx.
\end{align*}$
$S_n^+$ the upper hemisphere

\[ S_n^+ = \phi(B_{n-1}), \quad \phi = (\phi_1, \phi_2, \cdots, \phi_n) \]

\[ \phi_i(x_1, x_2, \cdots, x_{n-1}) = x_i, \quad i \in 1, n-1, \]

\[ \phi_n(x_1, x_2, \cdots, x_{n-1}) = \sqrt{1 - \sum_{i=1}^{n-1} x_i^2}. \]

The Jacobi matrix and its volume,

\[
J_{\phi} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-\frac{x_1}{x_n} & -\frac{x_2}{x_n} & \cdots & -\frac{x_{n-1}}{x_n}
\end{pmatrix}
\]

\[ \text{vol } J_{\phi} = \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{x_i}{x_n}\right)^2} = \frac{1}{|x_n|} = \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}}. \]
\( S_n^+ \) the upper hemisphere

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S_n^+ = \phi(B_{n-1}), \quad \phi = (\phi_1, \phi_2, \cdots, \phi_n)
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\phi_i(x_1, x_2, \cdots, x_{n-1}) = x_i, \quad i \in 1, n-1,
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\phi_n(x_1, x_2, \cdots, x_{n-1}) = \sqrt{1 - \sum_{i=1}^{n-1} x_i^2}.
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The Jacobi matrix and its volume,

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J_\phi = \begin{pmatrix}
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\vdots & \vdots & \ddots & \vdots \\
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\text{vol} J_\phi = \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{x_i}{x_n} \right)^2} = \frac{1}{|x_n|} = \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}}.
\]
$S_n^+ \text{ the upper hemisphere}$

$$S_n^+ = \phi (B_{n-1}), \quad \phi = (\phi_1, \phi_2, \cdots, \phi_n)$$

$$\phi_i(x_1, x_2, \cdots, x_{n-1}) = x_i, \ i \in 1, n-1,$$

$$\phi_n(x_1, x_2, \cdots, x_{n-1}) = \sqrt{1 - \sum_{i=1}^{n-1} x_i^2}.$$

The Jacobi matrix and its volume,

$$J_\phi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -x_1/x_n & -x_2/x_n & \cdots & -x_{n-1}/x_n \end{pmatrix}$$

$$\operatorname{vol} J_\phi = \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{x_i}{x_n} \right)^2} = \frac{1}{|x_n|} = \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}}.$$
$S_n^+$ the upper hemisphere

\[ S_n^+ = \phi(B_{n-1}), \quad \phi = (\phi_1, \phi_2, \ldots, \phi_n) \]

\[ \phi_i(x_1, x_2, \ldots, x_{n-1}) = x_i, \quad i \in \overline{1, n-1}, \]

\[ \phi_n(x_1, x_2, \ldots, x_{n-1}) = \sqrt{1 - \sum_{i=1}^{n-1} x_i^2}. \]

The Jacobi matrix and its volume,

\[
J_\phi = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-\frac{x_1}{x_n} & -\frac{x_2}{x_n} & \cdots & -\frac{x_{n-1}}{x_n}
\end{pmatrix}
\]

\[ \text{vol } J_\phi = \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{x_i}{x_n} \right)^2} = \frac{1}{|x_n|} = \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}}. \]
The area $A_n$ is twice the area of the “upper hemisphere”:

$$A_n = 2 \int_{\mathcal{B}_{n-1}} \frac{dx_1 dx_2 \cdots dx_{n-1}}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}} = 2 \int_0^1 \frac{dV_{n-1}(r)}{\sqrt{1 - r^2}}$$

$$= 2 \int_0^1 \frac{A_{n-1}(r) \, dr}{\sqrt{1 - r^2}} = 2 \int_0^1 \frac{A_{n-1} r^{n-2} \, dr}{\sqrt{1 - r^2}}$$

$$\therefore \frac{A_n}{A_{n-1}} = 2 \int_0^1 \frac{r^{n-2} \, dr}{\sqrt{1 - r^2}}$$

$$\therefore A_n = \frac{2 \pi \frac{n}{2}}{\Gamma\left(\frac{n}{2}\right)}$$

using well–known properties of the beta function,

$$B(p, q) := \int_0^1 (1 - x)^{p-1} x^{q-1} \, dx.$$
The unit sphere in $\mathbb{R}^n$ (cont’d)

The area $A_n$ is twice the area of the “upper hemisphere”:

\[
A_n = 2 \int_{B_{n-1}} \frac{dx_1 dx_2 \cdots dx_{n-1}}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}} = 2 \int_{r=0}^{1} \frac{dV_{n-1}(r)}{\sqrt{1 - r^2}}
\]

\[
= 2 \int_{r=0}^{1} \frac{A_{n-1}(r) dr}{\sqrt{1 - r^2}} = 2 \int_{r=0}^{1} \frac{A_{n-1} r^{n-2} dr}{\sqrt{1 - r^2}}
\]

\[
\therefore \quad \frac{A_n}{A_{n-1}} = 2 \int_{r=0}^{1} \frac{r^{n-2} dr}{\sqrt{1 - r^2}}
\]

\[
\therefore \quad A_n = \frac{2 \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}
\]

using well–known properties of the beta function,

\[
B(p, q) := \int_0^1 (1 - x)^{p-1} x^{q-1} dx.
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The area $A_n$ is twice the area of the “upper hemisphere”:

$$A_n = 2 \int_{B_{n-1}} dx_1 dx_2 \cdots dx_{n-1} \sqrt{1 - \sum_{i=1}^{n-1} x_i^2} = 2 \int_0^1 \frac{dV_{n-1}(r)}{\sqrt{1 - r^2}}$$

$$= 2 \int_0^1 \frac{A_{n-1}(r) dr}{\sqrt{1 - r^2}} = 2 \int_0^1 \frac{A_{n-1} r^{n-2} dr}{\sqrt{1 - r^2}}$$

$$\therefore \frac{A_n}{A_{n-1}} = 2 \int_0^1 \frac{r^{n-2} dr}{\sqrt{1 - r^2}}$$

$$\therefore A_n = \frac{2 \pi^{n/2}}{\Gamma \left( \frac{n}{2} \right)}$$

using well–known properties of the beta function,

$$B(p, q) := \int_0^1 (1 - x)^{p-1} x^{q-1} \, dx.$$
The unit sphere $S_n$ and an equatorial belt $T_n(\alpha)$, $\alpha > 0$

$$T_n(\alpha) = \{ x \in S_n : -\alpha \leq x_n \leq \alpha \}$$
\[ A(T_n(\alpha)) := \text{area of } T_n(\alpha) \]

\[ \text{Prob}\{X \in T_n(\alpha)\} = \frac{A(T_n(\alpha))}{A_n} \]

\[ A(T_n(\alpha)) = 2 \int_{(1-\alpha^2)^{1/2}}^{1} \frac{dv_{n-1}(r)}{\sqrt{1 - r^2}} \]

\[ = 2A_{n-1} \int_{(1-\alpha^2)^{1/2}}^{1} \frac{r^{n-2}}{\sqrt{1 - r^2}} \, dr , \]

\[ = A_{n-1} \int_{1-\alpha^2}^{1} x^{(n-3)/2} (1 - x)^{-1/2} \, dx , \text{ for } x = r^2 . \]

\[ \therefore \text{Prob}\{X \in T_n(\alpha)\} = \frac{A_{n-1}}{A_n} \int_{1-\alpha^2}^{1} x^{(n-3)/2} (1 - x)^{-1/2} \, dx , \]

\[ = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{1-\alpha^2}^{1} x^{(n-3)/2} (1 - x)^{-1/2} \, dx . \]
\[
A(T_n(\alpha)) := \text{area of } T_n(\alpha)
\]

\[
\Pr\{X \in T_n(\alpha)\} = \frac{A(T_n(\alpha))}{A_n}
\]

\[
A(T_n(\alpha)) = 2 \int_{(1-\alpha^2)^{1/2}}^1 \frac{dv_{n-1}(r)}{\sqrt{1-r^2}}
\]

\[
= 2A_{n-1} \int_{(1-\alpha^2)^{1/2}}^1 \frac{r^{n-2}}{\sqrt{1-r^2}} \, dr,
\]

\[
= A_{n-1} \int_{1-\alpha^2}^1 x^{(n-3)/2} (1-x)^{-1/2} \, dx, \text{ for } x = r^2.
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\[
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\]

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\]
\( \text{Prob}\{X \in T_n(\alpha)\}, \ X \sim \text{Unif}(S_n) \)

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\]
### Prob \( \{ X \in T_n \left( \frac{k}{\sqrt{n}} \right) \} \) for \( X \sim \text{Unif} (S_n) \)

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Values of Prob \( \{ X \in T_n \left( \frac{k}{\sqrt{n}} \right) \} \) for some \( k, n \)

### Theorem

Let \( u \in S_n \) be fixed, \( x \sim \text{Unif}(S_n) \). Then, as \( n \to \infty \),

\[
\text{Prob} \left\{ |\langle u, y \rangle| \leq \frac{k}{\sqrt{n}} \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-x^2/2} \, dx.
\]
Problems $\{X \in T_n\left(\frac{k}{\sqrt{n}}\right)\}$ for $X \sim \text{Unif}(S_n)$

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**Theorem**

Let $u \in S_n$ be fixed, $x \sim \text{Unif}(S_n)$. Then, as $n \to \infty$,

$$\text{Prob} \left\{ |\langle u, y \rangle| \leq \frac{k}{\sqrt{n}} \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-x^2/2} \, dx.$$
1 Motivation
2 Representations
3 Factorizations
4 A multilinear setting
5 Angles
6 Integrals
7 Concentration of measure
8 Probability
9 Application
10 References
Probability density of a function of RV's

Let \((X_1, \cdots, X_n)\) be RV's with a given joint density \(f_X(x_1, \cdots, x_n)\), and let \(h: \mathbb{R}^n \to \mathbb{R}\) be a well-behaved function, in particular \(\frac{\partial h}{\partial x_n} \neq 0\).

It is required to find the density of the RV

\[ Y = h(X_1, \cdots, X_n). \]

Solve for \(x_n\),

\[ x_n = h^{-1}(y|x_1, \cdots, x_{n-1}) \]

Change variables from \(\{x_1, \cdots, x_n\}\) to \(\{x_1, \cdots, x_{n-1}, y\}\), and use

\[ \det \left( \frac{\partial (x_1, \cdots, x_n)}{\partial (x_1, \cdots, x_{n-1}, y)} \right) = \frac{\partial h^{-1}}{\partial y} \]

to get the density of \(Y = h(X_1, \cdots, X_n)\)

\[ f_Y(y) = \int_{\mathbb{R}^{n-1}} f_X(x_1, \cdots, x_{n-1}, h^{-1}(y|x_1, \cdots, x_{n-1})) \left| \frac{\partial h^{-1}}{\partial y} \right| \, dx_1 \cdots dx_{n-1} \]
Probability density of a function of RV’s

\((X_1, \cdots, X_n)\) RV’s with a given joint density \(f_X(x_1, \cdots, x_n)\),
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\]
A surface integral on \( \mathbf{v}(y) \)

Let \( \mathbf{V}(y) \) be the surface in \( \mathbb{R}^n \) given by

\[
\begin{pmatrix}
  x_1 \\
  \\
  x_{n-1} \\
  x_n
\end{pmatrix}
= \begin{pmatrix}
  x_1 \\
  \\
  x_{n-1} \\
  h^{-1}(y|x_1, \cdots, x_{n-1})
\end{pmatrix}
= \phi \begin{pmatrix}
  x_1 \\
  \\
  x_{n-1}
\end{pmatrix}
\]

Then the surface integral of \( f_X \) over \( \mathbf{V}(y) \) is given by

\[
\int_{\mathbf{V}(y)} f_X = \\
\int_{\mathbb{R}^{n-1}} f_X(x_1, \cdots, x_{n-1}, h^{-1}(y|x_1, \cdots, x_{n-1})) \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{\partial h^{-1}}{\partial x_i} \right)^2} \, dx_1 \cdots dx_{n-1}
\]
A surface integral on $\mathbf{v}(y)$

Let $\mathbf{V}(y)$ be the surface in $\mathbb{R}^n$ given by

$$
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_{n-1} \\
  x_n
\end{pmatrix} = \begin{pmatrix}
  x_1 \\
  \vdots \\
  x_{n-1} \\
  h^{-1}(y| x_1, \cdots, x_{n-1})
\end{pmatrix} = \phi
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_{n-1}
\end{pmatrix}
$$

Then the surface integral of $f_X$ over $\mathbf{V}(y)$ is given by

$$
\int_{\mathbf{V}(y)} f_X = \int_{\mathbb{R}^{n-1}} f_X(x_1, \cdots, x_{n-1}, h^{-1}(y| x_1, \cdots, x_{n-1})) \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{\partial h^{-1}}{\partial x_i} \right)^2} \, dx_1 \cdots dx_{n-1}
$$
Theorem

If the ratio

\[ A := \frac{\frac{\partial h^{-1}}{\partial y}}{\sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial h^{-1}}{\partial x_i}\right)^2}} \]

does not depend on \( x_1, \cdots, x_{n-1} \), (C)

then

\[ f_Y(y) = A \int_{V(y)} f_X \]

Functions \( h : \mathbb{R}^n \to \mathbb{R} \) that satisfy (C).

\[
\begin{align*}
h(x_1, \cdots, x_n) &= \sum_{i=1}^{n} v_i x_i, \quad v_n \neq 0 \\
h(x_1, \cdots, x_n) &= \sum_{i=1}^{n} x_i^2
\end{align*}
\]
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If the ratio
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Functions \( h : \mathbb{R}^n \to \mathbb{R} \) that satisfy (C).

\[ h(x_1, \cdots, x_n) = \sum_{i=1}^{n} v_i x_i, \quad v_n \neq 0 \]
\[ h(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^2 \]
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If the ratio

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Proof

Compare

\[ f_Y(y) = \int_{\mathbb{R}^{n-1}} f_X(x_1, \ldots, x_{n-1}, h^{-1}(y|x_1, \ldots, x_{n-1})) \left| \frac{\partial h^{-1}}{\partial y} \right| dx_1 \cdots dx_{n-1} \]

\[
\int_{V(y)} f_X = \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{\partial h^{-1}}{\partial x_i} \right)^2} \ dx_1 \cdots dx_{n-1}
\]
Proof

Compare

\[ f_Y(y) = \int_{\mathbb{R}^{n-1}} f_X(x_1, \cdots, x_{n-1}, h^{-1}(y|x_1, \cdots, x_{n-1})) \left| \frac{\partial h^{-1}}{\partial y} \right| dx_1 \cdots dx_{n-1} \]

\[ \int f_X = \int_{\mathbf{V}(y)} \]

\[ \int_{\mathbb{R}^{n-1}} f_X(x_1, \cdots, x_{n-1}, h^{-1}(y|x_1, \cdots, x_{n-1})) \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{\partial h^{-1}}{\partial x_i} \right)^2} \, dx_1 \cdots dx_{n-1} \]
Check condition (C) for \( h(x_1, \cdots, x_n) := \sum_{i=1}^{n} v_i x_i, \ v_n \neq 0 \)

\[ x_n = h^{-1}(y| x_1, \cdots, x_{n-1}) := \frac{y}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \]

with \[ \frac{\partial h^{-1}}{\partial y} = \frac{1}{v_n}, \ \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{\partial h^{-1}}{\partial x_i} \right)^2} = \sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{v_i}{v_n} \right)^2} = \frac{\|v\|}{|v_n|} \]

The density of \( Y = \sum v_i X_i \) can be expressed as the integral of \( f_X \) on the hyperplane

\[ H(v, y) := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} v_i x_i = y \right\} \]

\[ \therefore f_Y(y) = \frac{1}{\|v\|} (Rf_X)(v, y) \]

\[ = \frac{1}{|v_n|} \int_{\mathbb{R}^{n-1}} f_X \left( x_1, \cdots, x_{n-1}, \frac{y}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \ dx_1 dx_2 \cdots dx_{n-1} \]
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The density of \( Y = \sum v_i X_i \) can be expressed as the integral of \( f_X \) on the hyperplane

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\[
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\]

\[
= \frac{1}{|v_n|} \int_{\mathbb{R}^{n-1}} f_X \left( x_1, \cdots, x_{n-1}, \frac{y}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i \right) \, dx_1 \, dx_2 \cdots \, dx_{n-1}
\]
Check condition (C) for \( h(x_1, \cdots, x_n) := \sum_{i=1}^{n} x_i^2 \)

Two solutions of \( y = h(x_1, \cdots, x_n) := \sum_{i=1}^{n} x_i^2 \) for \( x_n \),

\[
x_n = h^{-1}(y|x_1, \cdots, x_{n-1}) := \pm \sqrt{y - \sum_{i=1}^{n-1} x_i^2}
\]

with \( \frac{\partial h^{-1}}{\partial y} = \pm \frac{1}{2 \sqrt{y - \sum_{i=1}^{n-1} x_i^2}} \)

\[
\sqrt{1 + \sum_{i=1}^{n-1} \left( \frac{\partial h^{-1}}{\partial x_i} \right)^2} = \frac{\sqrt{y}}{\sqrt{y - \sum_{i=1}^{n-1} x_i^2}}
\]

Therefore the density of \( \sum X_i^2 \) is expressed in terms of the integral of \( f_X \) on the sphere \( S_n(\sqrt{y}) \) of radius \( \sqrt{y} \).
Corollary

Let \( (X_1, \cdots, X_n) \) have joint density \( f_X(x_1, \cdots, x_n) \). The density of

\[
Y = \sum_{i=1}^{n} X_i^2 \quad \text{is} \quad f_Y(y) = \frac{1}{2\sqrt{y}} \int_{S_n(\sqrt{y})} f_X
\]

the integral is over the sphere \( S_n(\sqrt{y}) \) of radius \( \sqrt{y} \),

\[
\int_{S_n(\sqrt{y})} f_X = \int_{B_{n-1}(\sqrt{y})} \left[ f_X \left( x_1, \cdots, x_{n-1}, \sqrt{y - \sum_{i=1}^{n-1} x_i^2} \right) + f_X \left( x_1, \cdots, x_{n-1}, -\sqrt{y - \sum_{i=1}^{n-1} x_i^2} \right) \right] \frac{\sqrt{y} \ dx_1 \cdots dx_{n-1}}{\sqrt{y - \sum_{i=1}^{n-1} x_i^2}}
\]

The factor \( 1/2\sqrt{y} \) is the width of the spherical shell bounded by the two spheres \( S_n(\sqrt{y}) \) and \( S_n(\sqrt{y+dy}) \), i.e. the difference of radii \( \sqrt{y+dy} - \sqrt{y} \approx \frac{dy}{2\sqrt{y}} \)
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\]

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The factor \(1/2\sqrt{y}\) is the width of the spherical shell bounded by the two spheres \(S_n(\sqrt{y})\) and \(S_n(\sqrt{y+dy})\), i.e. the difference of radii \(\sqrt{y+dy} - \sqrt{y} \approx \frac{dy}{2\sqrt{y}}\)
$Y = \{Y_1, Y_2, \cdots, Y_N\}$ a set of known faces;
$Y = \text{a face.}$

Question: $Y \in Y$?
Answer: Yes, if $\min_{i \in 1:N} \text{vol}(Y - Y_i) < \varepsilon$
No, otherwise.

VM = volume measure  
FD = Frpbenius distance  
YD = Yang distance  
AMD = assembled matrix distance  
BSM = boosted similarity measure
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THANKS FOR YOUR ATTENTION

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